# MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE National University "Zaporizhzhia Polytechnic" 

## LABORATORY GUIDELINES

## in the discipline

## "Introduction to multi-criteria decision making methods"

for students of specialty 123 "Computer Engineering" according to the educational program "Specialized Computer Systems" for all forms of education

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## 1 LABORATORY WORK № 1 DECISION-MAKING WITH LINEAR PROGRAMMING

The purpose of the work: to learn how to apply the methods of graphic and computer solving of linear programming problems.

### 1.1 Theoretical information and problem formulation

In this lab, the linear programming (LP) problem solution with two variables is discussed. Although two-variable problems are practically nonexistent in practice, their consideration provides an understanding of linear programming basic concepts.

All LP problems consist of three components:

1. The decision variables we need to determine.
2. The objective function that we need to optimize (maximize or minimize).
3. Constraints that the solution must satisfy.

The idea of linear programming methods is most understandable in its graphical interpretation. In this lab, we will demonstrate and learn a graphical method of LP problem solving and solving using the MS Excel package.

Hypothetical problem
Let some Manufacturer produce two types of products, for example, chairs and tables. These products can be sold at profits of $\$ 80$ and $\$ 150$, respectively. It takes 4 units of labour time, 3 units of wood and 5 units of varnish to produce one chair. Producing one table requires 6 units of labour time, 10 units of wood and 50 units of varnish. The manufacturer is limited in the possibility of using labour time and resources. Namely, the manufacturer has 132 units of labour time, 120 units of wood, and 500 units of varnish. The manufacturer would like to determine how many chairs and tables to produce in order to maximize profits.

Problem formulation
Let $x_{1}$ be the number of chairs produced and $x_{2}$ be the number of tables produced. Then the mathematical formulation of the linear programming problem will be:
maximize

$$
\begin{equation*}
f(x)=80 x_{1}+150 x_{2} \tag{1.1}
\end{equation*}
$$

subject to

$$
\begin{align*}
& 4 x_{1}+6 x_{2} \leq 132  \tag{1.2}\\
& 3 x_{1}+10 x_{2} \leq 120  \tag{1.3}\\
& 5 x_{1}+50 x_{2} \leq 500  \tag{1.4}\\
& x_{1} \geq 0, x_{2} \geq 0 \tag{1.5}
\end{align*}
$$

The restrictions (1.2)-(1.5) are referred to as constraints. The restrictions (1.5) are also referred to as nonnegativity constraints. Restriction (1.2) is the constraint by labour time, restriction (1.3) is the constraint by wood and restriction (1.4) is the constraint by varnish. Any values of $x_{1}$ and $x_{2}$ that satisfy all four constraints constitute a feasible solution. Otherwise, the solution is infeasible. The best feasible solution that maximizes $f(x)$ is the optimal solution.

### 1.2 Graphical LP problem solution

## Determination of the Feasible Solution Space

Consider the nonnegativity constraints $x_{1} \geq 0$ and $x_{2} \geq 0$. In Fig. 1.1, the horizontal axis $x_{1}$ and the vertical axis $x_{2}$ represent the chairs and tables, respectively. Thus, the nonnegativity constraints restrict the variables to the first quadrant (above the $x_{1}$-axis and to the right of the $x_{2}$-axis).

To account for the remaining four constraints, replace each inequality with an equation, and then draw the resulting straight line by locating two distinct points. For example, after replacing $4 x_{1}+6 x_{2} \leq 132$ with the straight line $4 x_{1}+6 x_{2}=132$, two distinct points are determined by setting $x_{1}=0$ to obtain $x_{2}=132 / 6=22$ and then by setting $x_{2}=0$ to obtain $x_{1}=33$. Thus the line $4 x_{1}+6 x_{2}=132$ passes through $(0 ; 22)$ and $(32 ; 0)$. The resulting straight line divides the plane into two half-planes. Only one of these two halves satisfies the inequality. To determine the correct side, designate any point not lying on the straight line as a reference point. If the chosen reference point satisfies the inequality, then its side is feasible.

A similar application of the reference-point procedure to the remaining constraints (1.3) and (1.4) produces the feasible solution space ABCDE in which all the constraints are satisfied (Fig. 1.1). All points outside the boundary of the area $A B C D E$ are infeasible.


Figure 1.1 - Determination of the feasible solution space

## Determination of the Optimum Solution

The number of solution points in the feasible space ABCDE in Fig. 1.1 is infinite, clearly precluding the use of exhaustive enumeration. Thus the procedure is needed to determine the optimum solution.

The direction in which the profit function $f(x)=80 x_{1}+150 x_{2}$ increases is determined by assigning arbitrarily increasing values to $f(x)$. In Fig. 1.2, the two lines $80 x_{1}+150 x_{2}=1000$ and $80 x_{1}+150 x_{2}=2000$ corresponding to $f(x)=1000$ and $f(x)=2000$ depict the direction in which $f(x)$ increases. Moving in that direction, the optimum solution occurs at $\mathbf{C}$ because it is the feasible point in the solution space beyond which any further increase will render an infeasible solution Fig. 1.2.


Figure 1.2 - Determination of the optimum solution
The values of $x_{1}$ and $x_{2}$ associated with the optimum point C are determined by solving the equations associated with lines (1.1) and (1.2):

$$
\begin{aligned}
& 4 x_{1}+6 x_{2}=132 \\
& 3 x_{1}+10 x_{2}=120
\end{aligned}
$$

The solution is $x_{1}=27$ and $x_{2}=4$ with associated profit $f(x)=\$ 2755$.

### 1.3 Graphical LP problem solution with Scilab

Let us demonstrate how graphically solve the LP problem in the Scilab mathematical package. For the convenience of presenting graphs, let us denote the number of chairs as $x$ and the number of tables as $y$. Then the objective function (1.1) and constraints (1.2)-(1.5) will take the form:

$$
\begin{align*}
& f(x, y)=80 x+150 y  \tag{1.6}\\
& 4 x+6 y \leq 132  \tag{1.7}\\
& 3 x+10 y \leq 120  \tag{1.8}\\
& 5 x+50 y \leq 500  \tag{1.9}\\
& x \geq 0, y \geq 0 \tag{1.10}
\end{align*}
$$

To graph straight lines on the plane $x 0 y$, we transform these inequalities for constraints into equalities

$$
\begin{align*}
& 4 x+6 y=132  \tag{1.11}\\
& 3 x+10 y=120  \tag{1.12}\\
& 5 x+50 y=500 \tag{1.13}
\end{align*}
$$

To implement straight lines in Scilab, we need an additional three variables $y 1, y 2, y 3$ that will take values along the $y$-axis when plotting straight lines (1.11), (1.12) and (1.13). With this in mind, we write the equations of these straight lines in the form:

$$
\begin{gathered}
4 x+6 y 1=132 \\
3 x+10 y 2=120 \\
5 x+50 y 3=500
\end{gathered}
$$

or after transformation we get:

$$
\begin{gather*}
y 1=(132-4 x) / 6  \tag{1.14}\\
y 2=(120-3 x) / 10  \tag{1.15}\\
y 3=(500-5 x) / 50 \tag{1.16}
\end{gather*}
$$

The code for a possible graph implementation of straight lines in Scilab is shown in Listing 1.1. The result of executing this code is shown in Fig. 1.3 and Fig. 1.4. The area inside the polygon ABCDE is the feasible solutions space.

| Listing 1.1 - The LP problem solution in the Scilab package |  |
| :---: | :---: |
| 1 | RX=35; h=1.0; //Range and step of variable $x$ change |
| 2 | RY=25; //Range of variables yl,y2,y3 change |
| 3 | $\mathrm{x}=0$ : $\mathrm{h}: ~ \mathrm{RX}$ |
| 4 | A11 $=4$; $A 12=6 ; ~ A 1=132 ;$ |
| 5 | A21 $=3.0$; $\mathrm{A} 22=10$; $\mathrm{A} 2=120$; |
| 6 | A31=5; $\mathrm{A} 32=50$; $\mathrm{A} 3=500$; |
| 7 |  |
| 8 | $\mathrm{y} 1=(\mathrm{A} 1-\mathrm{A} 11 * \mathrm{x}) / \mathrm{A} 12$; |
| 9 | $\mathrm{y} 2=(\mathrm{A} 2-\mathrm{A} 21 * \mathrm{x}) / \mathrm{A} 22$; |
| 10 | $\mathrm{y} 3=(\mathrm{A} 3-\mathrm{A} 31 * \mathrm{x}) / \mathrm{A} 32$; |
| 11 | //Draw straight lines of constraints |
| 12 | plot (x,y1,'-*r', x, y2,'-og', x,y3, '-^b'); |
| 13 | //Customizing the graphics window |
| 14 | g=gca () ; |
| 15 | g.font_size=4; |
| 16 | g.x_label.font_size=4; |
| 17 | g.y_label.font_size=4; |
| 18 | xlabel ('chairs'); |
| 19 | ylabel ('tables'); |
| 20 | g.data_bounds = [0;RX;0;RY]; |
| 21 | xgrid(); |
| 22 |  |
| 23 | //Draw the profit line $f=1000$ |
| 24 | C1=80; C2=150; |
| 25 | $\mathrm{f}=1000$; |
| 26 | $\mathrm{px}=\mathrm{f} / \mathrm{C} 1 ; \mathrm{py=f} / \mathrm{C} 2$; |
| 27 | plot ([px,0], [0, py],'--k','LineWidth', 2) ; |
| 28 | legend (['labour';'wood';'varnish';'profit'],1); |
| 29 | //Draw the profit line $f=2000$ |
| 30 | $\mathrm{f}=2000$; |
| 31 | $\mathrm{px}=\mathrm{f} / \mathrm{C} 1 ; \mathrm{py=f} / \mathrm{C} 2$; |
| 32 | plot ([px,0], [0, py],'--k', 'LineWidth', 2) ; |
| 33 | //Calc an optimal solution |
| 34 | A $=[\mathrm{A} 11, \mathrm{~A} 12$; $\mathrm{A} 21, \mathrm{~A} 22] ; \mathrm{B}=[-\mathrm{A} 1 ;-\mathrm{A} 2]$; |
| 35 | [x_opt] =linsolve (A, B) ; |
| 36 | //Calc the profit |
| 37 | $\mathrm{f}=\mathrm{C} 1$ *x_opt (1) + C2*x_opt (2); |
| 38 | ```printf("%d %d %d\n", round(x_opt(1)),round(x_opt(2)),round(f));``` |
| 39 |  |
| 40 | //Draw an optimal profit line |
| 41 |  |



Figure 1.3 - The feasible solution space
As seen in Fig. 1.4, the optimal solution is at the point of intersection of lines (1.15) and (1.16). Therefore, the solution of the equations for these lines is optimal. To find the values of $x$ and $y$ at this point, we solve the system of equations (1.15) and (1.16) using the Scilab linsolve (A, k) function. This function is intended for solving linear systems of the $\mathbf{A x}-\mathbf{k}=0$ type. In our case, matrix $\mathbf{A}$ and vector $\mathbf{k}$ are:

$$
\mathbf{A}=\left(\begin{array}{cc}
4 & 6  \tag{1.17}\\
3 & 10
\end{array}\right), \mathbf{k}=\binom{-132}{-120}
$$

In this case, the solution of the system of equations (1.15) and (1.16) gives: $y 1=27, y 2=4$. Substituting these values into the objective function (1.6), taking into account that $x=y 1$ (chairs) and $y=y 2$ (tables), gives the value of the manufacturer's maximum profit $f(x, y)=\$ 2755$.


Figure 1.4 - The profit lines

### 1.4 LP problem solution with MS Excel

In practice, typical LP problems can be solved by many computer tools. One of these tools is MS Excel Solver. Figure 1.5 shows the layout of the data for the LP problem, described above. Input data cells B3:B5 contain the stock of resources. Cells E3:E5 and F3:F5 represent coefficients from equations (1.2)-(1.4), i.e. the units of resource required for producing one chair and one table. In cells E6:F6 there are profits from the sale of product units. These values correspond to the coefficients of formula (1.1). Tab. 1.1 shows the formulas that must be entered in the spreadsheet cells.

Table 1.1 - Spreadsheet cells and formulae

| Cells | Formulae | Descriptions |
| :--- | :--- | :--- |
| C3 | E3*\$E\$9+F3*\$F\$9 | Constraint by labour |
| C4 | E4*\$E\$9+F4*\$F\$9 | Constraint by wood |
| C5 | E5*\$E\$9+F5*\$F\$9 | Constraint by varnish |
| F11 | E6*\$E\$9+F6*\$F\$9 | Objective function |
| D3, D4, D5 (Optional) | B3-C3, B4-C4, B5-C5 | Rest of resources |


| 4 | A | B | C | D | E | F |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Resource type | Stock | Used | Rest | Resource for unit manufa | required product acturing |
| 2 |  |  |  |  | Chairs | Tables |
| 3 | Labour | 132 |  |  | 4 | 6 |
| 4 | Wood | 120 |  |  | 3 | 10 |
| 5 | Varnish | 500 |  |  | 5 | 50 |
| 6 | Profit from the sale of a unit of product |  |  |  | 80 | 150 |
| 7 |  |  |  |  |  |  |
| 8 |  |  |  |  |  |  |
| 9 | The optimal number of product units |  |  |  |  |  |
| 10 |  |  |  |  |  |  |
| 11 | Profit |  |  |  |  |  |

Figure 1.5 - Input data of LP problem
Click Data $\rightarrow$ Solver from the spreadsheet menu bar to access the Solver parameters dialogue box (shown in Fig. 1.6). Next, update the dialogue box as follows:

- Set Target Cell:\$F\$11
- Equal To: Max
- By Changing Cells: \$E\$9:\$F\$9

This information tells Solver that the LP variables (cells $\$ \mathrm{E} \$ 9$ and $\$ \mathrm{~F} \$ 9$ ) are determined by maximizing the objective function in cell $\$ \mathrm{~F} \$ 11$.

To set up constraints, click $\boldsymbol{A d} \boldsymbol{d}$ in dialogue box to display the $\boldsymbol{A d d}$ Constraint box and then enter the left-hand side and right-hand side of the constraints as follows:
\$C $\$ 3: \$ \mathrm{C} \$ 5<=\$ \mathrm{~B} \$ 3: \$ \mathrm{~B} \$ 5$
\$E\$9:\$F\$9 => 0
\$E\$9:\$F\$9 = integer
Click Options in the Solver Parameters box to access Solver Options (see Fig. 1.7) and then check Assume Linear Model and click $\boldsymbol{O K}$.

Click Solve on Solver Parameters. A new dialogue box, Solver Results, then gives the status of the solution. If the LP problem setup is correct, the optimum value will appear in cell F11 and the values of numbers Chair and Tables will go to cells E9 and F9, respectively (Fig. 1.8).

## Solver Parameters



Figure 1.6 - Setting parameters of the LP problem


Figure 1.7 - Setting optional conditions and type of solver

| 4 | A | B | C | D | E | F |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Resource type | Stock | Used | Rest | Resource for unit manufa | required <br> product <br> acturing |
| 2 |  |  |  |  | Chairs | Tables |
| 3 | Labour | 132 | 132 | 0 | 4 | 6 |
| 4 | Wood | 120 | 120 | 0 | 3 | 10 |
| 5 | Varnish | 500 | 327 | 173 | 5 | 50 |
| 6 | Profit from the sale of a unit of product |  |  |  | 80 | 150 |
| 7 |  |  |  |  |  |  |
| 8 |  |  |  |  |  |  |
| 9 | The optimal number of product units |  |  |  | 27 | 4 |
| 10 |  |  |  |  |  |  |
| 11 | Profit |  |  |  |  | 2755 |

Figure 1.8 - The result of solving the LP problem

### 1.5 Tasks to complete

1. Implement the example given in this tutorial with tools (MS Excel, and Scilab). Analyse the results.
2. Solve the problem according to the variant from item 1.6 "Individual tasks", for which build a mathematical model and prepare data for processing in the Scilab environment and the Excel package. Compare the results obtained.
3. Create a progress report.

### 1.6 Individual tasks

## Problem statement.

To manufacture two types of products A and B, three types of raw materials are used: R1, R2, and R3. The denotations of raw material resources, their consumption rates per unit of production, and the profit received from a unit of product are given in Tab. 1.2.

Table 1.2 - The denotations of data

| Raw materials | Consumption rates per <br> unit of production |  | Resource Stock |
| :---: | :---: | :---: | :---: |
|  | A | B |  |
| R1 | A1 | B1 | S1 |
| R2 | A2 | B2 | S2 |
| R3 | A3 | B3 | S3 |
| Profit | CA | CB |  |

Determine the optimal plan for the release of products from the condition profit maximization

Problem variants are given in Tab. 1.3
Table 1.3 - Numerical data for problem variants No 1-4

| No | A1 | $\mathbf{B 1}$ | $\mathbf{A 2}$ | $\mathbf{B 2}$ | $\mathbf{A 3}$ | $\mathbf{B 3}$ | S1 | S2 | S3 | CA | CB |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{1}$ | 2 | 4 | 1 | 8 | 7 | 4 | 120 | 280 | 240 | 10 | 14 |
| $\mathbf{2}$ | 2 | 3 | 1 | 8 | 4.5 | 4 | 120 | 180 | 240 | 20 | 25 |
| $\mathbf{3}$ | 5 | 4 | 2 | 4 | 8 | 3 | 120 | 100 | 180 | 15 | 20 |
| $\mathbf{4}$ | 7 | 4 | 4 | 8 | 8 | 3 | 224 | 200 | 180 | 25 | 20 |

### 1.7 Contents of the progress report

The report should have the following components:
-title page;
-the purpose of the work;
-solution of the problem according to the variant of an individual task;
-the result of the tasks.

## 2 LABORATORY WORK №2 SOLVING THE TRANSPORTATION PROBLEM

The purpose of the work: to learn how to apply the methods of solving the linear programming problem, called the transportation problem.

### 2.1 Theoretical information and description of the transportation problem

In the presence of several suppliers and several consumers, there are always alternative plans for the transportation of products. These plans vary in cost. So, the problem arises of finding such a plan for the transportation of products, under which the total transportation costs would be the lowest. Problems of this type are called transportation problems. The transportation problem is a special type of linear programming problem.

The mathematical model of the transport-type problem was first formulated in 1941 by F. Hitchcock in his article "Distribution of a Product from Several Sources to Numerous Localities".

Problem statement
Some homogeneous product, which is concentrated in $m$ suppliers, $a_{i}$ units each $(i=1,2, \ldots, m)$, is needed by $n$ consumers in the number of $b_{j}$ units $(j=1,2, \ldots n)$. The cost of transportation of a product unit from the $i$-th suppliers to the $j$-th consumers is $c_{i j}$. It is necessary to find such a plan for the transportation of products (from where, where and how much) in which the total cost of all transportation was minimum. Denote by $x_{i j}$ the number units of product that transportation from the $i$-th suppliers to the $j$-th consumers.

Figure 2.1 shows a possible example of the transportation problem, where $A_{i}$ is the suppliers and $B_{j}$ is the consumers, and Fig. 2.2 shows a model of this problem.


Figure 2.1 - A possible example of the problem


Figure 2.2 - The model of trasportation problem

The mathematical statement of transportation problem Find such

$$
\begin{equation*}
x_{i j} \geq 0, i=1,2, \ldots, m ; j=1,2, \ldots n \tag{2.1}
\end{equation*}
$$

that satisfy the conditions

$$
\begin{align*}
& \sum_{j=1}^{n} x_{i j} \leq a_{i}  \tag{2.2}\\
& \sum_{i=1}^{m} x_{i j} \geq b_{j} \tag{2.3}
\end{align*}
$$

and minimize the total cost of product transportation

$$
\begin{equation*}
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i j} x_{i j} \rightarrow \min \tag{2.4}
\end{equation*}
$$

Inequality (2.2) is the constraint "exported from suppliers no more than available"; inequality (2.3) is the constraint "delivered to consumers no less than necessary"; the sum in expression (2.4) is the objective function.

In this laboratory work, we consider the so-called balanced transportation problem, in which there is another constraint

$$
\begin{equation*}
\sum_{i=1}^{m} a_{i}=\sum_{j=1}^{n} b_{j} \tag{2.5}
\end{equation*}
$$

which means that total demand equals the total supply.
It is convenient to represent the transportation problem in the form of a so-called transportation table (Fig. 2.3):


Figure 2.3 - Transportation table

### 2.2 An example of solving a transportation problem

The special structure of the transportation problem allows building the starting basic solution using several methods: Northwest-corner method,

Least-cost method, Vogel's approximation method, Russell's approximation method. In this laboratory work, the Least-cost method is demonstrated.

The idea of the Least-cost method is as follows: at each step, from the number of unfilled cells of the transportation table, we choose a cell with the minimum cost, fill it and exclude from further consideration the column or row in which this cell is located, depending on whether the corresponding needs are satisfied or exhausted appropriate stocks.

Let us consider an example. Let there be a transportation problem in which suppliers $A_{1}, A_{2}$ and $A_{3}$ supply equivalent type (homogeneous) products to consumers $B_{1}, B_{2}, B_{3}, B_{4}$ and $B_{5}$. Let us present the problem in the form of a transportation table (see Fig. 2.4)


Figure 2.4 - Example of transportation table

In the transportation table, in column $a_{i}$, product stocks at suppliers are indicated. Cells of row $b_{j}$ contain values of consumer needs.

In this example, the transportation problem is balanced, that is, the total number of product stocks is equal to the total number of consumer needs.

The cost of transportation is indicated in each cell of the table in the upper right corner. For example, in cell ( $A_{2}, B_{3}$ ), the number 6 corresponds to the conditional cost (or distance) of transporting a unit of manufactured product from supplier $A_{2}$ to consumer $B_{3}$. Column $a_{i}^{*}$ and row $b_{j}^{*}$ of this
table are optional, they are entered for the convenience of controlling the distribution of products in the transportation plan.

In the cells of the $a_{i}^{*}$ column, we will indicate the number of products left by the supplier (which have not yet been transported), and in the cells of the $b_{j}^{*}$ row, we will write down the number of products that the consumer has already received.

Consider several steps of the Least-cost method. Let us find the "cheapest" transportation. It is clear that this is cell $\left(A_{3}, B_{2}\right)$. Let us enter in this cell 40 units of product to satisfy the consumer's demand as much as possible (see Fig. 2.5). As a result of this action, the number of products in cell $a_{3}^{*}$ will decrease by 40 units. In this case, the consumer is completely satisfied, so column $B_{2}$ can be excluded from further consideration.


Figure 2.5 - Actions with cell $\left(\mathrm{A}_{3}, \mathrm{~B}_{2}\right)$

Let us find the next "cheapest" transportation. This is cell ( $A_{2}, B_{1}$ ). We will enter the maximum number of products into it (see Fig. 2.6). In this case, this value fully corresponds to the stock of supplier $A_{2}$, so the second row can be excluded from further consideration.

Continuing this process of distribution of products between transportation in a similar way (see Fig. 2.7 - Fig. 2.11), we will get the last basic plan of transportation.

| $\mathrm{A}_{1}$ | $\mathrm{B}_{1}$ | $\mathrm{B}_{2}$ | $\mathrm{B}_{3}$ | $\mathrm{B}_{4}$ | $\mathrm{B}_{5}$ | $\mathrm{a}_{\mathrm{i}}$ | $\mathrm{a}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 5 | 12 | 4 | 6 |  | 160 | 180 |
| $\mathrm{A}_{2}$ | $60^{2}$ | 9 | 6 | 5 |  | 0 | 60 |
| $\mathrm{A}_{3}$ |  | $40^{1}$ | 8 | 9 |  | 40 | 80 |
|  | 60 | 40 | 0 | 0 | 20 |  |  |
|  | 120 | 40 | 60 | 80 | 20 |  |  |

Figure 2.6 - Actions with cell $\left(\mathrm{A}_{2}, \mathrm{~B}_{1}\right)$

| $\mathrm{A}_{1}$ | $\mathrm{B}_{1}$ | $\mathrm{B}_{2}$ | $\mathrm{B}_{3}$ | $\mathrm{B}_{4}$ | $\mathrm{B}_{5}$ | $\mathrm{a}_{\mathrm{i}}{ }^{\text {a }}$ | $\mathrm{a}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 5 | 12 | 4 |  | $20^{3}$ | 160 | 180 |
| $\mathrm{A}_{2}$ | $60^{2}$ | 9 | 6 | 5 |  | 0 | 60 |
| $\mathrm{A}_{3}$ | 6 | $40^{1}$ | 8 | 9 | 4 | 40 | 80 |
| $\begin{aligned} & \mathbf{b}_{j}^{*} \\ & \mathbf{b}_{\mathrm{j}} \end{aligned}$ | 60 | 40 | 0 | 0 | 20 |  |  |
|  | 120 | 40 | 60 | 80 | 20 |  |  |

Figure 2.7 - Actions with cell ( $\mathrm{A}_{1}, \mathrm{~B}_{5}$ )

| $\mathrm{A}_{1}$ | $\mathrm{B}_{1}$ | $\mathrm{B}_{2}$ | $\mathrm{B}_{3}$ | $\mathrm{B}_{4}$ | $\mathrm{B}_{5}$ | $\mathrm{a}_{\mathrm{i}}{ }^{\text {a }}$ | $\mathrm{a}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 5 | 12 | $60^{4}$ |  | $20^{3}$ | 100 | 180 |
| $\mathrm{A}_{2}$ | $60^{2}$ | 9 | 6 | 5 | 7 | 0 | 60 |
| $\mathrm{A}_{3}$ | 6 | $40^{1}$ | 8 | 9 | 4 | 40 | 80 |
|  | 60 | 40 | 60 | 0 | 20 |  |  |
|  | 120 | 40 | 60 | 80 | 20 |  |  |

Figure 2.8 - Actions with cell $\left(\mathrm{A}_{1}, \mathrm{~B}_{3}\right)$

|  | $\mathrm{B}_{1}$ | $\mathrm{B}_{2}$ | $\mathrm{B}_{3}$ | $\mathrm{B}_{4}$ | $\mathrm{B}_{5}$ | $\mathrm{a}_{\mathrm{i}}{ }^{\text {a }}$ | $\mathrm{a}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{A}_{1}$ | $60^{5}$ |  | $60^{4}$ |  | $20^{3}$ | 40 | 180 |
| $\mathrm{A}_{2}$ | $60^{2}$ | 9 | 6 | 5 | 7 | 0 | 60 |
| $\mathrm{A}_{3}$ | 6 | $40^{1}$ | 8 | 9 |  | 40 | 80 |
| $\mathrm{b}_{j}{ }^{\text {* }}$ | 120 | 40 | 60 | 0 | 20 |  |  |
| $\mathrm{b}_{\mathrm{j}}$ | 120 | 40 | 60 | 80 | 20 |  |  |

Figure 2.9 - Actions with cell $\left(\mathrm{A}_{1}, \mathrm{~B}_{1}\right)$


Figure 2.10 - Actions with cell $\left(\mathrm{A}_{1}, \mathrm{~B}_{4}\right)$

|  | $\mathrm{B}_{1}$ | $\mathrm{B}_{2}$ | $\mathrm{B}_{3}$ | $\mathrm{B}_{4}$ | $\mathrm{B}_{5}$ | $\mathrm{a}_{\mathrm{i}}{ }^{\text {a }}$ | $\mathrm{a}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{A}_{1}$ | $4^{60}$ |  | $60^{4}$ |  | $20^{3}$ | 0 | 180 |
| $\mathrm{A}_{2}$ | $60^{2}$ | 9 | 6 | 5 | 7 | 0 | 60 |
| $\mathrm{A}_{3}$ | 6 $\leftarrow \quad+$ | $40^{1}$ |  | $\downarrow^{40}{ }^{9}$ | 4 | 0 | 80 |
| $\mathrm{b}_{\mathrm{j}}{ }^{\text { }}$ | 120 | 40 | 60 | 80 | 20 |  |  |
| $\mathrm{b}_{\mathrm{j}}$ | 120 | 40 | 60 | 80 | 20 |  |  |

Figure 2.11 - Actions with cell ( $\mathrm{A}_{3}, \mathrm{~B}_{4}$ )

Any basic plan is sometimes called a basis. The filled cells of the transportation table are called basic cells. The number of basic variables of the transportation problem is $m+n-1$, where $m$ is the number of suppliers, $n$ is the number of consumers. Unfilled cells of the transportation table are called free. The number of free cells in the transportation table is equal to the number of free variables of the transportation problem. The number of free variables of the transportation problem is equal to $(m-1)(n-1)$. This number can be used for control when action with the transportation table.

So, we got the basic transportation plan (see Fig. 2.11). According to this plan, the total cost of transportation is equal to $60 \cdot 5+60 \cdot 4+40 \cdot 6+20 \cdot 3+60 \cdot 2+40 \cdot 1+40 \cdot 9=1360$. This plan is not necessarily optimal.

Let us try to find a better solution. To do this, in the basic plan, it is necessary to find all free cells, starting from which it is possible to create a cycle in which only one cell (the first) is free, and the others are basic. That is, such a cycle, in which moving between the cells transported products, you can reduce the total cost of transportation.

In such cycles, we mark the cells where we transfer part of the transported product with a " + " sign and the cells from which we take this part with a "-" sign. If you calculate the algebraic sum of the cost of transportation (called the cost of the cycle) taking into account these signs, and the resulting value is negative, then the transfer of products between these cells will reduce the total cost of transportation.

As a result, we will get another basic plan, which will reduce the total cost of transportation. This improvement process continues until there are no cycles with a negative cost left in the plan. In this case, the resulting basic plan is optimal.

For our example, we find the following cycle (marked by arrows in Fig. 2.11) $\left(\mathrm{A}_{3} \mathrm{~B}_{4}\right) \rightarrow\left(\mathrm{A}_{3} \mathrm{~B}_{1}\right) \rightarrow\left(\mathrm{A}_{1} \mathrm{~B}_{1}\right) \rightarrow\left(\mathrm{A}_{1} \mathrm{~B}_{4}\right)$.

It has a negative value of $-9+6-5+6=-2$. Let us transfer the products in this cycle without disturbing the balance. As a result, we will get the optimal transportation plan (see Fig. 2.12), the transportation cost according to which is equal to $20 \cdot 5+60 \cdot 4+80 \cdot 6+20 \cdot 3+60 \cdot 2+40 \cdot 6+40 \cdot 1=1280$.

Note that the transportation problem, like other linear programming problems, can have several solutions. This depends on the solution method used, or on the sequence of selection of negative-cost cycles. However,
it will be an optimal solution (in our example, the optimum transportation cost is 1280).


Figure 2.12 - Optimum solution

### 2.3 An example of solving a transportation problem using MS Excel

Let us create a table in MS Excel and enter the input data of the transportation problem into it (see Fig. 2.13):

- cost of transportation, cells C4:G6;
- number of product units of each supplier, cells H4:H6;
- demand for the number of product units of each consumer, cells C7:G7.

|  | A | B | C | D | E | F | G | H |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |  | Cost of transportation |  |  |
| 2 |  | Suppliers | Consumers |  |  |  |  | Supply |
| 3 |  |  | B1 | B2 | B3 | B4 | B5 |  |
| 4 |  | A1 | 5 | 12 | 4 | 6 | 3 | 180 |
| 5 |  | A2 | 2 | 9 | 6 | 5 | 7 | 60 |
| 6 |  | A3 | 6 | 1 | 8 | 9 | 4 | 80 |
| 7 |  | Demand | 120 | 40 | 60 | 80 | 20 |  |

Figure 2.13 - Input data

Let us prepare a table for finding a solution to the problem (see Fig. 2.14):

- select an area for solving the problem, cells C9:H11;
- to calculate the number of product units transported from the respective suppliers, write into cell H 9 the formula for calculating the sum of values in cells C9:G9, in cell H10 - the sum of C10:G10, in cell H11 - the sum of C11:G11;
- to calculate the number of product units of products that received by the respective consumers, write into cell C 13 the formula for the sum of values in cells C9:C11, in cell D13 - the sum of D9:D11, in cell E13 - the sum of E9:E11, in cell F13 - the sum of F9:F11, in cell G13 - the sum of G9:G11;
- to calculate the cost of transportation between respective suppliers and consumers, write the formula $=\mathrm{C} 4 * \mathrm{C} 9$ in cell C 15 and copy this formula into cells C15:G17;
- to calculate the transportation total cost, write the formula SUM(C15:G17) into cell C19.

|  | A | B | C | D | E | F | G | H |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |  |  |  |  |
| 2 |  | Suppliers | Consumers |  |  |  |  | Supply |
| 3 |  |  | B1 | B2 | B3 | B4 | B5 |  |
| 4 |  | A1 | 5 | 12 | 4 | 6 | 3 | 180 |
| 5 |  | A2 | 2 | 9 | 6 | 5 | 7 | 60 |
| 6 |  | A3 | 6 | 1 | 8 | 9 | 4 | 80 |
| 7 |  | Demand | 120 | 40 | 60 | 80 | 20 |  |
| 8 |  |  |  |  |  |  |  |  |
| 9 |  |  |  |  |  |  |  | 0 |
| 10 |  | $\mathrm{X}=$ |  |  |  |  |  | 0 |
| 11 |  |  |  |  |  |  |  | 0 |
| 12 |  |  |  |  |  |  |  |  |
| 13 |  |  | 0 | 0 | 0 | 0 | 0 |  |
| 14 |  |  |  |  |  |  |  |  |
| 15 |  |  | 0 | 0 | 0 | 0 | 0 |  |
| 16 |  | C* $\mathrm{X}=$ | 0 | 0 | 0 | 0 | 0 |  |
| 17 |  |  | 0 | 0 | 0 | 0 | 0 |  |
| 18 |  |  |  |  |  |  |  |  |
| 19 |  | $\mathrm{F}=$ | 0 |  |  |  |  |  |

Figure 2.14 - Table for finding a solution to the problem

To find a solution to the problem, we will use the Excel Solver addin. Execute the Data $\rightarrow$ Solver command, which opens the Solver Parameters window.

In the Solver Parameters window, do the following (Fig. 2.15):

- in the Set Target Cell field, set the cell \$C\$19;
- check the Min option;
- enter the cell range $\$ \mathrm{C} \$ 9: \$ \mathrm{G} \$ 11$ into the By Changing Cells field;
- enter the constraints in the Subject to the Constraints field using the Add button:
\$C\$9:\$G\$11>=0
\$H\$9:\$H\$11 = \$H\$4:\$H\$6
\$C\$13:\$G\$13 = \$C\$7:\$G\$7
\$C\$9:\$G\$11= integer


Figure 2.15 - Actions in Solver Parameters window
After entering the data in the Solver parameters window, click the Solve button. If everything is done correctly, then in cells C9:H11 we get the optimal transportation plan, and in cell C19 - the value of the objective function (see Fig. 2.16).

|  | A | B | C | D | E | F | G | H |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 |  |  |  |  |  |  |  |  |
| 2 |  | Suppliers | Consumers |  |  |  |  | Supply |
| 3 |  |  | B1 | B2 | B3 | B4 | B5 |  |
| 4 |  | A1 | 5 | 12 | 4 | 6 | 3 | 180 |
| 5 |  | A2 | 2 | 9 | 6 | 5 | 7 | 60 |
| 6 |  | A3 | 6 | 1 | 8 | 9 | 4 | 80 |
| 7 |  | Demand | 120 | 40 | 60 | 80 | 20 |  |
| 8 |  |  |  |  |  |  |  |  |
| 9 |  |  | 40 | 0 | 60 | 80 | 0 | 180 |
| 10 |  | $\mathrm{X}=$ | 60 | 0 | 0 | 0 | 0 | 60 |
| 11 |  |  | 20 | 40 | 0 | 0 | 20 | 80 |
| 12 |  |  |  |  |  |  |  |  |
| 13 |  |  | 120 | 40 | 60 | 80 | 20 |  |
| 14 |  |  |  |  |  |  |  |  |
| 15 |  |  | 200 | 0 | 240 | 480 | 0 |  |
| 16 |  | $\mathrm{C}^{*} \mathrm{X}=$ | 120 | 0 | 0 | 0 | 0 |  |
| 17 |  |  | 120 | 40 | 0 | 0 | 80 |  |
| 18 |  |  |  |  |  |  |  |  |
| 19 |  | $\mathrm{F}=$ | 1280 |  |  |  |  |  |

Figure 2.16 - The result of solving the problem

### 2.4 Tasks to complete

1. Implement the example given in this tutorial manually and with MS Excel. Analyse the results.
2. Manually solve the transportation problem according to the variant from item 2.5 "Individual tasks", for which build the basic solution using the Least-cost method, and find the optimal solution based on the basic solution.
3. Solve the transportation problem using MS Excel.
4. Create a progress report.

### 2.5 Individual tasks

Each variant the transportation problem (Fig. 2.17, 2.18) contains such data as the supply number of product units, consumer demand for product units, and the cost of product unit transportation.

| Variants | Problems |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 5 | 4 | 5 | 6 | 3 | 110 |
|  | A2 | 9 | 6 | 4 | 7 | 8 | 150 |
|  | A3 | 7 | 4 | 8 | 5 | 1 | 160 |
|  | Demand | 120 | 90 | 50 | 100 | 60 |  |
| 2 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 2 | 4 | 5 | 7 | 3 | 280 |
|  | A2 | 5 | 6 | 3 | 7 | 8 | 160 |
|  | A3 | 9 | 4 | 3 | 5 | 12 | 170 |
|  | Demand | 60 | 150 | 160 | 80 | 160 |  |
| 3 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 2 | 4 | 7 | 3 | 9 | 220 |
|  | A2 | 6 | 8 | 1 | 5 | 7 | 160 |
|  | A3 | 5 | 12 | 9 | 4 | 6 | 180 |
|  | Demand | 120 | 80 | 160 | 110 | 90 |  |
| 4 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 7 | 10 | 4 | 6 | 4 | 180 |
|  | A2 | 2 | 8 | 6 | 5 | 3 | 60 |
|  | A3 | 6 | 11 | 8 | 7 | 4 | 150 |
|  | Demand | 120 | 50 | 60 | 100 | 60 |  |
| 5 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 4 | 5 | 6 | 11 | 3 | 180 |
|  | A2 | 3 | 7 | 9 | 5 | 5 | 350 |
|  | A3 | 7 | 3 | 4 | 1 | 8 | 20 |
|  | Demand | 110 | 90 | 120 | 80 | 150 |  |

Figure 2.17 - Variant the transportation problem from 1 to 5

| Variants | Problems |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 7 | 12 | 4 | 6 | 5 | 280 |
|  | A2 | 1 | 5 | 6 | 5 | 3 | 160 |
|  | A3 | 6 | 13 | 8 | 7 | 4 | 170 |
|  | Demand | 90 | 140 | 60 | 110 | 210 |  |
| 7 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 1 | 4 | 7 | 3 | 9 | 90 |
|  | A2 | 6 | 8 | 2 | 5 | 7 | 150 |
|  | A3 | 3 | 12 | 9 | 4 | 6 | 140 |
|  | Demand | 120 | 70 | 50 | 100 | 40 |  |
| 8 | Suppliers | Consumers |  |  |  |  | Supply |
|  |  | B1 | B2 | B3 | B4 | B5 |  |
|  | A1 | 9 | 4 | 5 | 5 | 3 | 180 |
|  | A2 | 5 | 6 | 11 | 4 | 8 | 160 |
|  | A3 | 2 | 4 | 3 | 5 | 2 | 70 |
|  | Demand | 60 | 50 | 60 | 80 | 160 |  |

Figure 2.18 - Variant the transportation problem from 6 to 8

### 2.6 Contents of the progress report

The report should have the following components:
-title page;
-the purpose of the work;

- solution of the problem according to the variant of an individual task;
-the result of the tasks.


## 3 LABORATORY WORK №3 DECISION-MAKING WITH ANALYTIC HIERARCHY PROCESS

The purpose of the work: to learn how to practically apply the Hierarchy Analysis Method for decision-making under many criteria.

### 3.1 Description of a hypothetical problem

To practical learn the Analytic Hierarchy Process (AHP) method, we will consider an example of decision-making and go through all the stages of the decision-making process using this method.

Consider the goal of a decision-maker, which is to buy a new laptop for tasks of medium complexity (e.g. student laboratory works). The purchase must be based on some criteria that are characteristics of the item being purchased. Let these criteria be: processor type, amount of RAM and hard drive, screen size, manufacturer's brand and colour. Let the decisionmaker become interested in four models of laptops.

As can be seen from the figure, each choice alternative depends on the characteristics inherent in this alternative, which we will call criteria. The importance of the criteria, in turn, depends on the goal. For example, a powerful processor (CPU) and a large and high-quality screen are needed for gaming purposes, the weight of a laptop is important for a mobile user, the processor capacity is needed for mathematical calculations, a large hard drive is necessary for working with databases, etc. Suppose a student will use the laptop for learning purposes only. Thus, the decision-maker (student) must understand which laptop model suits him best. The decision-maker can consult with his colleagues and, if possible, with experts, analyse the offers of specialized stores, and read reviews on the Internet about manufacturers.

The decision-maker, as a result of reflections and available information, decided to choose a laptop from four models, the characteristics (criteria) of which are given in Tab. 3.1

Let's assume that after the conducted market research, the decisionmaker has obtained an objective picture of the preferences of laptop buyers according to the considered criteria, based on the actual purchases made. Market research data are shown in Tab. 3.2.

Table 3.1 - Laptop Models and their Specifications

| Models | Processor | HDD | RAM | Screen | Brand | Colour |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Model 1 | I3 | 512 GB | 4 GB | 14 Inch | HP | Black |
| Model 2 | I5 | 1 TB | 4 GB | 15.6 Inch | Acer | Black |
| Model 3 | $\mathrm{I7}$ | 2 TB | 16 GB | 17.3 Inch | Asus | Silver |
| Model 4 | I5 | 1 TB | 8 GB | 15.6 Inch | HP | Silver |

Table 3.2 - Percentage of laptop buyers who preferred a particular model of laptop

|  | Processor | HDD | RAM | Screen | Brand | Colour |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Model 1 | 22.7 | 11.5 | 25.0 | 22.7 | 34.6 | 12.5 |
| Model 2 | 31.8 | 34.6 | 25.0 | 31.8 | 19.3 | 12.5 |
| Model 3 | 13.7 | 19.3 | 15.0 | 13.7 | 11.5 | 37.5 |
| Model 4 | 31.8 | 34.6 | 35.0 | 31.8 | 34.6 | 37.5 |

From Tab. 3.2 we can conclude that, for example, I3 processor was preferred by $22.7 \%$ of buyers, I5 - by $63.6 \%$ and I7 - by $13.7 \%$. In addition, we see that Model 2 and Model 4 have the same preference among buyers according to the Processor criterion, and Model 4 is preferable to Model 2 according to the RAM criterion.

### 3.2 Decision-making by steps

## Step 1. Creating the model of problem

The decision-making problem can be represented as a hierarchy Goal - Criteria - Alternatives, shown in Fig. 3.1.

## Step 2. Pairwise comparison of criteria

Suppose that based on a survey of more than 100 users and their preferences, the decision-maker compiled a table of pairwise comparison of criteria (Tab. 3.4) using the T. Saaty scale (Tab. 3.3).

From Tab. 3.4, for example, we can see that for the decision-maker, the criterion Processor is five times more preferable than the criterion HDD. Therefore, the advantage of the hard disk criterion over the processor criterion is inverse, i.e. $1 / 5$.


Figure 3.1 - Creating a new script

Table 3.3 - Saaty's scale

| Verbal judgment | Numeric Value |
| :---: | :---: |
| Extremely Important | 9 |
|  | 8 |
| Very Strong or Demonstrated Importance | 7 |
|  | 6 |
| Strong Importance | 5 |
|  | 4 |
| Moderate Importance | 3 |
|  | 2 |
| Equal Importance | 1 |
| $8,6,4,2$ Intermediate values when |  |
| compromise is needed |  |

Table 3.4 - Pairwise comparisons of criteria

| Comparisons | Processor | HDD | RAM | Screen | Brand | Colour |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Processor | 1 | 5 | 3 | 2 | 7 | 9 |
| HDD | $1 / 5$ | 1 | $1 / 3$ | $1 / 3$ | 2 | 5 |
| RAM | $1 / 3$ | 3 | 1 | 2 | 3 | 9 |
| Screen | $1 / 2$ | 3 | $1 / 2$ | 1 | 3 | 7 |
| Brand | $1 / 7$ | $1 / 2$ | $1 / 3$ | $1 / 3$ | 1 | 5 |
| Colour | $1 / 9$ | $1 / 5$ | $1 / 9$ | $1 / 7$ | $1 / 5$ | 1 |

Step 3. Normalization of the pair-wise comparison matrix Denote the pairwise comparisons matrix as follows:

$$
\mathbf{A}=\left(\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 n}  \tag{3.1}\\
a_{21} & a_{22} & \ldots & a_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n 1} & a_{n 2} & \ldots & a_{n n}
\end{array}\right)
$$

In this matrix, the elements $a_{i j}$ are judgments about the preference of the criterion $i$ relative to the criterion $j$.

In order for all elements of the matrix to correspond to the same scale of values, from 0 to 1 , the matrix must be normalized. There are many techniques to normalize datasets (sum, max, geometric, etc.). We will apply the simplest one - normalization by sum. To do this, we divide each element of each column by the sum of the elements of this column. As a result, we obtain a normalized matrix, which we denote by $\mathbf{R}$ :

$$
\mathbf{R}=\left(\begin{array}{cccc}
r_{11} & r_{12} & \ldots & r_{1 n}  \tag{3.2}\\
r_{21} & r_{22} & \ldots & r_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
r_{n 1} & r_{n 2} & \ldots & r_{n n}
\end{array}\right) \text {, where } r_{i j}=\frac{a_{i j}}{\sum_{i} a_{i j}},
$$

Using the data from Tab. 3.4, we obtain Tab. 3.5.
Table 3.5 - Normalized pairwise comparison matrix

|  | Processor | HDD | RAM | Screen | Brand | Colour |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- |
| Processor | 0.437 | 0.394 | 0.568 | 0.344 | 0.432 | 0.250 |
| HDD | 0.087 | 0.079 | 0.064 | 0.057 | 0.123 | 0.139 |
| RAM | 0.146 | 0.236 | 0.189 | 0.344 | 0.185 | 0.250 |
| Screen | 0.219 | 0.236 | 0.095 | 0.172 | 0.185 | 0.194 |
| Brand | 0.062 | 0.039 | 0.063 | 0.057 | 0.062 | 0.139 |
| Colour | 0.049 | 0.016 | 0.021 | 0.025 | 0.012 | 0.028 |

## Step 4. Vector of priorities calculating

There are several methods for calculating the vector of priorities $\mathbf{w}$. Here we use as priorities the average values of the rows of the normalized matrix of pairwise comparisons

$$
\begin{gather*}
w_{i}=\frac{1}{n} \sum_{j} r_{i j}, i=1,2, \ldots, n  \tag{3.3}\\
\mathbf{w}=(0.404,0.092,0.225,0.184,0.07,0.025)^{T} \tag{3.4}
\end{gather*}
$$

## Step 5. Consistency calculation

Let us calculate the consistency of the criteria pairwise comparisons matrix. There is an exact algorithm for calculating the matrix consistency and many approximate methods used in practical calculations. In this lab, we will use a simplified algorithm:

1) Calculate the weighted sum vector:

$$
\begin{equation*}
\mathbf{w s}=\mathbf{A} \cdot \mathbf{w}, \mathbf{w s}=\left(w s_{1}, w s_{2}, \ldots, w s_{n}\right)^{T} \tag{3.5}
\end{equation*}
$$

2) Calculate the consistency of each criterion:

$$
\begin{equation*}
\lambda_{i}=\frac{w s_{i}}{w_{i}}, i=1,2, \ldots, n \tag{3.6}
\end{equation*}
$$

3) Calculate average consistency:

$$
\begin{equation*}
\lambda_{\max }=\frac{1}{n} \sum_{i} \lambda_{i} \tag{3.7}
\end{equation*}
$$

4) Calculate the Consistency Index (CI):

$$
\begin{equation*}
C I=\frac{\lambda_{\max }-n}{n-1} \tag{3.8}
\end{equation*}
$$

5) Calculate Consistency Ratio (CR):

$$
\begin{equation*}
C R=C I / R I \tag{3.9}
\end{equation*}
$$

where RI is the Randomly Generated Index shown in Tab. 3.6.

Table 3.6 - Randomly Generated Indices

| $\boldsymbol{n}$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\boldsymbol{R I}$ | 0.58 | 0.9 | 1.12 | 1.24 | 1.32 | 1.41 | 1.45 | 1.49 |

If the consistency ratio $(\mathrm{CR})$ is less than 0.1 , then the judgments of the decision-maker are acceptable, and the pairwise comparison matrix is consistent. The CR of the pairwise comparison matrix in our example is 0.048 .

## Step 6. Local alternative priority calculation

The next stage of the AHP is to obtain the local priorities of each alternative for each criterion. To do this, the decision-maker compiles the matrices of pairwise comparison of alternatives for each criterion. An example of such a matrix for the Processor criterion is given in Tab. 3.7.

Table 3.7 - An example pairwise comparison matrix

| Processor | Model 1 | Model 2 | Model 3 | Model 4 |
| :--- | :--- | :--- | :--- | :--- |
| Model 1 | 1 | 0.71 | 1.66 | 0.71 |
| Model 2 | 1.4 | 1 | 2.32 | 1 |
| Model 3 | 0.6 | 0.43 | 1 | 0.43 |
| Model 4 | 1.4 | 1 | 2.32 | 1 |

If such matrices are compiled based on the decision-maker's judgments, then it is necessary to verify the consistency of each matrix and calculate the priority vectors. For example, for the criterion Processor the priority vector is $\mathbf{w}_{\operatorname{Pr}}=(0.227,0.318,0.137,0.318)^{T}$. But these procedures are unnecessary for our example, as we already have these priority vectors. They are the columns of Tab. 3.2 (given as a percentage). Thus, the vectors of local priorities are:

$$
\begin{aligned}
& \mathbf{w}_{\mathrm{Pr}}=(0.227,0.318,0.137,0.318)^{T}, \\
& \mathbf{w}_{\mathrm{HDD}}=(0.115,0.346,0.193,0.346)^{T}, \\
& \mathbf{w}_{\mathrm{RAM}}=(0.25,0.25,0.15,0.35)^{T}, \\
& \mathbf{w}_{\mathrm{Scr}}=(0.227,0.318,0.137,0.318)^{T},
\end{aligned}
$$

$$
\begin{gather*}
\mathbf{w}_{\mathrm{Br}}=(0.346,0.193,0.115,0.346)^{T}, \\
\mathbf{w}_{\mathrm{Col}}=(0.125,0.125,0.375,0.375)^{T} . \tag{3.10}
\end{gather*}
$$

Step 7. Global alternative priority calculation
The global priority vector of alternatives is calculated as a weighted sum of their local priorities (3.10). In this case, the weights of the weighted sum are the elements of the vector of criteria priorities (3.4):

$$
\mathbf{w}_{G}=\mathbf{w}_{\mathrm{Pr}} \cdot w_{1}+\mathbf{w}_{\mathrm{HDD}} \cdot w_{2}+\mathbf{w}_{\mathrm{RAM}} \cdot w_{3}+\mathbf{w}_{\mathrm{Scr}} \cdot w_{4}+\mathbf{w}_{\mathrm{Br}} \cdot w_{5}+\mathbf{w}_{\mathrm{Col}} \cdot w_{6} .
$$

Substitution of numerical values from (3.4) and (3.10) into this formula gives:

$$
\begin{equation*}
\mathbf{w}_{G} \approx(0.23,0.29,0.15,0.33)^{T} . \tag{3.11}
\end{equation*}
$$

## Step 8. Making a Final Decision

Now, based on the global priority vector (3.11) decision-maker can make a final ranking (Tab. 3.8) of alternatives and choose the best one.

Table 3.8 - Alternative ranking

| Alternatives | Priorities | Ranks |
| :--- | :--- | :--- |
| Model 4 | 0.33 | 1 |
| Model 2 | 0.29 | 2 |
| Model 1 | 0.23 | 3 |
| Model 3 | 0.15 | 4 |

### 3.3 Tasks to complete

1. Implement the example given in this tutorial. Analyse the results.
2. Solve the problem according to the variant from item 3.4 "Individual tasks" including:

- calculate the consistency of matrices for tables:
var. 1 - for Tab. 3.9, also for criteria from Tab. 3.10 (flexibility, and design); var. 2 - for Tab. 3.11, also for criteria from Tab. 3.12 (brand, and safety);
- calculate the global priority vector of alternatives X, Y, and Z;
- to do the ranking of the alternatives $\mathrm{X}, \mathrm{Y}$ and Z .

3. Create a progress report.

### 3.4 Individual tasks

## Variant 1. Problem statement

The firm wants to buy one new piece of equipment (for example, a drilling machine) and has four criteria in mind that will determine its choice when buying: C - cost; P - performance, F - flexibility; D - design. Three equipment options are considered: $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$. The chief engineer judged the relative importance of four criteria ( $\mathrm{C}, \mathrm{P}, \mathrm{F}$ and D ) and compiled the following pairwise comparison matrix (Tab. 3.9).

Table 3.9 - Pairwise comparison of criteria C, P, F and D

|  | C | P | F | D |
| :--- | :--- | :--- | :--- | :--- |
| C | 1 | 0.5 | 2 | 2 |
| P | 2 | 1 | 4 | 3 |
| F | 0.5 | 0.25 | 1 | 2 |
| D | 0.5 | 0.33 | 1 | 1 |

Further, the buyer compiled four matrices of pairwise comparison of $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ in terms of four criteria (Tab. 3.10): cost, performance, flexibility, and design.

Table 3.10 - Pairwise comparison of $\mathrm{X}, \mathrm{Y}$ and Z
in terms of cost

| $\mathbf{C}$ | X | Y | Z |
| :--- | :--- | :--- | :--- |
| X | 1 | 0.5 | 2 |
| Y | 2 | 1 | 4 |
| Z | 0.5 | 0.25 | 1 |

in terms of performance

| $\mathbf{P}$ | X | Y | Z |
| :--- | :--- | :--- | :--- |
| X | 1 | 2 | 0.5 |
| Y | 0.5 | 1 | 0.25 |
| Z | 2 | 4 | 1 |

in terms of flexibility

| $\mathbf{F}$ | X | Y | Z |
| :--- | :--- | :--- | :--- |
| X | 1 | 3 | 5 |
| Y | 0.33 | 1 | 4 |
| Z | 0.2 | 0.25 | 1 |

in terms of design

| $\mathbf{D}$ | X | Y | Z |
| :--- | :--- | :--- | :--- |
| X | 1 | 0.11 | 0.2 |
| Y | 9 | 1 | 3 |
| Z | 5 | 0.33 | 1 |

## Variant 2. Problem statement

The buyer wants to purchase a household appliance (for example, a washing machine) and has four criteria in mind that will determine its choice
when buying: C - cost; B - brand; E - energy consumption, S - safety. Three washing machines are considered: $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$. The buyer judged the relative importance of four criteria ( $\mathrm{C}, \mathrm{B}, \mathrm{E}$ and S ) and compiled the following pairwise comparison matrix (Tab. 3.11).

Table 3.11 - Pairwise comparison of criteria C, B, E and S

|  | C | B | E | S |
| :--- | :--- | :--- | :--- | :--- |
| C | 1 | 0.5 | 0.4 | 2 |
| B | 2 | 1 | 0.5 | 5 |
| E | 2.5 | 2 | 1 | 4 |
| S | 0.5 | 0.2 | 0.25 | 1 |

Further, the buyer compiled four matrices of pairwise comparison of $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ in terms of four criteria (Tab. 3.12): cost, brand, energy consumption, and safety.

Table 3.12 - Pairwise comparison of X, Y and Z

| in terms of cost |  |  |  | in terms of brand |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| C | X | Y | Z | B | X | Y | Z |
| X | 1 | 2 | 0.5 | X | 1 | 3 | 5 |
| Y | 0.5 | 1 | 0.25 | Y | 0.33 | 1 | 3 |
| Z | 2 | 4 | 1 | Z | 0.2 | 0.33 | 1 |
| in terms of energy consumption |  |  |  | in terms of safety |  |  |  |
| E | X | Y | Z | S | X | Y | Z |
| X | 1 | 2 | 5 | X | 1 | 7 | 5 |
| Y | 0.5 | , | 2.5 | Y | 0.14 | 1 | 0.5 |
| Z | 0.2 | 0.4 | 1 | Z | 0.2 | 2 | 1 |

### 3.5 Contents of the progress report

The report should have the following components:
-title page;
-the purpose of the work;
-solution of the problem according to the variant of an individual task;
-the result of the tasks.

## LITERATURE

1 Taha H.A. (2022). Operations Research: An Introduction, 11th Edition. Pearson.

2 Hillier F. and Lieberman G. (2019). Introduction to Operations Research, 11th Edition. McGraw-Hill Higher Education.

3 Brunnelli, M. Introduction to the analytic hierarchy process. Cham: Springer. https://doi.org/10.1007/978-3-319-12502-2

4 Mu E., Pereyra-Rojas M. Practical Decision Making using Super Decisions v3. Cham: Springer. https://doi.org/10.1007/ 978-3-319-68369-0

5 Saaty, T. L. Decision making for leaders: The analytic hierarchy process for decisions in a complex world (3rd Rev. ed.). Pittsburgh: RWS Publications.

6 Aragon, T. J., Dalnoki-Veress, F. \& Shiu, K. Deriving Criteria Weights for Health Decision Making: A Brief Tutorial. UC Berkeley: Center for Infectious Diseases \& Emergency Readiness. Retrieved from https://escholarship.org/uc/item/4366g5pv

