MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National University “Zaporizhzhia Polytechnic”

METHODOLOGICAL INSTRUCTIONS
to the performance of practical work in the discipline

OPTIMIZATION OF TECHNICAL SOLUTIONS IN MECHANICAL ENGINEERING AND NUMERICAL MATHEMATICAL METHODS

For students of specialties 133 Industrial engineering and
131 Applied mechanics of all forms of education

2023

Methodological instructions to the performance of practical work on the discipline “Optimization of technical solutions in mechanical engineering and numerical mathematical methods”. For students of specialties: 133 Industrial engineering, 131 Applied mechanics of all forms of education / Comp.: Mykhaylo FROLOV. – Zaporizhzhia: National University “Zaporizhzhia Polytechnic”, 2023. – 41 p.

Методичні вказівки до виконання практичних робіт з дисципліни «Оптимізація технічних рішень в машинобудуванні та чисельні математичні методи» для студентів спеціальностей: 133 Галузеве машинобудування, 131 Прикладна механіка усіх форм навчання (англ) / Укл.: Михайло Фролов. – Запоріжжя: НУ «Запорізька політехніка», 2023. – 41 с.

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Approved at a sitting of the Department of Metal Cutting Machines and Tools. Protocol No. 1 dated 08/09/23

Recommended to the publication by Scientific Methodical Commission of the Mechanical Engineering Faculty. Protocol No. 2 dated 14/09/23

CONTENT

INTRODUCTION

Optimization tasks or extreme tasks are found where there are multivariate solutions concerning the phenomenon or object under investigation. Then the task is to find optimal solutions, that is, the best in some sense. The search criterion is the optimality criterion. To solve optimization tasks, a goal (objective) function is defined, which is a quantitative expression of the optimization criterion.

The essence of optimization tasks is to find the extremum (maximum or minimum) of the goal function. Analytical solution of optimization tasks by differentiation of functions is completely solved in mathematical analysis, but when studying technical objects, which are known to belong to "poorly organized" systems, the functions describing certain characteristics of them are not always differentiated or their differentiation causes some difficulties are related to the type of functions themselves. In addition, numerous restrictions may be imposed on the factors included in the goal function, which complicates the process of optimization or finding the best solution. Therefore, the best solution that meets the existing constraints will not always coincide with the extremum of the objective function. In the general case, the objective function may not have extrema at all - for example, it may have a linear form.

The above considerations for solving optimization tasks require the use of numerical methods, special calculation procedures and algorithms, familiarization with which and the acquisition of practical skills are devoted to practical works.

Note that if the continuous function $f(x)$ has a minimum at the point x^*, then the function $-f(x)$ has a maximum at x*. Therefore, the same methods are used to find the maximum as for finding the minimum. Based on this, we will talk only about the task of finding the minimum of the function $f(x)$ that is continuous on the segment [a, b].

1. PRACTICAL WORK #1. GENERAL INFORMATION ON OPTIMIZATION AND COMPARISON OF NUMERICAL METHODS OF ONE-DIMENSIONAL UNCONDITIONAL OPTIMIZATION

1.1 Purpose of work

Learn what the task of optimization is;

- Learn theoretical information about optimization models, their structure and construction principles; optimality criteria and objective functions.
- To study methods of numerical unconditional optimization of functions and construction of algorithms based on them;
- Familiarize yourself with the efficiency criteria of numerical methods of unconditional optimization and learn how to choose the most convenient algorithm for solving the given task.

1.2 Tasks for preparation for practical work

To prepare for practical work, it is necessary to learn: theoretical material devoted to the formulation of the optimization task; basic terms and definitions; optimality criteria and goal functions; the structure of optimization models and their construction. Pay special attention to numerical methods of narrowing the interval.

Recommended sources: course lectures; [1] – P.4 – 25

1.3 Control questions for self-checking and control of readiness for work

1. What solution is called optimal?
2. What can a set of solutions consist of when optimizing a technical object?
3. What features of optimization exist when using computer technology?
4. What prerequisites are necessary for setting the optimization task?
5. What is the mathematical description of the optimization task?

6. What is the composition of the optimization algorithm?
7. What is modelling based on the principle of optimization?
8. What elements are present in the optimization model?
9. What are factors?
10. Define the term "parameters".
11. What is an optimality criterion?
12. What is an objective function?
13. According to what principles should the target function be built?
14. What function is called unimodal?
15. What is the principle of unambiguity?
16. What is the principle of compliance?
17. What is the principle of controllability?
18. What is the task of optimizing the objective function?
19. What characteristics are usually accepted as optimality criteria?
20. Describe the requirements for optimality criteria.
21. What is the difference between static and dynamic optimization?
22. What optimization is called one-dimensional?
23. What is a stationary point?
24. What is one-dimensional unconditional optimization?
25. Why is function minimization considered?
26. Why is it necessary to use numerical methods?
27. What is a localization segment?
28. What three numbers are called lucky?
29. What is iteration?
30. How is the accuracy of a numerical method determined?
31. What groups of methods are used in one-dimensional optimization?
32. What interval is called the interval of uncertainty?
33. What is the general minimum search method?
34. What is the method of half division (dichotomy)?
35. What is the golden section method?
36. What is the "golden ratio"?
37. Name the method and algorithm for establishing the initial uncertainty interval.

1.4 Tasks, content of work and report

For the function given under the option of Table 1.1, determine the minimum on the given localization segment with the accuracy of ε by the following numerical methods:

a) general search;
b) dichotomies;
c) golden ratio.

Compare the effectiveness of the applied methods by the number of iterations to achieve the specified accuracy; by the number of arithmetic operations in iterations; universality concerning the goal function.

During the work, the following actions and information must be performed and displayed in the report:

d) Topic and purpose of the work;
e) Given data;
f) Search algorithm with calculation formulas;
g) Iterations and solutions for each of the methods;
h) Computational scheme;
i) Conclusion regarding the comparison of numerical methods;

Calculations can be performed using any tools and software, such as *Microsoft Excel* spreadsheets

N.B. After the first iteration, it is necessary to ensure the existence of a "successful triple of numbers" and the fulfilment of the condition of unimodality for the segment of localizations. In their absence, the segment (interval) of localization must be increased towards the smallest value of the function at its edges. In turn, a significant increase in the localization segment will lead to an increase in the number of iterations.

Table 1.1 - Given data for practical work

Variant	Function *f(x)*	[a, b]	ε
1	$f(x)=x\sin x + 2\cos x$	-5; -4	0,02
2	$f(x) = x^4 + 8x^3 - 6x^2 - 72x + 90$	1,5; 2	0,05
3	$f(x) = 3x^4 - 10x^3 + 21x^2 + 12x$	-0.5; 0,5	0,01
4	$f(x) = x^6 + 3x^2 + 6x - 1$	-1; 0	0,1
5	$f(x) = 10 \cdot x \cdot \ln(x) - x^2/2$	0,2; 1	0,02
6	$f(x) = x^4 + 2x^2 + 4x + 1$	-1; 0	0,1
7	$f(x) = x^5 - 5x^3 + 10x^2 - 5x$	-3; -2	0,1
8	$f(x) = x^2 + 3x \cdot \ln(x) - 1$	0,5; 1	0,05
9	$f(x) = x^2 - 2x - 2\cos(x)$	0,5; 1	0,05
10	$(x) = (x + 1)^4 - 2x^2$	-3; -2	0,05
11	$f(x) = (1 - e^x \sin(x))^2 - 2x^2$	0; 1	0.01
12	$f(x) = \dfrac{x}{x^2 + 1}$	-2; 0	0,01
13	$f(x) = 2x^2 + 3e^{-x}$	0; 1	0,02
14	$f(x) = -e^{-x}\ln(x)$	0; 1	0,01
15	$f(x) = -x \cdot sin(\ln(x))$	10; 12	0,01

1.5 Calculation example

Function: $f(x) = 90 + 10 * (1 + \ln(2x))^2 * \cos(x)$
Uncertainty interval: [2, 4], ε = 0.001

1.5.1 General search

The number of partition intervals is N = 20.
For an explanation of the method, see Section 5.1.1 and Fig. 5.3 [1].
The main results of the calculation are summarized in Table. 1.2

Table 1.2 - Results of calculations by the general search method

Iteration	Uncertainty interval	Uncertainty interval length	Step δ	Computation points a x^* b	f(x)	Error $\varepsilon = \lvert x_i^* - x_{i-1}^* \rvert$
1	[2,4]	2	0,1	2	66,3029	-
				3,3	7,6919	
				4	28,0152	
2	[3.2, 3.4]	0.2	0.01	3.2	8,5547	0,04
				3.34	7,6000	
				3.4	7,74058	
3	[3.33, 3.35]	0.02	0.001	3.33	7,6092	0,005
				3,345	7,5989	
				3.35	7,6001	
4	[3,344; 3,346]	0,002	0,0001	3,344	7,59897	0,0001
				3.3449	7,59893	
				3,346	7,59899	

Solution: the minimum value of the function at *x = 3.34489* - *f(x)=7.598934*

The minimum value of the function and the corresponding value of the argument in each iteration are highlighted in green

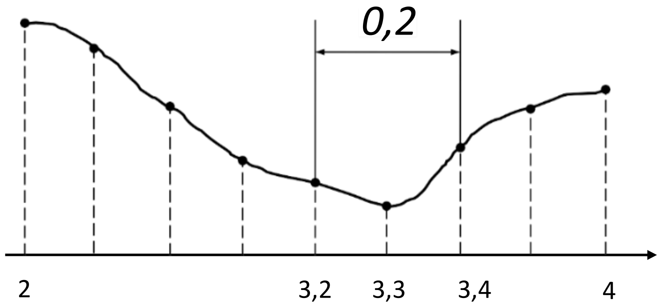

Figure 1.1 - Computational scheme to the general search method

The number of iterations is 4
Number of arithmetic operations in iterations: calculation of points – 21, calculation of function value – 21. In total: (21+21)*4 = 210

1.5.2 Dichotomy method

For an explanation of the method, see Section 5.1.2 and Fig. 5.4 [1]
Based on the specified accuracy, we take $\delta = 0.001$
The calculation results are summarized in Table. 1.3.

The minimum value of the function and the corresponding value of the argument are highlighted in green; In gray are the values that will enter the next iteration as an uncertainty interval

Table 1.3 - Calculation results by dichotomy method

Iter.	Uncertainty interval and/or its length	Computation points		*f(x)*	Error
1	[2,4] / 2	X1	2	66,302931	
		X3-𝛿/2	2,999995	12,84091	
		X3	3	12,840768	-
		X3+δ/2	3,000005	12,84062	
		X2	4	28,015236	
2	1,00005	X1	2,999995		
		X3-𝛿/2	3,499993	8,730558	
		X3	3,499998	8,730632	0,5
		X3+δ/2	3,500003	8,730705	
		X2	4		
3	0,500008	X1	2,999995		
		X3-𝛿/2	3,249994	8,011717	
		X3	3,249999	8,011674	0,25
		X3+δ/2	3,250004	8,01163	
		X2	3,500003		
4	0,250009	X1	3,249994		
		X3-𝛿/2	3,374993	7,641097	
		X3	3,374998	7,641111	0,125
		X3+δ/2	3,375003	7,641126	
		X2	3,500003		
5	0,125009	X1	3,249994		
		X3-𝛿/2	3,312493	7,647407	
		X3	3,312498	7,647392	0,062
		X3+δ/2	3,312503	7,647377	
		X2	3,375003		
6	0,06251	X1	3,312493		
		X3-𝛿/2	3,343743	7,59899496	

Iter.	Uncertainty interval and/or its length	Computation points		f(x)	Error
		X3	3,343748	7,5989944	0,031
		X3+δ/2	3,343753	7,5989939	
		X2	3,375003		
7	0,03126	X1	3,343743		
		X3-δ/2	3,359368	7,608674	
		X3	3,359373	7,608681	0,015
		X3+δ/2	3,359378	7,608688	
		X2	3,375003		
8	0,015635	X1	3,343743		
		X3-δ/2	3,351556	7,600999	
		X3	3,351561	7,601002	0,008
		X3+δ/2	3,351566	7,601005	
		X2	3,359378		
9	0,007822	X1	3,343743		
		X3-δ/2	3,34765	7,599289	
		X3	3,347655	7,59929	0,004
		X3+δ/2	3,34766	7,599291	
		X2	3,351566		
10	0,003916	X1	3,343743		
		X3-δ/2	3,345696	7,598965	
		X3	3,345701	7,598965	0,002
		X3+δ/2	3,345706	7,599291	
		X2	3,34766		
11		X1	3,343743		
		X3-δ/2	3,344675	7,598937	
		X3	3,344725	7,598936	0,0009
		X3+δ/2	3,344775	7,598935	
		X2	3,345706		

Solution: the minimum value of the function at $x = 3.344775$ - $f(x)=7.59894$

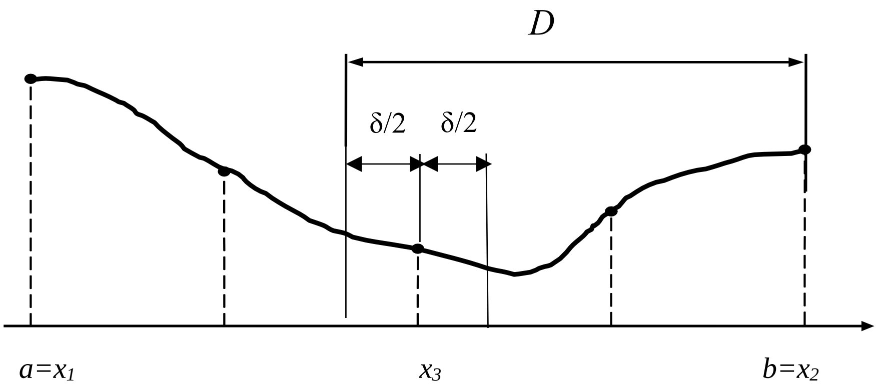

Figure 1.2 - Computational scheme to the dichotomy method

The number of iterations is 11
Number of arithmetic operations:
iteration 1 – 8 operations
iterations 2 ... 10 – 10x6 = 60 operations
in total: 68

1.5.3 Golden ratio method

For an explanation of the method, see Section 5.1.3 and Fig. 5.5 [1].

The calculation results are summarized in Table. 1.4.

The minimum values of the function and the corresponding argument at each iteration are highlighted in green, as well as the previous value of the argument that will enter the next iteration as one *of the limits of the uncertainty interval; in yellow - the maximum values of the function and the corresponding argument.*

The error is calculated as:

$$\text{the } |x_i^* - x_{i-1}^*| \leq \varepsilon$$

Also, take into account:

$$|f(x_i^*) - f(x_{i-1}^*)| \leq \varepsilon$$

Table 1.4 - Calculation by the method of the golden section

Iter.	D	Computation points		*f(x)*	Error
1	2	X1	2	66,302931	
		X3	2,763906	21,74558	
		X4	3,2360939	8,1405105	
		X2	4	28,015236	
2	1,236094	X1	2,763906		
		X3	3,236036	8,141084	
		X4	3,5278702	9,1770991	
		X2	4		
3	0,763964	X1	2,763906		
		X3	3,055704	11,32125	
		X4	3,2360718	8,1407297	
		X2	3,5278702		
4	0,472166	X1	3,055704		
		X3	3,23605	8,140949	
		X4	3,3475251	7,5992575	0,111
		X2	3,5278702		
5	0,291821	X1	3,23605		
		X3	3,347511	7,599254	0,000014
		X4	3,4164084	7,8378499	
		X2	3,5278702		

Solution: the minimum value of the function at $x = 3.347511$ - $f(x)=7.599254$

The number of iterations is 5
Number of arithmetic operations:
Iteration 1 – 6
Iterations 2...5 - 4x4 = 16
In total: 22

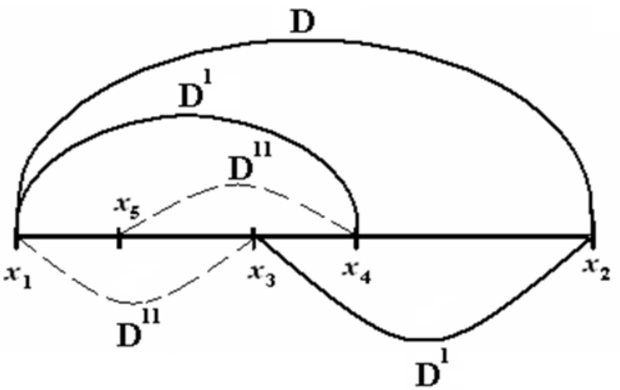

Figure 1.3 - Computational scheme to the method of the golden section

2. PRACTICAL WORK #2. UNCONDITIONAL MULTIDIMENSIONAL OPTIMIZATION OF THE FUNCTION BY THE METHOD OF COORDINATE DESCENT (GAUSSIAN)

2.1 Purpose of the work

a) Learn theoretical information about the relief of the function of many variables and its optimization;
b) To learn what the coordinate descent method consists of;
c) To acquire practical skills in unconditional optimization of the function of many variables by the method of coordinate descent.

2.2 Tasks for preparation for practical work

To prepare for practical work, it is necessary to learn: theoretical material devoted to the optimization of the function of many variables; basic terms and definitions; relief of functions and their varieties; and the sequence of optimization by the coordinate descent method. Repeat optimization methods by narrowing the uncertainty interval;

Recommended sources: course lectures; [1] – P.20 – 27, P.31 – 35

2.3 Control questions for self-testing and control of readiness for work

1. Why is it necessary to use numerical methods?
2. What is a localization segment?
3. What three numbers are called the best-case?
4. What is iteration?
5. How is the accuracy of the numerical method determined?
6. What groups of methods are used in one-dimensional optimization?
7. What interval is called the interval of uncertainty?

8. Name the method and algorithm for establishing the initial uncertainty interval.
9. How are level lines formed?
10. What types of relief features are there?
11. How is the relief of a function depicted on a plane?
12. What is a real gully?
13. What is a real ridge?
14. What is a soluble gully or ridge?
15. What characterizes the disordered type of relief?
16. What is the Gaussian method?
17. Is it always possible to find the minimum using the Gaussian method?
18. For what types of relief is the Gaussian method most suitable?
19. What is the Gaussian method most often used for?

2.4 Tasks, content of work and report

For the function of two variables, given following the option of Table 2.1, determine the unconditional minimum on the given localization segment with accuracy ε by the Gaussian method, using your choice: the general search method or the dichotomy method or the golden section method

During the work, the following actions and information must be performed and displayed in the report:

a) Topic and purpose of the work;
b) Given data;
c) Search algorithm with calculation formulas;
d) Iterations;
e) Solution;
f) Computational scheme;
g) Conclusion;

Table 2.1 - Given data for practical work

Вар.	f(x)	[a1; b1] [a2; b2]	ε
1	$f(x) = x_1^2 + 2x_2^2 + e^{x_1^2+x_2^2} - x_1 + 2x_2$	-1; 1 -1; 1	0,001
2	$the\ f(x) = \sqrt{x_1^2 + (x_2 - 1)^2 + 1} + \frac{x_1}{2} - \frac{x_2}{3}$	-2; 0 0; -2	0,002
3	$f(x) = x_1^4 + x_2^4 + x_1^2 x_2^2 + 2x_1 + x_2$	-2; 2 -2; 2	0,002
4	$f(x) = x_1^2 + 3x_2^2 + \cos(x_1 + x_2)$	-2; 2 -2; 2	0,001
5	$the\ f(x) = \sqrt{2x_1^2 + x_2^2 + 1} + e^{x_1^2+x_2^2} - x_1 - x_2$	-2; 2 -2; 2	0,005
6	$f(x) = 2x_1 \cos(x_1 + x_2) + x_2^2 + 4\sqrt{x_1^2 + x_2^2}$	-2; 2 -3; 1	0,002
7	$f(x) = 2x_1 - 5x_2 + e^{x_1^2+\frac{1}{2}x_2^2}$	-3; 1 -2; 2	0,001
8	$f(x) = x_1^2 + 4x_2^2 - e^{x_1^2+x_2^2} - 2x_1 + x_2$	-2; 2 -2; 2	0,005
9	$f(x) = 4x_1 + 3x_2 + 2e^{\frac{1}{3}x_1^2+\frac{1}{2}x_2^2}$	-3; 1 -2; 2	0,002
10	$f(x) = 2x_1^4 + x_2^4 - x_1^2 x_2^2 + 3x_1 + 5x_2$	-3; 1 -3; 1	0,001

2.5 Calculation example

Function: $f(x) = 3x_1^2 + \frac{1}{4}x_2^2 + 2e^{-\left(\frac{1}{3}x_1 + x_2\right)}$
Uncertainty interval: *x1 [-2, 2], x2 [-2, 2], ε = 0.001*

We choose the general search method N = 20

The search algorithm for each of the variables:

- The length of the uncertainty interval:

  $D_i = b_i - a_i$

- Breakdown step in iteration i:

  $\delta_i = \frac{D_i}{N}$

- Determination of the value of the factor xi:

  $x_{ij} = a_i + (j-1) \cdot \delta_i, \qquad j = \overline{1,N}$

- Calculation of the values of the function f(x_ij) and selection of the minimum:

  $x_i^* = x_{ij}\ for\ which\ f(x_i^*) = \min(f(x_{ij})),\ \ j = \overline{1,N}$

- The length of the uncertainty interval after the first iteration:

  $D_{i+1} = 2\delta_i$

- The limits of the new interval relative to the minimum point in the previous iteration i - x_i^*:

  $a_{i+1} = x_i^* - \delta_i, \quad b_{i+1} = x_i^* + \delta_i$

- Checking accuracy and repeating the cycle if the condition is not met:

$the\ |x_i^* - x_{i-1}^*| \leq \varepsilon, \quad i = 2, 3, \ldots$ or $D_i \leq \varepsilon$

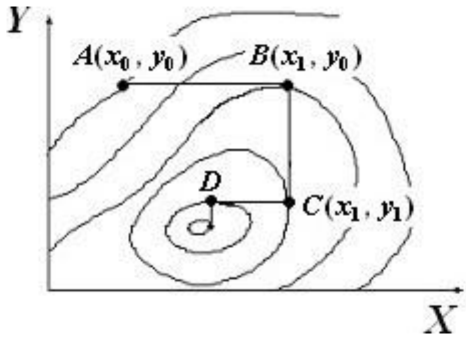

Figure 2.1 - Computational scheme to the finding the optimum using the Gaussian method

Calculations are performed in Microsoft Excel. The results of calculations according to the algorithm are summarized in Table. 2.1

Notes:

- ✓ *Note that if the relief of the function is different from the vally, minimum of the function, by the Gaussian method may not be found;*
- ✓ *For the first iteration, the selected initial value is taken as x_1*;*
- ✓ *The uncertainty range in Tab. 1 is specified for that variable whose value is not fixed*
- ✓ ***At each of the iterations, check for a best-case number trio and, if necessary, adjust the uncertainty interval, expanding it towards the smallest value of the function***

Solution: $x_1 = 0{,}03322;\ x_2 = 1{,}19614; f(x_1;\ x_2) = 0{,}929057$

Table 2.2 - Iterations and calculation results

It.	X1				X2				F(x)	ε
	[a, b]	D	δ	x_1^*	[a, b]	D	δ	x_2^*		
1				1	-2;2	4	0,2	1	3,7770	4
	-2;2	4	0.2	0				1	0.9858	4
2				0	0.8; 1.2	0.4	0.02	1.2	0.9624	0.4
	-0.2; 0.2	0.4	0.02	0.04				1.2	0.9592	0.4
3				0.04	1.18 1.22	0.04	0.002	1.194	0.9516	0.04
	0.02 0.06	0.04	0.002	0.034				1.194	0.959060	0.04
4				0.034	1.192; 1.196	0.004	0.0002	1.196	0.9590 59	0.004
	0.032 0,036	0.004	0.0002	0.03322				1.196	0.959057	0.004
5				0.03322	1.1958 1.1965	0.0004	0.00002	1.19614	0.95905 7	0.0004
	0.03318 0.03322	0.0004	0.00002	0.03322				1.19614	0.959057	0.0004

3. PRACTICAL WORK #3. CONDITIONAL OPTIMIZATION OF A FUNCTION BY THE METHOD OF PENALTY FUNCTIONS

3.1 Purpose of work

a) Learn theoretical information about conditional optimization, types of restrictions and ways of taking them into account;
b) Familiarize yourself with the typical limitations that arise during cutting;
c) To learn what the penalty function method consists of, in particular, the external and internal point methods;
d) To acquire practical skills in conditional optimization of a function of many variables by the method of penalty functions.

3.2 Tasks for preparation for practical work

To prepare for practical work, it is necessary to learn: methods, general information about conditional optimization; restrictions in the form of equalities; constraints in the form of inequalities; method and Lagrange multipliers; method of penalty functions and its varieties; typical limitations arising from cutting; the concept of non-rigid constraints. It is recommended to repeat the optimization methods by narrowing the uncertainty interval.

Recommended sources: course lectures; [1] – P.36 – 53

3.3 Control questions for self-checking and control of readiness for work

1. What is conditional optimization?
2. What is a stationary point?
3. What is the unconditional optimum?
4. What is a conditional optimum?
5. What restrictions are called active?

6. What restrictions are called inactive?
7. What does the condition of regularity of the limiting functions at the point x* mean?
8. In what situations is the condition of regularity of limiting functions not fulfilled?
9. How are restrictions in the form of equalities taken into account?
10. How is the Lagrange function written?
11. Name the condition of the global minimum of the Lagrange function.
12. Which matrix is positive definite?
13. What actions make sense to start solving the task of conditional optimization with restrictions in the form of inequalities?
14. How are constraints in the form of inequalities transformed into constraints - equalities?
15. What is the essence of the Kuhn-Tucker theorem?
16. What conditions for the Lagrange function must be fulfilled at a stationary point?
17. What are the varieties of the method of penalty functions?
18. What is the external point method?
19. Which constraints can be interpreted as loose?
20. What is the form of non-rigid constraints of factors?
21. How is the quadratic penalty function written?
22. Give an example of a penalty function for an equality constraint using the interior point method?
23. What is the internal point method?
24. What functions are called barriers?
25. Give an example of a barrier function.
26. What difficulties can arise when applying barrier functions?
27. What is the minimum search algorithm by the method of penalty functions?
29. By what indicators is the cutting speed limit determined?
30. Name additional limitations in cutting

3.4 Tasks, content of work and report

For the function of two variables and constraints given according to the option of Table 3.1, by the method of the external penalty function, using the quadratic form of the penalty function, determine the conditional minimum by the Gaussian method, using your choice: the general search method or the dichotomy method or the golden section method.

During the work, the following actions and information must be performed and displayed in the report:

a) Topic and purpose of the work;
b) Given data;
c) Definition of the auxiliary function that is optimized and includes penalty functions.
d) Search algorithm with calculation formulas;
e) Determination of localization segment and uncertainty interval;
f) Iterations and solutions;
g) Graphs of the function to be optimized, the auxiliary function, and the constraints from one of the variables;
h) Conclusion;

Table 3.1 - Given data for practical work

Var	Function	Constrictions	ε
1	$f(x) = x_1^2 + 2x_2^2 - 16x_1 - 20x_2$	$2x_1 + 5x_2 \leq 40$ $2x_1 + x_2 \leq 16$ $x_1, x_2 \geq 0$	0,3
2	$f(x) = x_1^2 + x_2^2 - 18x_1 - 20x_2$	$2x_1 + 5x_2 \leq 60$ $3x_1 + x_2 \leq 30$ $x_1 + x_2 \leq 15$ $x_1, x_2 \geq 0$	0,2
3	$f(x) = (x_1 - 16)^2 + (x_2 - 9)^2$	$5x_1 + 2x_2 \leq 60$ $x_1 + 4x_2 \leq 40$ $x_1 + x_2 \leq 15$ $x_1 \geq 0$	0,3
4	$f(x) = x_1^2 + x_2^2 - x_1 x_2 - 3x_2$	$x_1 + x_2 \leq 4$ $x_1, x_2 \geq 0$	0,2
5	$f(x) = x_1^2 + x_2^2 - 4x_1 - 2x_2$	$0 \leq x_1 \leq 1$ $0 \leq x_2 \leq 1$	0,2
6	$f(x) = x_1^2 + 2x_2^2 - 2x_1 - 4x_2$	$x_1 + 2x_2 \leq 2$ $2x_1 - x_2 \leq 3$ $x_1, x_2 \geq 0$	0,25
7	$f(x) = x_1^2 + \frac{x_2^2}{4} - 4x_1 + x_2$	$4 \leq x_1 \leq 8$ $-5 \leq x_2 \leq 1$	0,3
8	$f(x) = x_1^2 + 9x_2^2 - 12x_1 - 36x_2$	$-1 \leq x_1 \leq 8$ $1 \leq x_2 \leq 4$	0,2
9	$f(x) = 2\sqrt{x_1^2 + 2x_2^2 + 1} + x_1 + x_2$	$5 \leq x_1 \leq 8$ $1 \leq x_2 \leq 10$	0,2
10	$f(x) = x_1^2 + x_2^2 - 6x_1 - 3x_2 + 5$	$x_1 + x_2 \leq 3$ $2x_1 + x_2 \leq 4$ $x_1, x_2 \geq 0$	0,1

3.5 Calculation example

Given: function $f(x) = x_1^2 + x_2^2 - 5x_1 - 10x_2 \to min$

And the restrictions, which we will transform into the form $g_i(x) \le 0$:

$$\begin{cases} 9x_1 + 8x_2 \le 72 \\ x_1 + 2x_2 \le 10 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases} \Rightarrow \begin{cases} g_1(x) = 9x_1 + 8x_2 - 72 \le 0 \\ g_2(x) = x_1 + 2x_2 - 10 \le 0 \\ g_3(x) = -x_1 \le 0 \\ g_4(x) = -x_2 \le 0 \end{cases}$$

Let's create an auxiliary function using the quadratic form of the penalty function. Let's take the initial value of the coefficient r = 100. Thus, the auxiliary function will have the form:

$$Q(x) = f(x) + 100 \cdot \left\{\frac{1}{4}[g_1(x) + |g_1(x)|]^2 + \frac{1}{4}[g_2(x) + |g_2(x)|]^2 + \frac{1}{4}[g_3(x) + |g_3(x)|]^2 + \frac{1}{4}[g_4(x) + |g_4(x)|]^2\right\}$$

Taking into account the constraints $x_1, x_2 \ge 0$ and using Sven's method, we determine the uncertainty intervals for x_1 [0;6] and x_2 [0;6] in such a way that a best-case trio of numbers for each variable is obtained. Let's take $\varepsilon = 0{,}15$

Important!

- ✓ *When performing calculations at the stage of finding the uncertainty interval, the values of the functions f(x),* $g_i(x)$ *and Q(x) should be calculated separately. When comparing the obtained values of f(x) and Q(x), it is necessary to increase the value of the coefficient r as necessary until, when* $g_i(x) > 0$*, their minimum values are at the same x.*
- ✓ *When searching for a minimum, follow all recommendations for unconditional optimization*

Using the general search method, we minimize the function *Q(x)* on the obtained uncertainty intervals. Let's take N=20. The calculation is performed according to the following algorithm:

- The length of the uncertainty interval:

$$D_i = b_i - a_i$$

- Breakdown step in iteration i:

$$\delta_i = \frac{D_i}{N}$$

- Determination of the value of the factor xi:

$$x_{ij} = a_i + (j-1) \cdot \ \delta_i, \qquad j = \overline{1, N}$$

- Calculation of the values of the function f(x_ij) and selection of the minimum:

$$x_{ij} = a_i + (j-1) \cdot \ \delta_i, \qquad j = \overline{1, N}$$

- The length of the uncertainty interval after the first iteration:

$$D_{i+1} = 2\delta_i$$

The limits of the new interval relative to the minimum point in the previous iteration i - x_i^*:

$$a_{i+1} = x_i^* - \delta_i, \quad b_{i+1} = x_i^* + \delta_i$$

Checking accuracy and repeating the cycle if the condition is not met:

$$the\ |x_i^* - x_{i-1}^*| \le \varepsilon, \quad i = 2, 3, \dots \text{ or } D_i \le \varepsilon$$

The results of the calculations are summarized in Table 3.2.

Table 3.2 - Calculation results

Iter.	X1				X2				Q(x)	ε
	[a, b]	D	δ	x_1^*	[a, b]	D	δ	x_2^*		
1	0;6	6	0.3	2.4				0	-6.2400	
				2.4	0;6	6	0.3	3.6	-29,2800	
2	2.1; 2.7	0.6	0.03	2.49				3.6	-29,2899	3,6
				2.49	3.3; 3.9	0.6	0.03	3,75	-29.6874	0,15
3	2.46; 2.52	0.06	0.003	2.499				3.75	-29.6875	0,15
				2.499	3.72; 3.78	0.06	0.003	3.75	-29.6875	0

Solution: $x_1 = 2.499, x2 = 3.75, f(x)= - 29.6875$

Let's plot graphs of functions for x1 = 2.499 and x2 in the range [0;6] - Fig. 3.1. As can be seen from the graphs and calculations:

a) when the restrictions are fulfilled, the graphs of the functions f(x) and Q(x) coincide; as soon as the restrictions are violated, the value of the auxiliary function Q(x) increases sharply;
b) The conditional and unconditional minima of the function f(x) do not coincide.

c) The minimum of the function Q(x) coincides with the smallest value of f(x) for which the constraints are fulfilled

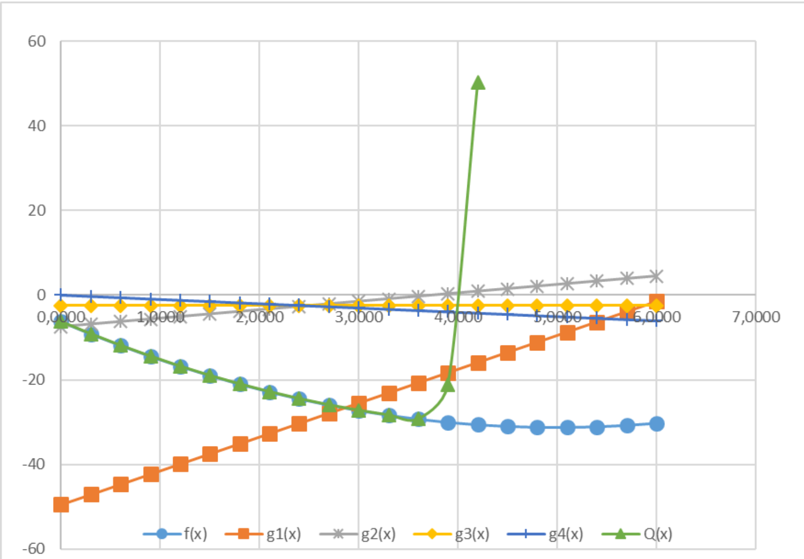

Figure 3.1 - Graphs of minimizable functions and constraints

4. PRACTICAL WORK #4. LINEAR PROGRAMMING TASK

4.1 Purpose of work

a) Learn theoretical information about linear programming and concepts about its main task;
b) Familiarize yourself with some typical tasks of linear programming and ways to solve them;
c) To acquire practical skills for solving the task of linear programming.

4.2 Tasks for preparation for practical work

To prepare for practical work, it is necessary to learn: the concept of linear programming and the main task of linear programming; forms of recording the main task of linear programming in the form of a system of equations and inequalities; content of typical tasks of linear programming.

Recommended sources: course lectures; [1] – P.54 – 65

4.3 Control questions for self-checking and control of readiness for work

1. What are linear programming tasks?
2. Where are the solutions of linear programming tasks concerning constraints?
3. What is the form of the main task of linear programming?
4. What decision is called admissible?
5. What is the outline of a linear programming task?
6. Which solution to the task of linear programming is called optimal?
7. When there is a single solution to the task of linear programming - is it optimal?
8. In what case does the task of linear programming have no solution?
9. How can each task of linear programming be presented as basic?

10. What is a solution polygon?
11. What is the essence of the transport task?
12. What does the simplified version of the transport task involve?

4.4 Tasks, content of work and report

The content of the linear programming task is selected depending on the option in Table. 4.1 or Table. 4.2.

During the work, the following actions and information must be performed and displayed in the report:

a) Topic and purpose of the work;
b) Given data;
c) Writing the conditions of the task in the form of the main task of linear programming;
d) Record the target function;
e) Graphic interpretation of the linear programming task;
f) Solution of the task;
g) Conclusion;

4.4.1 The task of using raw materials

Production of two types of products - P1 and P2 require four types of raw materials - *S1, S2, S3, S4*. The amount of raw materials is limited and is, respectively, *b1, b2, b3, b4*. The number of units of raw materials required to produce a unit of production is known and given in the form of a table. Income from product sales is C1 and C2, respectively. It is necessary to draw up such a plan for the manufacturing of products that the income from its sale would be maximum. The conditions of the task are presented in Table. 4.1.

Table 4.1 – Given data for the task of using raw materials

Raw material	1			2			3			4			5		
	Raw material stock	Products type		Raw material stock	Products type		Raw material stock	Products type		Raw material stock.	Products type		Raw material stock	Products type	
		P_1	P_2		P_1	P_2		P_1	P_2		P_1	P_2		P_1	P_2
S_1	*18*	*2*	*0*	*20*	*2*	*4*	*18*	*2*	*1*	*24*	*2*	*3*	*24*	*3*	*2*
S_2	*14*	*0*	*3*	*15*	*1*	*3*	*60*	*3*	*6*	*16*	*0*	*2*	*18*	*1*	*3*
S_3	*18*	*3*	*6*	*20*	*0*	*5*	*21*	*3*	*0*	*20*	*4*	*2*	*16*	*1*	*1*
S_4	*12*	*4*	*2*	*24*	*4*	*2*	*22*	*0*	*2*	*10*	*1*	*0*	*22*	*0*	*2*
Incom		*10*	*8*		*16*	*12*		*5*	*7*		*9*	*6*		*8*	*11*

4.4.2 The task of using equipment productivity

In time T, the enterprise must produce N1 units of the P1 product and N2 units of the P2 product. Both types of products can be produced on equipment A and B, which have different productivity (number of units of products produced per time unit) and cost of production per time unit. The given data are summarized in Table. 4.2

Table 4.2 - Given data for the tasks of using equipment productivity

Var.	Equipment	Productivity per time unit		Production costs per unit of time		Time for program production		Program, pcs.		Total time T
	Prod.	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	
6	A	*5*	*25*	*5*	*40*	x_1	x_2	*40*	*80*	*8*
	B	*14*	*14*	*15*	*30*	x_3	x_4			
7	A	*8*	*14*	*6*	*20*	x_1	x_2	*110*	*200*	*16*
	B	*10*	*12*	*10*	*24*	x_3	x_4			
8	A	*6*	*18*	*4*	*20*	x_1	x_2	*140*	*121*	*12*
	B	*10*	*12*	*16*	*30*	x_3	x_4			
9	A	*20*	*5*	*15*	*10*	x_1	x_2	*180*	*100*	*10*
	B	*30*	*15*	*35*	*40*	x_3	x_4			
10	A	*12*	*48*	*8*	*70*	x_1	x_2	*70*	*220*	*8*
	B	*20*	*22*	*26*	*52*	x_3	x_4			

4.5 Calculation example

4.5.1 The task of using raw materials

The initial design data are given in Table. 4.3

Table 4.3 - Given data

Raw material	Raw material stock	Products type	
		P_1	P_2
S_1	*19*	*2*	*3*
S_2	*13*	*2*	*1*
S_3	*15*	*0*	*3*
S_4	*18*	*3*	*0*
Income		*7*	*5*

Conditions of Table. 4.3 will be written in the form of the systems of equations:

$$\begin{cases} 2x_1 + 3x_2 \le 19 \\ 2x_1 + x_2 \le 13 \\ 3x_2 \le 15 \\ 3x_1 \le 18 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{cases} \Rightarrow \begin{cases} x_1 \ge 0 & (I) \\ x_2 \ge 0 & (II) \\ 19 - 2x_1 - 3x_2 \ge 0 & (III) \\ 13 - 2x_1 - x_2 \ge 0 & (IV) \\ 15 - 3x_2 \ge 0 & (V) \\ 18 - 3x_1 \ge 0 & (VI) \end{cases} \Rightarrow \begin{cases} x_1 \ge 0 \\ x_2 \ge 0 \\ 19 - 2x_1 - 3x_2 = x_3 \\ 13 - 2x_1 - x_2 = x_4 \\ 15 - 3x_2 = x_5 \\ 18 - 3x_1 = x_6 \end{cases}$$

At the same time, the matrix of the system, composed of coefficients with unknowns, will have the form:

$$I = \begin{pmatrix} 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The system cannot be solved unambiguously, because the number of unknowns is greater than the number of equations: the rank of the matrix r = 4, the number of unknowns n= 6. The number of degrees of freedom k = 6 – 4 = 2

The goal function has the form:

$$F = 7x_1 + 5x_2 \rightarrow max$$

That can be transformed:

$$F = -7x_1 - 5x_2 \rightarrow min$$

To solve the task, we will use its geometric interpretation. For this, we introduce the x_1Ox_2 coordinate system. It is known that the locus of points on the plane, the coordinates of which satisfy the system of linear inequalities, form a convex polygon, which we will call *the polygon of solutions* of the system of inequalities. The sides of this polygon are located on straight lines, the equations of which are obtained if the inequality signs are replaced by exact equalities in the inequalities of the system. This polygon itself is the intersection of the half-planes into which each of the indicated lines divides the plane. In our case, these lines are shown in Fig. 4.1.

$$\begin{cases} x_1 = 0 & (I) \\ x_2 = 0 & (II) \\ 19 - 2x_1 - 3x_2 = 0 & (III) \\ 13 - 2x_1 - x_2 = 0 & (IV) \\ 15 - 3x_2 = 0 & (V) \\ 18 - 3x_1 = 0 & (VI) \end{cases}$$

The arrows in the figure indicate which half-planes give the polygon of solutions.

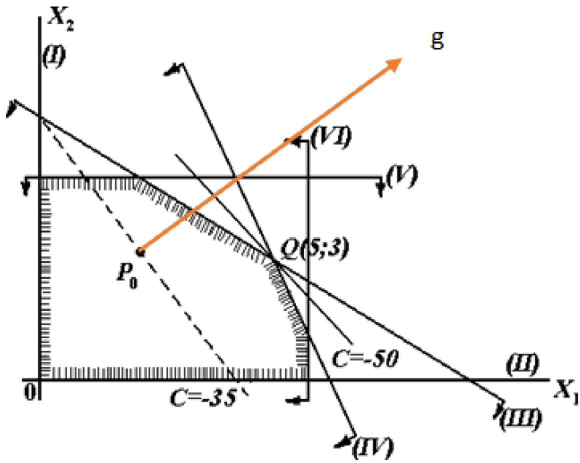

Figure 4.1 - Polygon of solutions.

Along with this, we will consider the objective function F mentioned above and write it in the form:

$$-7x_1 - 5x_2 = C$$

By changing C, we get a set of parallel lines. Fig. 4.1 shows a straight line for C=-35, and the vector g indicates the direction of movement to decrease the values of C. Therefore, it is obvious that the solution will lie on the side or top of the polygon of solutions furthest in the direction of the vector g. Such a vertex is Q formed at the intersection of lines (III) and (IV). The solution of the system of the corresponding equations gives the solution of the task:

$$\begin{cases} 19 - 2x_1 - 3x_2 = 0 & (III) \\ 13 - 2x_1 - x_2 = 0 & (IV) \end{cases} \Rightarrow \begin{cases} x_1 = 5 \\ x_2 = 3 \end{cases}$$

Therefore, the most profitable solution is the production of 5 units of P1 and 3 units of P2. At the same time, the value of the objective function is F = -50.

4.5.2 The task of using equipment productivity

The initial design data are given in Table. 4.4

Table 4.4 - Given data

Equipment	Productivity per time unit		Production costs per unit of time		Time for program production		Program, pcs.		Total time T
Products	P_1	P_2	P_1	P_2	P_1	P_2	P_1	P_2	
A	*6*	*24*	*4*	*47*	x_1	x_2	*30*	*96*	6
B	*13*	*13*	*13*	*26*	x_3	x_4			

Assuming by default that all variables are non-negative, let's write down the system of constraints and the goal function:

$$\begin{cases} x_1 + x_2 \leq 6 & (I') \\ x_3 + x_4 \leq 6 & (II') \\ 6x_1 + 13x_3 = 30 & (III') \\ 24x_2 + 13x_4 = 96 & (IV') \end{cases}$$

$$F = 4x_1 + 13x_3 + 47x_2 + 26x_3$$

The constraint system matrix will look like this:

$$I = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 6 & 13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 13 & 0 & 0 \end{pmatrix}$$

Let's choose x_1 and x_2 as free variables and express x_3 and x_4 from equations (III') and (IV') through them:

$$x_3 = \frac{1}{13}(30 - 6x_1) = \frac{5}{13}(5 - x_1) \geq 0 \Rightarrow x_1 \leq 5$$

$$x_4 = \frac{1}{13}(96 - 24x_2) = \frac{24}{13}(4 - x_2) \geq 0 \Longrightarrow x_2 \leq 4$$

By substituting the obtained expressions into equation (II'), we obtain a system of inequalities or line equations for constructing a polygon of solutions and a goal function:

$$\begin{cases} x_1 \geq 0 & (I) \\ x_2 \geq 0 & (II) \\ x_1 \leq 5 & (III) \\ x_2 \leq 4 & (IV) \\ x_1 + x_2 \leq 6 & (V) \\ x_1 + 4x_2 \geq 8 & (VI) \end{cases} \Rightarrow \begin{cases} x_1 = 0 & (I) \\ x_2 = 0 & (II) \\ x_1 = 5 & (III) \\ x_2 = 4 & (IV) \\ x_1 + x_2 = 6 & (V) \\ x_1 + 4x_2 = 8 & (VI) \end{cases}$$

$$F = 222 - 2x_1 - x_2 \rightarrow min$$

Thus, the geometric interpretation of the solution will have the form (Fig. 4.2):

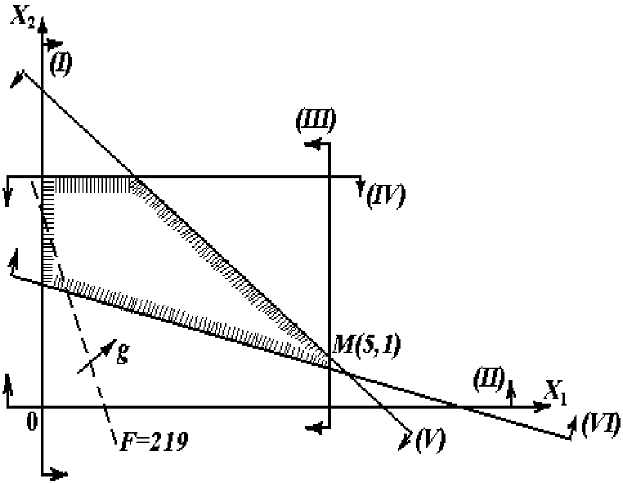

Figure 4.2 - Geometric interpretation of the solution – polygon of solutions

It can be seen from Fig. 4.2 that the solution will be the point farthest in the direction of vector g - this is point M, the coordinates of which are determined by the combined solution of equations (III) and (V):

$$\begin{cases} x_1 = 5 & (III) \\ x_1 + x_2 = 6 & (V) \end{cases} \Rightarrow \begin{cases} x_1 = 5 \\ x_2 = 1 \end{cases}$$

So:

- On equipment A, product P1 must be manufactured during $x_1 = 5$ hours. At the same time, 5x6 = 30 units will be produced. Product P2 - $x_2 = 1\ hour$ and 1x24 = 24 units will be produced.

- On equipment B, product P1 must be manufactured: $x_3 = \frac{5}{13}(5-5) = 0$ h. and 0 units will be produced. Product P2 $x_4 = \frac{24}{13}(4-1) = \frac{72}{13}$ h. and 72/13·13=72 units will be produced.

The maximum value of the goal function is F=211.

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