MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National University Zaporizhzhia Polytechnic

WORKBOOK
ON HIGHER MATHEMATICS
(3rd module)

for students majoring in
141 Power Engineering, Electrical Engineering
and Electrical Mechanics

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INTRODUCTION

One of the types of unaided work of students is the execution of calculation tasks during the semester. Their purpose is to develop practical skills. They are designed to help students more deeply capture theoretical material and learn how to apply the acquired knowledge to solve practical problems. The offered workshop with examples of calculation tasks corresponds to the course "Higher Mathematics" taught to students majoring in Power Engineering, Electrical Engineering and Electrical Mechanics full-time in the third module of the two-semester course. This workbook can also be used for students in other technical majors who study higher mathematics in a two-semester course.

This workbook covers virtually all major sections of the third module of the course on higher mathematics; it contains the necessary theoretical information, methodological instructions for completing the tasks, and examples of solving problems with a detailed explanation. A list of recommended literature is also provided. The manual also contains the necessary background material (tables, etc.). For ease of use, in this manual the numbering of examples corresponds to the numbering of tasks in the task book [12].

Calculation tasks are executed during the semester, which provides students with a systematic study of the course. In carrying out these tasks, students work with recommended textbooks and manuals, independently search for necessary literary sources and materials, analyze them and summarize, independently research and make written presentation of practical assignments.

The materials of this practicum can be considered as part of a synopsis of lectures on higher mathematics, in which the theoretical statements are illustrated in practical examples. That is why the materials in this manual can be used not only in the performance of calculation tasks, but also in preparation for the exam.

1 FUNCTIONS OF COMPLEX VARIABLE

There are several forms of writing a ***complex number***:

$z = x + iy$ – ***rectangular form***;

$z = re^{i\varphi}$ –***power form***;

$z = r(\cos\varphi + i\sin\varphi)$ – ***trigonometric form***,

where $x = Re\, z$, $y = Im\, z$ (Cartesian coordinates), $r = |z|$, $\varphi = arg\, z$ (polar coordinates). There are the following relations:

$\varphi = arctg\frac{y}{x}$, if z belongs to the I or IV quadrants,

$\varphi = arctg\frac{y}{x} \pm \pi$, if z belongs to the II or III quadrants, (1.1)

$$r = \sqrt{x^2 + y^2}, \tag{1.2}$$

$$x = r\cos\varphi, \qquad y = r\sin\varphi.$$

Raising a complex number $z = r(\cos\varphi + i\sin\varphi)$ ***to a power*** are given by ***DeMoivre's law***:

$$z^n = r^n(\cos n\varphi + i\sin n\varphi). \tag{1.3}$$

The root of degree n of a complex number z has n different values, which are calculated by the formula

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right), \quad k = 0, \ldots, n-1. \tag{1.4}$$

The points on the complex plane corresponding to the values $\sqrt[n]{z}$ are the vertices of a regular n-sided polygon inscribed in a circle of radius $R = \sqrt[n]{r}$ centered at the origin.

Task 1.1. Express a complex number in trigonometric and power forms. Make a graph.

a) $z = -1 - \sqrt{3}\, i$;

b) $z = -7^{\sqrt{7}}$;

c) $z = \sqrt[10]{6}\, i$.

Solution. a) We find the modulus and the argument of the number z. Number z belongs to the III quadrants because $x = -1 < 0$, $y = -\sqrt{3} < 0$;

$$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2;$$

$$\varphi = arctg\frac{-\sqrt{3}}{-1} + \pi = arctg\sqrt{3} + \pi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}.$$

Therefore, the number z can be represented as:

$$z = 2\left(cos\frac{4}{3}\pi + i\,sin\frac{4}{3}\pi\right);$$

$$z = 2e^{i\frac{4}{3}\pi}.$$

Fig. 1.1 shows complex number and all calculated quantities.

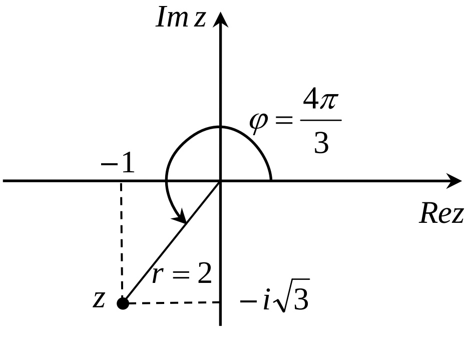

Figure 1.1

b) $r = 7^{\sqrt{7}}$, $\varphi = \pi$, because radius-vector forms an angle π with the positive direction of the axis OX. So,

$$z = 7^{\sqrt{7}}\left(\cos\pi + i\sin\pi\right);$$

$$z = 7^{\sqrt{7}}\, e^{i\pi}.$$

Fig. 1.2 shows complex number and all calculated quantities.

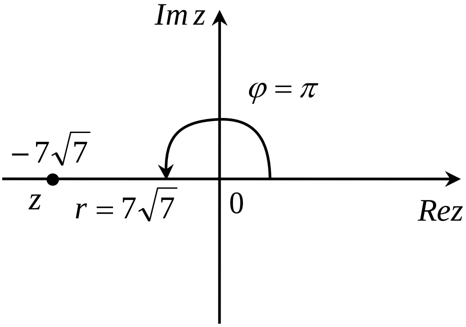

Figure 1.2

c) We find the modulus and the argument of the number z. $\varphi = \dfrac{\pi}{2}$, because radius-vector forms an angle $\dfrac{\pi}{2}$ with the positive direction of the axis OX. Further,, $r = \sqrt[10]{6}.$ Therefore, the number z can be represented in trigonometric and power forms as:

$$z = \sqrt[10]{6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right); \qquad z = \sqrt[10]{6}e^{i\frac{\pi}{2}}.$$

Fig. 1.3 shows complex number and all calculated quantities.

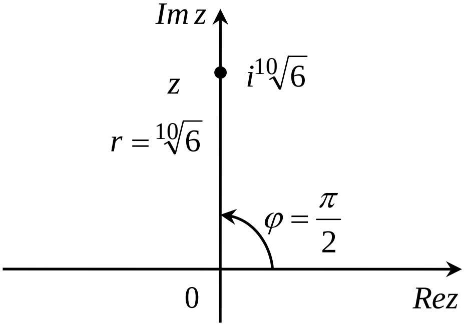

Figure 1.3

Answer: a) $z=2\left(cos\frac{4}{3}\pi+i\,sin\frac{4}{3}\pi\right)=2e^{i\frac{4}{3}\pi}$;

b) $z_3=7^{\sqrt{7}}\left(cos\,\pi+i\,sin\,\pi\right)=7^{\sqrt{7}}e^{i\pi}$;

c) $z_2=\sqrt[10]{6}\left(cos\frac{\pi}{2}+i\,sin\frac{\pi}{2}\right)=\sqrt[10]{6}e^{i\frac{\pi}{2}}$.

Task 1.2. Express a complex number in rectangular form $a+ib$. Make a graph.

а) $w=\left(1+i\sqrt{3}\right)\left(-i-1\right)$;

б) $w=\dfrac{1-2i}{\frac{1}{3}+\frac{1}{2}i}$;

в) $w^4+1=0$.

Solution. a) We perform multiplication:

$$w=\left(1+i\sqrt{3}\right)\left(-i-1\right)=-i-1+\sqrt{3}-i\sqrt{3}=\left(\sqrt{3}-1\right)-i\left(\sqrt{3}+1\right).$$

Here we have taken into account that $i\cdot i=-1$.

Fig. 1.4 shows all complex numbers.

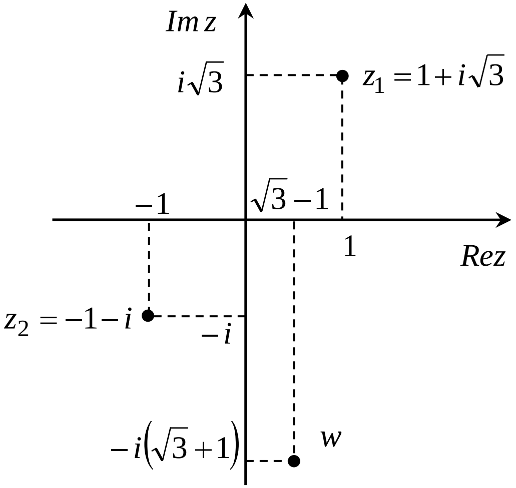

Figure 1.4

b) To reduce the fraction to the rectangular form, we multiply the numerator and denominator by the value conjugated to the denominator. In our case, this value is $\frac{1}{3} - \frac{1}{2}i$:

$$w = \frac{1-2i}{\frac{1}{3}+\frac{1}{2}i} = \frac{(1-2i)\left(\frac{1}{3}-\frac{1}{2}i\right)}{\left(\frac{1}{3}+\frac{1}{2}i\right)\left(\frac{1}{3}-\frac{1}{2}i\right)} = \frac{\frac{1}{3}-\frac{1}{2}i-\frac{2}{3}i-1}{\frac{1}{9}+\frac{1}{4}} =$$

$$= \frac{-\frac{2}{3}-\frac{7}{6}i}{\frac{13}{36}} = -\frac{24}{13} - \frac{42}{13}i\,.$$

Fig. 1.5 shows all complex numbers.

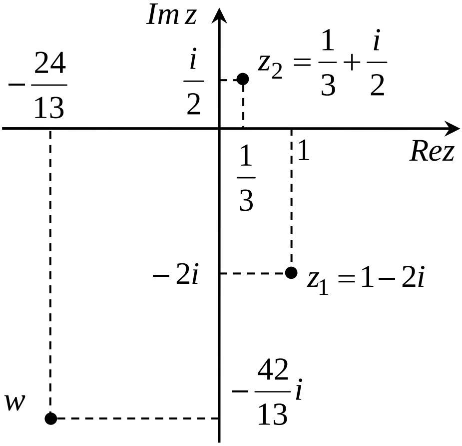

Figure 1.5

c) We use the formula (1.4). As $w^4 + 1 = 0$, then $w = \sqrt[4]{-1}$; $|-1| = 1$, $arg(-1) = \pi$. According to the formula (1.4) there are four roots:

$$w = cos\frac{\pi + 2\pi k}{4} + i\,sin\frac{\pi + 2\pi k}{4}, \quad k = 0,1,2,3.$$

$$k = 0: \qquad w_1 = cos\frac{\pi}{4} + i\,sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2};$$

$$k = 1: \qquad w_2 = cos\frac{3\pi}{4} + i\,sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2};$$

$$k = 2: \qquad w_3 = cos\frac{5\pi}{4} + i\,sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2};$$

$$k = 3: \qquad w_4 = cos\frac{7\pi}{4} + i\,sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

Geometrically numbers w_k are the vertices of a square inscribed in a unit circle. (fig. 1.6).

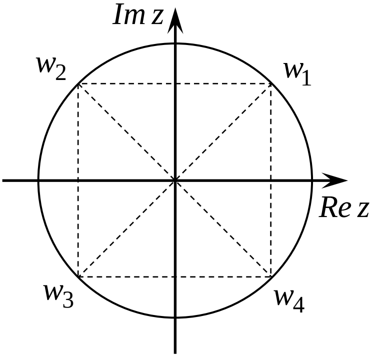

Figure 1.6

Answer: a) $\left(\sqrt{3}-1\right)-i\left(\sqrt{3}+1\right)$;

b) $-\dfrac{24}{13}-\dfrac{42}{13}i$;

c) $\dfrac{\sqrt{2}}{2}+i\dfrac{\sqrt{2}}{2}$, $-\dfrac{\sqrt{2}}{2}+i\dfrac{\sqrt{2}}{2}$, $\dfrac{\sqrt{2}}{2}-i\dfrac{\sqrt{2}}{2}$, $-\dfrac{\sqrt{2}}{2}-i\dfrac{\sqrt{2}}{2}$.

Task 1.3. Using DeMoivre's law compute the given quantity: $\left(-1+i\sqrt{3}\right)^{60}$.

Solution. We express a complex number $z=-1+i\sqrt{3}$ in trigonometric form (taking into account that $|z|=2$, $arg\,z=\dfrac{2\pi}{3}$):

$$-1+i\sqrt{3}=2\left(cos\frac{2\pi}{3}+i\,sin\frac{2\pi}{3}\right).$$

Applying DeMoivre's low (1.3) gives:

$$\left(-1+i\sqrt{3}\right)^{60}=2^{60}\left(cos\left(60\cdot\frac{2\pi}{3}\right)+i\,sin\left(60\cdot\frac{2\pi}{3}\right)\right)=$$

$$=2^{60}\left(cos\,40\pi+i\,sin\,40\pi\right)=2^{60}.$$

Answer: 2^{60}.

Task 1.4. Find the set of points on complex plane defined by the condition $\left|z-\frac{1}{2}\right|\le\left|1-\frac{1}{2}\bar{z}\right|$. Make a graph.

Solution. Let $z=x+iy$. Then the inequality has the form:

$$\left|\left(x-\frac{1}{2}\right)+iy\right|\le\left|\left(1-\frac{1}{2}x\right)+\frac{1}{2}iy\right|.$$

By definition of modulus we have:

$$\sqrt{\left(x-\frac{1}{2}\right)^2+y^2}\le\sqrt{\left(1-\frac{1}{2}x\right)^2+\frac{1}{4}y^2}.$$

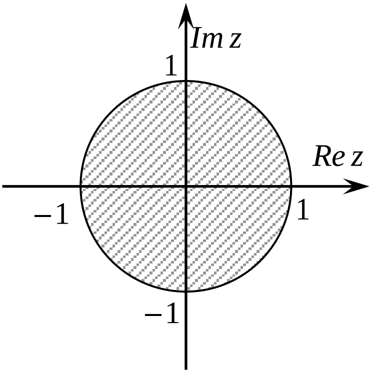

Figure 1.7

Solving the inequality we get:

$$x^2+y^2\le 1.$$

This is the inner domain of the unit circle, including the boundary (fig. 1.7).

Answer: unit circle $x^2+y^2\le 1$.

When solving problems, you can use the Euler's identity:

$$e^{iz}=\cos z+i\sin z\,, \tag{1.5}$$

from which it follows:

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}\,,\qquad \cos z=\frac{e^{iz}+e^{-iz}}{2}\,. \tag{1.6}$$

We also give the definition of another elementary function Lnz:

$$Lnz=\ln|z|+i\arg z+2\pi ki\,,\quad k\in Z\,. \tag{1.7}$$

The function $w=f(z)$ maps the points of the complex z-plane to the corresponding points of the complex w-plane. Let $z=x+iy$ and $w=u+iv$. Then the dependency $w=f(z)$ can be described using two real functions u and v of two real variables x and y:

$$\begin{cases} u=u(x,y)\\ v=v(x,y)\end{cases}.$$

The Cauchy-Riemann conditions are used to investigate the analytic function:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\,,\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\,. \tag{1.8}$$

The existence of partial derivatives of the functions $u(x,y)$ and $v(x,y)$, as well as the fulfillment of the equations (1.8) is a necessary condition for the analyticity of the function $f(z)$, and with the additional requirement of differentiability of functions $u(x,y)$ and $v(x,y)$ we get a sufficient condition of analyticity of the function $f(z)$.

Task 1.5. Find the analytic function $f(z)$ having its imaginary part $v(x,y)=2\cos x\ chy-x^2+y^2$, if $f(0)=2$.

Solution. If the function $f(z)$ is analytic, then its imaginary and real parts satisfy the Cauchy-Riemann conditions (1.8). Using the first condition, we obtain:

$$u(x,y)=\int\frac{\partial v}{\partial y}dx+\psi(y)=\int(2\,cos\,xshy+2y)dx+\psi(y)=$$

$$=2\,sin\,xshy+2xy+\psi(y).$$

From the second condition we obtain equality:

$$2\,sin\,xchy+2x+\psi'(y)=-(-2\,sin\,xchy-2x),$$

from which we get $\psi'(y)=0$ and $\psi(y)=C$. Consequently:

$$u(x,y)=2\,sin\,xshy+2xy+C\;;$$

$$f(z)=2\,sin\,xshy+2xy+C+i\left(2\,cos\,xchy-x^2+y^2\right)=$$

$$=2i(\cos x\cos iy-\sin x\sin iy)-i\left(x^2+2xyi-y^2\right)+C=$$

$$=2i\,cos(x+iy)-i(x+iy)^2+C=2i\,cos\,z-iz^2+C\,.$$

We find the constant C from the condition $f(0)=2$: $2i+C=2$, then $C=2-2i$. We finally have:

$$f(z)=2i(\cos z-1)-iz^2+2\,.$$

Answer: $f(z)=2i(\cos z-1)-iz^2+2$.

2 DIFFERENTIAL EQUATIONS

The equation of the next forms

$$f_1(x)\cdot g_1(y)dx + f_2(x)\cdot g_2(y)dy = 0 . \qquad (2.1)$$

or

$$y' = f_1(x)f_2(y). \qquad (2.2)$$

are called ***equations with separable variables.***

To solve equation (3.1), we separate the variables so that dx is multiplied by the function of x alone, and dy is multiplied by the function of y alone. For this (2.1) is divided by $g_1(y)\cdot f_2(x)$ ($f_2(x)\neq 0$, $g_1(y)\neq 0$). We get

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0 .$$

The general integral of the equation (2.1) is:

$$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = C .$$

To solve equation (2.2) we separate the variables, assuming that $f_2(y)\neq 0$:

$$\frac{dy}{dx} = f_1(x)f_2(y),$$

$$\frac{dy}{f_2(y)} = f_1(x)dx .$$

The general integral of the equation (2.2):

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C .$$

Task 2.1. Find the general solution of the separable differential equations.

1) $(x+1)\frac{dy}{dx} = y+2$;

2) $(x^2+4)y' - y^2 = 16$;

3) $(y+yx^2)dx + (x-xy^2)dy = 0$.

***Solution.* 1)** $(x+1)\frac{dy}{dx} = y+2$.

We separate the variables and integrate this equation:

$$\frac{dy}{y+2} = \frac{dx}{x+1},$$

$$\int \frac{dy}{y+2} = \int \frac{dx}{x+1} + lnC,$$

$$ln|y+2| = ln|x+1| + ln|C|,$$

$$y+2 = C(x+1),$$

$$y = C(x+1) - 2.$$

2) $(x^2+4)y' - y^2 = 16$.

We separate the variables and integrate this equation:

$$(x^2+4)\frac{dy}{dx} = 16 + y^2,$$

$$\frac{dy}{y^2+16} = \frac{dx}{x^2+4}, \qquad \int \frac{dy}{y^2+16} = \int \frac{dx}{x^2+4},$$

$$\frac{1}{4} arctg \frac{y}{4} = \frac{1}{2} arctg \frac{x}{2} + C.$$

3) $(y+yx^2)dx + (x-xy^2)dy = 0$.

We separate the variables and integrate this equation:

$$x(y^2-1)dy = y(1+x^2)dx,$$

$$\left(y - \frac{1}{y}\right)dy = \left(\frac{1}{x} + x\right)dx, \qquad \int \left(y - \frac{1}{y}\right)dy = \int \left(\frac{1}{x} + x\right)dx,$$

$$\frac{y^2}{2} - ln|y| = ln|x| + \frac{x^2}{2} + C,$$

$$\frac{y^2}{2} = ln|x\,y| + \frac{x^2}{2} + C .$$

Answer: **1)** $y = C(x+1) - 2$;

2) $\frac{1}{4} arctg \frac{y}{4} = \frac{1}{2} arctg \frac{x}{2} + C$;

3) $\frac{y^2}{2} = ln|x\,y| + \frac{x^2}{2} + C$.

A function $f(x, y)$ is called a ***homogeneous function of the m-th degree*** with respect to arguments x and y if equality is satisfied

$$f(tx,ty) = t^m f(x,y) \;\; \forall t \in R ,\; \alpha \in R .$$

Differential equation in a norrmal form

$$y' = f(x, y)$$

is called ***homogeneous*** with respect to variables x and y , if function $f(x, y)$ is a homogeneous function of zero degree with respect to its arguments.

Differential equation in a differential form

$$P(x, y)dx + Q(x, y)dy = 0$$

is homogeneous one, if functions $P(x, y)$, $Q(x, y)$ are homogeneous functions of the same degree α , i.e. $P(tx,ty) = t^\alpha P(x,y)$, $Q(tx,ty) = t^\alpha Q(x,y)$.

Homogeneous equations by substitution

$$u = \frac{y}{x} ,\; y = u \cdot x ,$$

$$y' = u + x u'$$

can be reduced to equations with separable variables with respect to a new function $u = u(x)$.

Task 2.2. Find the general solution of the homogeneous differential equation.

1) $xy' = y + \sqrt{4x^2 + y^2}$;

2) $\left(y^2 + x^2\right)dx + \left(xy - 2x^2\right)dy = 0$.

Solution. **1)** $xy' = y + \sqrt{4x^2 + y^2}$.

Write the equation in the normal form:

$$y' = \frac{y}{x} + \sqrt{4 + \left(\frac{y}{x}\right)^2}.$$

This equation is homogeneous because the function $\frac{y}{x} + \sqrt{4 + \left(\frac{y}{x}\right)^2}$ is a homogeneous function of zero degree.

Let's make the substitution $u = \frac{y}{x}$, then $y = u\,x$, $y' = u + x\,u'$. We have:

$$u + x u' = u + \sqrt{4 + u^2},$$

$$\frac{du}{\sqrt{4+u^2}} = \frac{dx}{x}, \qquad \int \frac{du}{\sqrt{4+u^2}} = \int \frac{dx}{x},$$

$$ln\left|u + \sqrt{4+u^2}\right| = ln|x| + ln\,C,$$

$$u + \sqrt{4+u^2} = Cx.$$

In the last expression, we return to the variable y:

$$y + \sqrt{4x^2 + y^2} = Cx^2.$$

2) $\left(y^2 + x^2\right)dx + \left(xy - 2x^2\right)dy = 0$.

Functions $y^2 + x^2$ and $xy - 2x^2$ are homogeneous function of two degree. Therefore, this equation is homogeneous one. Let's make the substitution $u = \frac{y}{x}$, then $y = u\,x$, $y' = u + x\,u'$. We have

$$y' = -\frac{y^2+x^2}{xy-2x^2} = -\frac{\left(\frac{y}{x}\right)^2+1}{\frac{y}{x}-2},$$

$$u'x+u = -\frac{u^2+1}{u-2},$$

$$\frac{u-2}{2u^2-2u+1}du = -\frac{dx}{x}.$$

We integrate the left-hand part of the equation separately:

$$\int\frac{u-2}{2u^2-2u+1}du = \frac{1}{4}\int\frac{(4u-2)-6}{2u^2-2u+1}du = \frac{1}{4}\int\frac{d\left(2u^2-2u+1\right)}{2u^2-2u+1} -$$

$$-\frac{3}{2}\int\frac{1}{2u^2-2u+1}du = \frac{1}{4}ln\left|2u^2-2u+1\right| - \frac{3}{4}\int\frac{1}{(u-1/2)^2+1/4}du =$$

$$= \frac{1}{4}ln\left|2u^2-2u+1\right| - \frac{3}{2}arctg(2u-1).$$

The general integral of the equation is:

$$\frac{1}{4}ln\left|2u^2-2u+1\right| - \frac{3}{2}arctg(2u-1) = -ln|x| + C.$$

In the last expression, we return to the variable y:

$$\frac{1}{4}ln\left|\frac{2y^2-2xy+x^2}{x^2}\right| - \frac{3}{2}arctg\left(\frac{2y}{x}-1\right) = -ln|x| + C.$$

Answer: **1)** $y+\sqrt{4x^2+y^2} = Cx^2$;

2) $\frac{1}{4}ln\left|\frac{2y^2-2xy+x^2}{x^2}\right| - \frac{3}{2}arctg\left(\frac{2y}{x}-1\right) = -ln|x| + C$.

A first-order differential equation is called ***linear*** if it can be written in the form

$$y' + P(x)y = Q(x), \qquad (2.3)$$

where $P(x)$, $Q(x)$ are defined functions.

Bernoulli's method of solving first-order linear differential equations. Equation (2.3) can be reduced to two differential equations with separable variables by substitution $y = uv$ (Bernoulli's substitution), where $u = u(x)$ and $v = v(x)$ are some function. Then:

$$u'v + uv' + P(x)uv = Q(x),$$

$$u'v + u[v' + P(x)v] = Q(x). \tag{2.4}$$

Let us choose a function $v = v(x)$ such as the expression in square brackets be equal to zero:

$$v' + P(x)v = 0.$$

We find the particular solution of this equation $v(x) = e^{-\int P(x)dx}$, substitute it in the equation (2.4) and obtain the second differential equation with separable variables (with respect to the function $u(x)$):

$$u'v = Q(x), \qquad \text{i.e. } u' = Q(x)e^{\int P(x)dx} \tag{2.5}$$

We find the general solution of the equation (2.5) $u(x) = \int Q(x)e^{\int P(x)dx}dx + C$. Therefore, we find the general solution of the original equation (2.3):

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right].$$

Differential equation

$$y' + P(x)y = Q(x)y^{\alpha},$$

where $\alpha = const \in R$, $\alpha \neq 1$, $\alpha \neq 0$, is called ***Bernoulli's equation***. It can also be solved with the Bernoulli's substitution.

Task 2.3. Find the general solution of the linear differential equation of the first order or Bernoulli’s equation.

1) $xy' - 2y = \frac{1}{x}$;

2) $y' - \frac{1}{x}y = y^2(ln\, x + x)$.

Solution. **1)** $xy' - 2y = \frac{1}{x}$.

This equation is linear of the first order. We divide the equation by $x \quad (x \neq 0)$:

$$y' - \frac{2y}{x} = \frac{1}{x^2} \qquad \left(P(x) = -\frac{2}{x},\ Q(x) = \frac{1}{x^2} \right).$$

We make Bernoulli's substitution: $y = uv$, then

$$u'v + v'u - \frac{2}{x}uv = \frac{1}{x^2},$$

$$u'v + u\left(v' - \frac{2v}{x} \right) = \frac{1}{x^2}.$$

Step 1: $\quad v' - \frac{2v}{x} = 0 \quad \Rightarrow \quad \frac{dv}{dx} - \frac{2v}{x} = 0,$

$$\frac{dv}{v} = \frac{2}{x}dx, \qquad \int \frac{dv}{v} = 2\int \frac{dx}{x},$$

and we get $v = x^2$.

Step 2: $\quad u'x^2 = \frac{1}{x^2} \ \Rightarrow \ \frac{du}{dx} = \frac{1}{x^4},$

$$du = \frac{1}{x^4}dx, \qquad \int du = \int \frac{1}{x^4}dx,$$

and we get $u = -\frac{1}{3x^3} + C$.

Thus, the general solution of this equation is:

$$y = uv = -\frac{1}{3x} + Cx^2.$$

2) $y' - \frac{1}{x}y = y^2(ln\, x + x)$.

This is Bernoulli's equation. We make Bernoulli's substitution: $y = uv$, then

$$u'v + v'u - \frac{1}{x}uv = u^2v^2(\ln x + x),$$

$$u'v + u\left(v' - \frac{v}{x}\right) = u^2v^2(\ln x + x).$$

Step 1: $v' - \frac{v}{x} = 0 \quad \Rightarrow \quad \frac{dv}{dx} - \frac{v}{x} = 0,$

$$\frac{dv}{v} = \frac{1}{x}dx, \qquad \int\frac{dv}{v} = \int\frac{dx}{x},$$

and we get $v = x$.

Step 2: $u'x = u^2x^2(\ln x + x) \Rightarrow \frac{du}{dx} = u^2x(\ln x + x),$

$$\frac{du}{u^2} = x(\ln x + x)dx, \qquad \int\frac{du}{u^2} = \int x(\ln x + x)dx,$$

and we get $-\frac{1}{u} = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{1}{3}x^3 - \frac{C}{12}$, or:

$$u = \frac{12}{C - 6x^2 \ln x + 3x^2 - 4x^3}.$$

Thus, the general solution of this equation is:

$$y = uv = \frac{12x}{C - 6x^2 \ln x + 3x^2 - 4x^3}.$$

Answer: **1)** $y = Cx^2 - \frac{1}{3x}$;

2) $y = \frac{12x}{C - 6x^2 \ln x + 3x^2 - 4x^3}$.

Differential equation of the first order

$$P(x,y)dx + Q(x,y)dy = 0. \tag{2.6}$$

is called ***exact differential equation***, if in the domain of definition of functions $P(x,y)$, $Q(x,y)$ equality holds

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}. \tag{2.7}$$

The general integral of the equation (2.6) is determined by one of the next formulas:

$$\int_{x_0}^{x} P(x, y_0)dx + \int_{y_0}^{y} Q(x, y)dy = C, \qquad (2.8)$$

$$\int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy = C, \qquad (2.9)$$

where $(x_0; y_0) \in D$.

Task 2.4. Find the general solution of the exact differential equation $\left(2x - 4y^2 + 6xy^2\right)dx + \left(2 - 8xy + 6x^2 y\right)dy = 0$.

Solution. Let's check the condition (2.7):

$$P(x, y) = 2x - 4y^2 + 6xy^2; \quad Q(x, y) = 2 - 8xy + 6x^2 y.$$

$$\frac{\partial P}{\partial y} = -8y + 12xy, \qquad \frac{\partial Q(x, y)}{\partial x} = -8y + 12xy.$$

Condition (2.7) is satisfied, so this equation is the exact differential equation. The solution of this equation can be found by the formulas (2.8) or (2.9). The point $(x_0; y_0) \in D$ should be selected so that the functions $P(x, y)$, $Q(x, y)$ are as simple as possible. Take the point $(0;0)$ as a point $(x_0; y_0)$.

According to the formula (2.8) we get:

$$\int_0^x (2x - 4 \cdot 0 + 6x \cdot 0)dx + \int_0^y \left(2 - 8xy + 6x^2 y\right)dy = C,$$

$$x^2 + 2y - 4xy^2 + 3x^2 y^2 = C.$$

Answer: $x^2 + 2y - 4xy^2 + 3x^2 y^2 = C$.

Differential equations of order higher than the first. Reduction of order.

1) Differential equation

$$y^{(n)} = f(x) \qquad (2.10)$$

can be solved by sequential integration n times. The general solution of the equation (2.10):

$$y=\varphi(x)+C_1\frac{x^{(n-1)}}{(n-1)!}+C_2\frac{x^{(n-2)}}{(n-2)!}+\ldots+C_n .$$

2) If we have the equation of the form

$$F\left(x,y^{(k)},y^{(k+1)},\ldots,y^{(n)}\right)=0, \qquad (2.11)$$

which does not contain explicitly the unknown function and its derivatives up to the $(k-1)$-th order, then in order to reduce the order of equation (2.11), a substitution $y^{(k)}=z(x)$ must be made, and $y^{(k+1)}=z'(x)$, …, $y^{(n)}=z^{(n-k)}(x)$. Equation (2.11) reduces to the $(n-k)$-th order equation with respect to the function $z(x)$:

$$F\left(x,z',z'',\ldots,z^{(n-k)}\right)=0 .$$

3) If we have the equation of the form

$$F\left(y,y',y'',\ldots,y^{(n)}\right)=0, \qquad (2.12)$$

which does not contain explicitly independent variable x, then in order to reduce the order of equation (2.12), a substitution $y'=p(y)$ must be made. Further we get

$$y''=p\cdot\frac{dp}{dy}=p\cdot p',$$

$$y'''=p\left(\left(\frac{dp}{dy}\right)^2+p\cdot\frac{d^2p}{dy^2}\right)=p\left((p')^2+p\cdot p''\right)$$

and so on.

We obtain the equation of the form:

$$F\left(y,p,\frac{dp}{dy},\frac{d^2p}{dy^2},\ldots,\frac{d^{(n-1)}p}{dy^{(n-1)}}\right)=0.$$

Task 2.5. Find the general solution of the differential equation of order higher than the first.

1) $y'''=x^3+2$;

2) $x^2y''=2y'^2$**;**

3) $1+y'^2-2yy''=0$.

***Solution.* 1)** $y'''=x^3+2$**.**

This equation is of the type (2.10). We integrate the given equation three times by the variable x:

$$y''=\int\left(x^3+2\right)dx=\frac{x^4}{4}+2x+C_1,$$

$$y'=\int\left(\frac{x^4}{4}+2x+C_1\right)dx=\frac{x^5}{20}+x^2+C_1x+C_2,$$

$$y=\int\left(\frac{x^5}{20}+x^2+C_1x+C_2\right)dx=\frac{x^6}{120}+\frac{x^3}{3}+\frac{1}{2}C_1x^2+C_2x+C_3.$$

2) $x^2y''=2y'^2$**.**

This equation is of the type (2.11), it does not contain explicitly the unknown function $y(x)$. Let us reduce its order. Let $z=y'$, then $z'=y''$. We have:

$$x^2\cdot z'=2z^2 \qquad (2.13)$$

Separate variable equation is obtained. Let's find his solution:

$$x^2\cdot\frac{dz}{dx}=2z^2, \qquad x^2\cdot\frac{dz}{dx}=2z^2,$$

$$\int\frac{dz}{z^2}=\int\frac{2}{x^2}dx, \qquad -\frac{1}{z}=-\frac{2}{x}-C_1,$$

$$z=\frac{x}{2+C_1x}.$$

Since $z=y'$, then $y'=\dfrac{x}{2+C_1x}$. We integrate:

$$y=\int\frac{xdx}{2+C_1x}=\frac{1}{C_1}\int\left(1-\frac{2}{2+C_1x}\right)dx=\frac{1}{C_1}\left(x-\frac{2}{C_1}\ln\left|2+C_1x\right|\right)+C_2,$$

if $C_1\neq 0$,

$$y=\int\frac{xdx}{2}=\frac{x^2}{4}+C\text{, if } C_1=0$$

3) $1+y'^2-2yy''=0$.

This equation is of the type (2.12), it does not contain explicitly independent variable x. Let us reduce its order. Let $y'=p(y)$, then $y''=pp'$. We have:

$$1+p^2-2ypp'=0, \qquad 2ypp'=p^2+1,$$

$$\frac{2pdp}{p^2+1}=\frac{dy}{y}, \qquad \int\frac{2pdp}{p^2+1}=\int\frac{dy}{y},$$

$$ln\left|p^2+1\right|=ln|y|+lnC_1, \qquad p^2=C_1y-1,$$

$$y'=\pm\sqrt{C_1y-1}, \qquad \frac{dy}{\sqrt{C_1y-1}}=\pm dx,$$

$$\int\frac{dy}{\sqrt{C_1y-1}}=\pm\int dx, \qquad \frac{2}{C_1}\sqrt{C_1y-1}=\pm x+C_2$$

Answer: **1)** $y=\frac{x^6}{120}+\frac{x^3}{3}+\frac{1}{2}C_1x^2+C_2x+C_3$;

2) $y=\frac{1}{C_1}\left(x-\frac{2}{C_1}\ln|1+C_1x|\right)+C_2$ $(C_1\neq 0)$,

$y=\frac{x^2}{4}+C$;

3) $\frac{2}{C_1}\sqrt{C_1y-1}=\pm x+C_2$.

Task 2.6. Solve the Cauchy's problem.

1) $y''=\frac{1}{x^3}$, $y(1)=\frac{3}{2}$, $y'(1)=-\frac{1}{2}$.

2) $2y(y')^3+y''=0$, $y(0)=0$, $y'(0)=-2$;

3) $y'-xy''=2(1+x^2y'')$, $y(1)=3$, $y'(1)=2$.

***Solution.* 1)** $y'' = \frac{1}{x^3}$, $y(1) = \frac{3}{2}$, $y'(1) = -\frac{1}{2}$.

We integrate the given equation:

$$y' = \int \frac{1}{x^3} dx = -\frac{1}{2x^2} + C_1.$$

Using the second initial condition we find C_1:

$$y'(1) = -\frac{1}{2} + C_1 = -\frac{1}{2}, \; C_1 = 0.$$

Given that $C_1 = 0$, we get

$$y = -\int \frac{1}{2x^2} dx = \frac{1}{2x} + C_2.$$

Using the first initial condition we find C_2:

$$y(1) = \frac{1}{2} + C_2 = \frac{3}{2}, \; C_2 = 1.$$

Finally we get the solution of the Cauchy's problem:

$$y = \frac{1}{2x} + 1.$$

2) $2y(y')^3 + y'' = 0$, $y(0) = 0$, $y'(0) = -2$.

This equation is of the type (2.12), it does not contain explicitly independent variable x. Let $y' = p(y)$, then $y'' = pp'$. We have

$$2yp^3 + pp' = 0, \qquad p' = -2yp^2, \qquad p' = -2yp^2,$$

$$\frac{dp}{dy} = -2yp^2, \qquad \frac{dp}{p^2} = -2ydy,$$

$$\int \frac{dp}{p^2} = -2\int ydy, \qquad -\frac{1}{p} = -y^2 + C_1,$$

$$-\frac{1}{y'} = -y^2 + C_1. \tag{2.14}$$

Using the both initial conditions we find C_1:

$$-\frac{1}{-2} = -0^2 + C_1, \; C_1 = \frac{1}{2}.$$

Let's substitute $C_1 = \frac{1}{2}$ in (2.14):

$$\frac{1}{y'} = y^2 - \frac{1}{2}, \qquad y' = \frac{2}{2y^2 - 1},$$

$$\left(y^2 - \frac{1}{2}\right)dy = dx, \qquad \int\left(y^2 - \frac{1}{2}\right)dy = \int dx,$$

$$\frac{y^3}{3} - \frac{y}{2} = x + C_2.$$

Using the first initial condition we find $C_2 = 0$.

Solution of the Cauchy's problem:

$$\frac{y^3}{3} - \frac{y}{2} = x.$$

3) $y' - xy'' = 2\left(1 + x^2 y''\right),\ y(1) = 3,\ y'(1) = 2$.

This equation is of the type (2.11), it does not contain explicitly the unknown function $y(x)$. Let $z = y'$, then $z' = y''$. We have

$$z'\left(2x^2 + x\right) = z - 2, \qquad \frac{dz}{z-2} = \frac{dx}{x(2x+1)},$$

$$\frac{dz}{z-2} = \frac{dx}{x(2x+1)}, \qquad \int\frac{dz}{z-2} = \int\left(\frac{1}{x} - \frac{2}{2x+1}\right)dx,$$

$$\ln|z-2| = \ln|x| - \ln|2x+1| + \ln C_1, \qquad z = \frac{C_1 x}{2x+1} + 2.$$

Let's return to the variable y:

$$y' = \frac{C_1 x}{2x+1} + 2.$$

Using the second initial condition we find $C_1 = 0$, then

$$y' = 2,\ y = 2x + C_2.$$

Using the first initial condition we find C_2:

$$y(1) = 2 + C_2 = 3, \qquad C_2 = 1.$$

Finally we get the solution of the Cauchy's problem:

$$y = 2x + 1.$$

Answer: **1)** $y = \dfrac{1}{2x} + 1$;

2) $\dfrac{y^3}{3} - \dfrac{y}{2} = x$;

3) $y = 2x + 1$,

Differential equation

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = f(x), \qquad (2.15)$$

(where a_1, a_2, …, a_n are constants, $f(x)$ is a given continuous function) is called ***non-homogeneous linear differential equation (NLDE) of the*** n ***th order with constant coefficients.***

Differential equation

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = 0, \qquad (2.16)$$

(where a_1, a_2, …, a_n are constants) is called ***homogeneous linear differential equation (HLDE) of the*** n ***th order with constant coefficients.***

Solution of HLDE of the n ***th order with constant coefficients.*** We construct a characteristic equation and find its roots:

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0. \qquad (2.17)$$

1) If all the roots of equation (2.17) are real and simple (different), then we write the general solution of equation (2.16) as:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x}.$$

2) If all the roots of the equation (2.17) are real, but among them there are the roots of multiplicities m $(m > 1)$, then each simple root λ corresponds to a particular solution $e^{\lambda x}$, and each root λ of multiplicity m corresponds to m particular solutions

$$e^{\lambda x},\ xe^{\lambda x},\ x^2 e^{\lambda x},\ \ldots,\ x^{m-1} e^{\lambda x}.$$

3) If among the roots of equation (2.17) there are complex conjugated ones, then each pair of simple roots $\alpha \pm i\beta$ corresponds to two particular solutions $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$, and each pair of

conjugated roots $\alpha \pm i\beta$ of multiplicity m corresponds to $2m$ particular solutions:

$$e^{\alpha x} \cos \beta x,\ x \cdot e^{\alpha x} \cos \beta x,\ x^2 \cdot e^{\alpha x} \cos \beta x,\ \ldots,\ x^{m-1} \cdot e^{\alpha x} \cos \beta x,$$

$$e^{\alpha x} \sin \beta x,\ x \cdot e^{\alpha x} \sin \beta x,\ x^2 \cdot e^{\alpha x} \sin \beta x,\ \ldots,\ x^{m-1} \cdot e^{\alpha x} \sin \beta x.$$

The structure of the general solution NLDE of the n ***th order***. The general solution of (2.15) consists of the sum of particular solution y_{nh} of non-homogeneous equation and the general solution y_0 of the corresponding homogeneous equation.

NLDE of the n***th order with constant coefficients and the right-hand side of the special form.*** Let the right part of the ***NLDE*** (2.15) have a special form:

$$f(x) = e^{ax}\left(P_r(x)\cos bx + Q_s(x)\sin bx\right), \tag{2.18}$$

where $P_r(x)$, $Q_s(x)$ are polynomials of degrees r and s respectively, a, b are any numbers.

For differential equations with the right-hand side of the form (2.18), the partial solution has a similar structure:

$$y_{nh} = x^k e^{ax}\left(\tilde{P}_m(x) \cos bx + \tilde{Q}_m(x) \sin bx\right),$$

where $\tilde{P}_m(x)$, $\tilde{Q}_m(x)$ are polynomials of degree $m = \max\{r, s\}$, k – the number of roots of the characteristic equation (given their multiplicities), which coincide with the number $z = a + ib$. We define the structure of the general solution of equation (2.15), in which only the coefficients of polynomials $\tilde{P}_m(x)$, $\tilde{Q}_m(x)$ are unknown, then we substitute the particular solution y_{nh} and its derivatives in equation (2.15), and equate the coefficients of similar terms on the left-hand side and right-hand side. We obtain the required number of linear algebraic equations to find the unknown coefficients.

Superposition principle. If the right-hand side of equation (2.15) is the sum of two functions $f(x) = f_1(x) + f_2(x)$, and y_{1nh}, y_{2nh} are particular solutions of the equations

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = f_1(x),$$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = f_2(x),$$

then the function $y_{nh} = y_{1nh} + y_{2nh}$ is the solution of the equation (2.15).

Task 2.7. Find the general solutions of the linear differential equations using method of undetermined coefficients.

1) $y'' + 2y' + 2y = 40\,e^{3x}\cos x$;

2) $y'' + 2y' = (2x+1)\cos 2x + \sin 2x$;

3) $y^{IV} + 8y'' + 16y = 2x^3 + 4x^2 + 3x + 1$;

4) $y'' - 6y' + 9y = e^{3x}\left((2x+1)\cos x + \sin x\right)$;

5) $y'' - 4y' + 5y = xe^x + \sin 2x$.

Solution. **1)** $y'' + 2y' + 2y = 40\,e^{3x}\cos x$.

Find the roots of the characteristic equation:

$$\lambda^2 + 2\lambda + 2 = 0,$$
$$\lambda_{1,2} = -1 \pm i.$$

Solution of homogeneous equation: $y_0 = e^{-x}\left(C_1\cos x + C_2\sin x\right)$.

For a given equation we have $a = 3$, $b = 1$, $z = 3 + i$, therefore $k = 0$. A particular solution looks like this:

$$y_{nh} = e^{3x}\left(A\cos x + B\sin x\right).$$

Let's find y'_{nh} and y''_{nh}:

$$y'_{nh} = e^{3x}\left((3A + B)\cos x + (-A + 3B)\sin x\right),$$
$$y''_{nh} = e^{3x}\left((8A + 6B)\cos x + (-6A + 8B)\sin x\right).$$

We substitute y'_{nh} and y''_{nh} into the original equation and divide both parts of the equation by e^{3x}, then we obtain:

$$(16A + 8B)\cos x + (-8A + 16B)\sin x = 40\cos x.$$

Equating the coefficients at $\cos x$ and $\sin x$ in the left-hand side and the right-hand side of the last equality, we obtain a system of algebraic equations with respect to A and B:

$$\begin{array}{r|l} \cos x & \quad 16A + 8B = 40, \\ \sin x & \quad -8A + 16B = 0. \end{array}$$

We have $A=2$, $B=1$. So, $y_{nh}=e^{3x}(2\,cos\,x+sin\,x)$.

The general solution of the equation:

$$y=e^{-x}(C_1\cos x+C_2\sin x)+e^{3x}(2\cos x+\sin x).$$

2) $y''+2y'=(2x+1)\cos 2x+\sin 2x$.

Find the roots of the characteristic equation:

$$\lambda^2+2\lambda=0,$$

$$\lambda_1=0,\qquad \lambda_2=-2.$$

Solution of homogeneous equation: $y_0=C_1+C_2e^{-2x}$.

For a given equation we have $a=0$, $b=2$, $z=2i$, therefore $k=0$. A particular solution looks like this:

$$y_{\div}=(Ax+B)\cos 2x+(Cx+D)\sin 2x.$$

Let's find y'_{nh} and y''_{nh}:

$$y'_{nh}=(2C\,x+A+2D)cos\,2x+(-2Ax-2B+C)sin\,2x,$$

$$y''_{nh}=(-4A\,x-4B+4C)cos\,2x+(-4Cx-4A-4D)sin\,2x.$$

We substitute y'_{nh} and y''_{nh} into the original equation:

$$((-4A+C)x+2A-4B+4C+4D)\cos 2x+((-4A-4C)x-$$

$$-4A-4B+2C-4D)\sin 2x=(2x+1)\cos 2x+\sin 2x.$$

Equating the coefficients at $\cos x$, $\sin x$, $x\cos x$ and $x\sin x$ in the left-hand side and the right-hand side of the last equality, we obtain a system of algebraic equations with respect to A, B, C and D:

$$\begin{array}{l|l}
x\cos 2x & -4A+C=2, \\
\cos 2x & 2A-4B+4C+4D=1, \\
x\sin 2x & -4A-4C=0, \\
\sin 2x & -4A-4B+2C-4D=1.
\end{array}$$

From the obtained system we find that $A=-\dfrac{1}{4}$, $B=0$, $C=\dfrac{1}{4}$, $D=\dfrac{1}{8}$.

So, $y_{nh}=-\dfrac{1}{4}x\,cos\,2x+\left(\dfrac{1}{4}x+\dfrac{1}{8}\right)sin\,2x$.

The general solution of the equation:

$$y = C_1 + C_2 e^{-2x} - \frac{1}{4} x \cos 2x + \left(\frac{1}{4} x + \frac{1}{8} \right) \sin 2x .$$

3) $y^{IV} + 8y'' + 16y = 2x^3 + 4x^2 + 3x + 1$.

Find the roots of the characteristic equation:

$$\lambda^4 + 8\lambda^2 + 16 = 0 ,$$

$$\lambda_{1,2} = 2i , \qquad \lambda_{3,4} = -2i .$$

Solution of homogeneous equation:

$$y_0 = (C_1 + C_2 x) \cos 2x + (C_3 + C_4 x) \sin 2x .$$

For a given equation we have $a = 0$, $b = 0$, $z = 0$, therefore $k = 0$. A particular solution looks like this:

$$y_{nh} = Ax^3 + Bx^2 + Cx + D ,$$

$$y'_{nh} = 3Ax^2 + 2Bx + C ,$$

$$y''_{nh} = 6Ax + B , \; y'''_{nh} = 6A , \; y^{IV}{}_{nh} = 0 .$$

We substitute $y_{nh}{}^{IV}$ and y''_{nh} into the original equation, then we obtain:

$$8(6Ax + B) + 16\left(Ax^3 + Bx^2 + Cx + D\right) = 2x^3 + 4x^2 + 3x + 1 .$$

We equate the coefficients at equal degrees of x in the left-hand side and the right-hand side of the last equality, so we obtain a system of algebraic equations with respect to A, B, C and D:

$$\begin{array}{c|l} x^3 & 16A = 2, \\ x^2 & 16B = 4, \\ x^1 & 48A + 16C = 3, \\ x^0 & 16B + 16D = 1. \end{array}$$

we find that $A = \frac{1}{8}$, $B = \frac{1}{4}$, $C = D = -\frac{3}{16}$. So,

$$y_{nh} = \frac{1}{8} x^3 + \frac{1}{4} x^2 - \frac{3}{16} x - \frac{3}{16} .$$

The general solution of the equation:

$$y = (C_1 + C_2 x) \cos 2x + (C_3 + C_4 x) \sin 2x + \frac{1}{8} x^3 + \frac{1}{4} x^2 - \frac{3}{16} x - \frac{3}{16} .$$

4) $y'' - 6y' + 9y = e^{3x}\left(\left(2x+1\right)\cos x + \sin x\right)$.

Find the roots of the characteristic equation:

$$\lambda^2 - 6\lambda + 9 = 0\,,$$

$$\lambda_{1,2} = 3.$$

Solution of homogeneous equation: $y_0 = \left(C_1 + C_2\, x\right)e^{3x}$.

For a given equation we have $a = 3$, $b = 1$, $z = 3 + i$, therefore $k = 0$. A particular solution looks like this:

$$y_{nh} = e^{3x}\left(\left(Ax + B\right)cos\, x + \left(Cx + D\right)sin\, x\right).$$

Let's find y'_{nh} and y''_{nh}:

$$y'_{nh} = e^{3x}\left(\left(\left(3A + C\right)x + A + 3B + D\right)cos\, x + \right.$$
$$\left. + \left(\left(-A + 3C\right)x - B + C + 3D\right)\sin x\right),$$

$$y''_{nh} = e^{3x}\left(\left(\left(8A + 6C\right)x + 6A + 8B + 2C + 6D\right)cos\, x + \right.$$
$$\left. + \left(\left(-6A + 8C\right)x - 2A - 6B + 6C + 8D\right)\sin x\right).$$

We substitute y'_{nh} and y''_{nh} into the original equation and divide both parts of the equation by e^{3x}, then we obtain:

$$\left(-2Ax - B + 2C\right)\cos x + \left(-Cx - 2A - D\right)\sin x = \left(2x+1\right)\cos x + \sin x\,.$$

Equating the coefficients at $\cos x$, $\sin x$, $x\cos x$ and $x\sin x$ in the left-hand side and the right-hand side of the last equality, we obtain a system of algebraic equations with respect to A, B, C and D:

$$\begin{array}{r|l} x\cos x & -A = 2, \\ \cos x & -B + 2C = 1, \\ x\sin x & -C = 0, \\ \sin x & -2A - D = 1. \end{array}$$

We have $A = -2$, $B = -1$, $C = 0$, $D = 3$. So, $y_{nh} = e^{3x}\left(\left(-2x - 1\right)cos\, x + 3\,sin\, x\right)$.

The general solution of the equation:

$$y = e^{3x}\left(C_1 + C_2 x\right) + e^{3x}\left(\left(-2x - 1\right)\cos x + 3\sin x\right).$$

5) $y'' - 4y' + 5y = xe^x + \sin 2x$.

Find the roots of the characteristic equation:

$$\lambda^2 - 4\lambda + 5 = 0\,,$$

$$\lambda_{1,2} = 2 \pm i\,.$$

Solution of homogeneous equation: $y_0 = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$.

According to the superposition principle, a particular solution has a form: $y_{nh} = y_{1nh} + y_{2nh}$.

a) $f_1(x) = xe^x$, $a = 1$, $b = 0$, $z = 1$, $k = 0$, $y_{1nh} = (Ax + B)e^x$.

b) $f_2(x) = \sin 2x$, $a = 0$, $b = 2$, $z = 2i$, $k = 0$, $y_{2nh} = C\,sin\,2x + D\,cos\,2x$.

A particular solution looks like this:

$$y_{nh} = (Ax + B)e^x + C\,sin\,2x + D\,cos\,2x\,.$$

Let's find y'_{nh} and y''_{nh}:

$$y'_{nh} = (Ax + A + B)e^x - 2C\,sin\,2x + 2D\,cos\,2x\,,$$

$$y''_{nh} = (Ax + 2A + B)e^x - 4C\,sin\,2x - 4D\,cos\,2x\,.$$

We substitute y'_{nh} and y''_{nh} into the original equation:

$$(2Ax - 2A + 2B)e^x + (C + 8D)sin\,2x + (-8C + D)cos\,2x =$$

$$= x\,e^x + sin\,2x\,.$$

Equating the coefficients at xe^x, e^x, $\cos 2x$ and $\sin 2x$ in the left-hand side and the right-hand side of the last identity, we obtain a system of algebraic equations with respect to A, B, C and D:

$$\begin{array}{r|c}
x\,e^x & 2A = 1, \\
e^x & -2A + 2B = 0, \\
\sin 2x & C + 8D = 1, \\
\cos 2x & -8C + D = 0.
\end{array}$$

We have $A = B = \frac{1}{2}$, $C = \frac{1}{65}$, $D = \frac{8}{65}$. So,
$y_{nh} = \left(\frac{1}{2}x + \frac{1}{2}\right)e^{x} + \frac{1}{65}sin\,2x + \frac{8}{65}cos\,2x$.

The general solution of the equation:

$$y = e^{2x}\left(C_1\cos x + C_2\sin x\right) + \left(\frac{1}{2}x + \frac{1}{2}\right)e^{x} + \frac{1}{65}\sin 2x + \frac{8}{65}\cos 2x.$$

Answer: **1)** $y = e^{-x}\left(C_1\cos x + C_2\sin x\right) + e^{3x}\left(2\cos x + \sin x\right)$;

2) $y = C_1 + C_2e^{-2x} - \frac{1}{4}x\cos 2x + \left(\frac{1}{4}x + \frac{1}{8}\right)\sin 2x$;

3) $y = \left(C_1 + C_2x\right)cos\,2x + \left(C_3 + C_4x\right)sin\,2x +$
$+\frac{1}{8}x^3 + \frac{1}{4}x^2 - \frac{3}{16}x - \frac{3}{16}$;

4) $y = e^{3x}\left(C_1 + C_2x + \left(-2x - 1\right)\cos x + 3\sin x\right)$;

5) $y = e^{2x}\left(C_1\,cos\,x + C_2\,sin\,x\right) + \left(\frac{1}{2}x + \frac{1}{2}\right)e^{x} +$
$+\frac{1}{65}sin\,2x + \frac{8}{65}cos\,2x$.

Task 2.8. Find the general solutions of the linear differential equations using method of undetermined coefficients (without finding the numerical values of the coefficients).

1) $y''' + 4y'' + 3y' = e^x\left(x\cos 2x - \sin 2x\right)$;

2) $y'' - 16y' + 64 = 5x^2e^{8x} + x\sin x$;

3) $4y'' - 5y' + y = 5x^2 + x + 2 + \cos x$.

Solution. **1)** $y''' + 4y'' + 3y' = e^x\left(x\cos 2x - \sin 2x\right)$.

Find the roots of the characteristic equation:

$$\lambda^3 + 4\lambda^2 + 3\lambda = 0,$$
$$\lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -3.$$

Solution of homogeneous equation: $y_0 = C_1 + C_2 e^{-x} + C_3 e^{-3x}$.

Since $a = 1$, $b = 2$, then $z = 1 + 2i$, $k = 0$. A particular solution looks like this:

$$y_{nh} = e^x \left((Ax + B) cos\, 2x + (A_1 x + B_1) sin\, 2x \right).$$

So, we obtain:

$$y = C_1 + C_2 e^{-x} + C_3 e^{-3x} + e^x \left((Ax + B)\cos 2x + (A_1 x + B_1)\cos 2x \right).$$

2) $y'' - 16y' + 64 = 5x^2\, e^{8x} + x \sin x$.

Find the roots of the characteristic equation:

$$\lambda^2 - 16\lambda + 64 = 0,$$

$$\lambda_1 = \lambda_2 = 8.$$

Solution of homogeneous equation: $y_0 = (C_1 + C_2 x) e^{8x}$.

Using the principle of superposition, we get that a particular solution has the form: $y_{nh} = y_{1nh} + y_{2nh}$.

a) $f_1(x) = 5x^2 e^{8x}$, $a = 8$, $b = 0$, $z = 8$, $k = 2$, $y_{1nh} = x^2 \left(Ax^2 + Bx + C \right) e^{8x}$.

b) $f_2(x) = x \sin x$, $a = 0$, $b = 1$, $z = i$, $k = 0$, $y_{2nh} = (A_1 x + B_1) sin\, x + (A_2 x + B_2) cos\, x$.

The general solution of the equation:

$$y = y_0 + y_{nh} = (C_1 + C_2 x) e^{8x} + x^2 \left(Ax^2 + Bx + C \right) e^{8x} +$$
$$+ (A_1 x + B_1) \sin x + (A_2 x + B_2) \cos x.$$

3) $4y'' - 5y' + y = 5x^2 + x + 2 + \cos x$.

Find the roots of the characteristic equation:

$$4\lambda^2 - 5\lambda + 1 = 0,$$

$$\lambda_1 = 1, \qquad \lambda_2 = \frac{1}{4}.$$

Solution of homogeneous equation: $y_0 = C_1 e^x + C_2 e^{x/4}$.

Particular solution:

a) $f_1(x) = 5x^2 + x + 2$, $a = 0$, $b = 0$, $z = 0$, $k = 0$, $y_{1nh} = Ax^2 + Bx + C$.

b) $f_2(x) = \cos x$, $a = 0$, $b = 1$, $z = i$, $k = 0$, $y_{2nh} = D\sin x + E\cos x$.

The general solution of the equation:

$$y = y_0 + y_{nh} = C_1 e^x + C_2 e^{x/4} + Ax^2 + Bx + C + D\sin x + E\cos x.$$

Answer: **1)** $y = C_1 + C_2 e^{-x} + C_3 e^{-3x} +$

$$+ e^x\left((Ax + B)\cos 2x + (A_1 x + B_1)\cos 2x\right);$$

2) $y = (C_1 + C_2 x)e^{8x} + x^2\left(Ax^2 + Bx + C\right)e^{8x} +$

$$+ (A_1 x + B_1)\sin x + (A_2 x + B_2)\cos x;$$

3) $y = C_1 e^x + C_2 e^{x/4} + Ax^2 + Bx + C +$

$$+ D\sin x + E\cos x.$$

3 OPERATIONAL CALCULUS

When finding Laplace transforms by given originals, we usually use ***Laplace transforms*** table for elementary functions. This is given in Appendix D. You can also use the ***Laplace transformation properties***. They are listed in Appendix C.

Task 3.1. Find a Laplace transforms for a given functions.

1) $f(t)=\sin 3t\cos 5t$;

2) $f(t)=(t-3)^2\eta(t-3)$.

Solution. **1)** $f(t)=sin\, 3t\, cos\, 5t$.

Let's make trigonometric transformations:

$$f(t)=\sin 3t\cos 5t=\frac{1}{2}\sin 8t-\frac{1}{2}\sin 2t\ .$$

We find according to the table (see Appendix D):

$$\sin 2t \leftrightarrow \frac{2}{p^2+4}, \qquad \sin 8t \leftrightarrow \frac{8}{p^2+64}.$$

Using the linearity of the Laplace transformation (see Appendix C), we obtain:

$$f(t)=\frac{1}{2}\sin 8t-\frac{1}{2}\sin 2t \leftrightarrow$$

$$\leftrightarrow \frac{1}{2}\frac{8}{p^2+64}-\frac{1}{2}\frac{2}{p^2+4}=\frac{4}{p^2+64}-\frac{1}{p^2+4}=F(p).$$

2) $f(t)=(t-3)^2\eta(t-3)$.

For the function $t^2\eta(t)$ we have (see Appendix D):

$$t^2\eta(t)\leftrightarrow\frac{2}{p^3}\ .$$

By the delay theorem (see Appendix C) for the function $(t-3)^2\eta(t-3)$we have:

$$(t-3)^2\eta(t-3)\leftrightarrow e^{-3p}\,\frac{2}{p^3}=F(p).$$

In this case it is important that we look for the transform exactly for function $(t-3)^2\eta(t-3)$, that is, it is a function that is zero if $t<3$. However, if you consider the function $(t-3)^2\eta(t)$, i. e. $\left(t^2-6t+9\right)\eta(t)$, that using the linearity of the Laplace transformation we get:
:

$$(t-3)^2\eta(t)=\left(t^2-6t+9\right)\eta(t)\leftrightarrow\frac{2}{p^3}-\frac{6}{p^2}+\frac{9}{p}.$$

Answer: **1)** $F(p)=\dfrac{4}{p^2+64}-\dfrac{1}{p^2+4}$;

2) $F(p)=e^{-3p}\,\dfrac{2}{p^3}$.

In addition to using tables the following technique can be used to ***find the original*** $f(t)$ ***for a given Laplace transform*** $F(p)$. If $F(p)=\dfrac{Q(p)}{R(p)}$ is a rational fraction, then it is decomposed into the sum of partial fractions and we can find original for each partial fraction, using the Laplace transformation properties and tables (see Appendices C, D).

Task 3.2. Find an original functions for a given Laplace transforms.

1) $F(p)=\dfrac{1}{p(p-2)\left(p^2+4\right)}$;

2) $F(p)=\dfrac{e^{-p}}{p+1}$.

Solution. **1)** $F(p)=\dfrac{1}{p(p-2)\left(p^2+4\right)}$.

We decompose $F(p)$ into the sum of partial fractions:

$$\frac{1}{p(p-2)\left(p^2+4\right)}=\frac{A}{p}+\frac{B}{p-2}+\frac{Cp+D}{p^2+4}=.$$

$$=\frac{A(p-2)\left(p^2+4\right)+Bp\left(p^2+4\right)+(Cp+D)p(p-2)}{p(p-2)\left(p^2+4\right)}=$$

$$=\frac{(A+B+C)p^3+(-2A-2C+D)p^2+(4A+4B-2D)p-8A}{p(p-2)\left(p^2+4\right)}.$$

We equate the coefficients at equal degrees of p in numerators, so we obtain a system of algebraic equations with respect to A, B, C and D. We get:

$$A=-\frac{1}{8},\ B=\frac{1}{16},\ C=\frac{1}{16},\ D=-\frac{1}{8}.$$

So,

$$F(p)=-\frac{1}{8}\frac{1}{p}+\frac{1}{16}\frac{1}{p-2}+\frac{1}{16}\frac{p}{p^2+4}-\frac{1}{8}\frac{1}{p^2+4}.$$

The originals for each partial fraction of the right-hand side of equality are given in the tables (see Appendix D). Using the linearity property of the Laplace transformation, we obtain:

$$f(t)=-\frac{1}{8}+\frac{1}{16}e^{2t}+\frac{1}{16}cos\,2t-\frac{1}{16}sin\,2t.$$

2) $F(p)=\dfrac{e^{-4p}}{p+17}$.

The presence of the multiplier e^{-4p} indicates the need to apply the delay theorem. We have $\tau=4$, $\dfrac{1}{p+17}\leftrightarrow e^{-17t}$, therefore:

$$\frac{e^{-4p}}{p+17}\leftrightarrow e^{-17(t-4)}\eta(t-4).$$

Answer: **1)** $f(t)=-\dfrac{1}{8}+\dfrac{1}{16}e^{2t}+\dfrac{1}{16}cos\,2t-\dfrac{1}{16}sin\,2t$;

2) $f(t)=e^{-17(t-4)}\eta(t-4)$.

The operating method allows solving ***Cauchy's problems for ordinary differential equations with constant coefficients***. This uses, in particular, the original differentiation theorem (see Appendix C).

Task 3.3. Solve the Cauchy's problem.

$$x'' + x = 2\cos t; \quad x(0) = 0,\ x'(0) = -1;$$

Solution. Let $x(t) \leftrightarrow X(p)$, then by the differentiation theorem for the original we have:

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p),$$

$$x''(t) \leftrightarrow p^2X(p) - px(0) - x'(0) = p^2X(p) + 1.$$

In addition, we find in the tables $\cos t \leftrightarrow \dfrac{p}{p^2+1}$. We get the operator equation:

$$p^2X(p) + 1 + X(p) = \frac{2p}{p^2+1}.$$

We solve the equation and find the unknown function $X(p)$:

$$\left(p^2+1\right)X(p) = \frac{2p}{p^2+1} - 1,$$

$$X(p) = \frac{2p}{\left(p^2+1\right)^2} - \frac{1}{p^2+1}.$$

This is an operator solution. Let's find the original function for it. We find according to the tables (see Appendix D):

$$\frac{1}{p^2+1} \leftrightarrow \sin t, \qquad \frac{2p}{\left(p^2+1\right)^2} \leftrightarrow t\sin t.$$

So, $X(p) \leftrightarrow t\sin t - \sin t = (t-1)\sin t = x(t)$.

Answer: $x(t) = (t-1)\sin t$.

BIBLIOGRAPHY

1. Анпілогов Д.І. Диференціальні рівняння: навч. посібник / Д.І.Анпілогов, Н.В.Сніжко. – Запоріжжя: НУ "Запорізька політехніка", 2019. – 176 с.
2. Сніжко Н.В. Тримовний математичний словник / Н.В.Сніжко, Д.І.Анпілогов, Є.Д.Анпілогова. – Запоріжжя: НУ "Запорізька політехніка", 2023. – 184 с.
3. Овчинников П.П., Михайленко В.М. Вища математика: Підручник. У 2 ч. Ч. 2: Диференціальні рівняння. Операційне числення. Ряди та їх застосування. / За ред. П.П. Овчинникова. – К.: Техніка, 2004. – 792 с.
4. Ahlfors L.V. Complex Analysis / L.V. Ahlfors. – McGrah Hill, 1966. – 317 p.
5. Beerends R. J. Fourier and Laplace Transforms / Beerends R. J., et al. – Cambridge University Press, 2003. – 458 p.
6. Bird J. Higher Engineering Mathematics / J. Bird. – Amsterdam-Tokyo : Newnes, 2010. – 705 p.
7. Chasnov J. R. Introduction to Differential Equations. Lecture notes for MATH / Chasnov J. R. – The Hong Kong University of Science and Technology, 2009. – 128 p.
8. Kodaira K. Introduction to Complex Analysis / K. Kodaira. – Cambridge University Press, 2007. – 400 p.
9. Trench W. F. Elementary Differential Equations / Trench W. F. – Texas, USA, 2013. – 662 p.
10. Тексти (конспект) лекцій з дисципліни „Математичний аналіз” для студентів спеціальності 141 «Електроенергетика, електротехніка та електромеханіка» денної форми навчання (англійською мовою), частина перша / Укл. В. П. Чумаченко. – Запоріжжя: ЗНТУ, 2019. – 82 с.
11. Тексти (конспект англійською мовою) лекцій з дисципліни «Математичний аналіз» для студентів напряму підготовки 6.050702 «Електромеханіка» (частина II) // Укл.: В. П. Чумаченко. – Запоріжжя: ЗНТУ, 2015. – 118 с.

12. Calculation tasks on higher mathematics (3rd module) for students majoring in 141 Power Engineering, Electrical Engineering and Electrical Mechanics / Compiler: Snizhko Nataliia. – Zaporizhzhia: National University Zaporizhzhia Polytechnic, 2025. – 29 p.
13. Lectures on Complex Analysis and Operational Calculus for students majoring in 141 Power Engineering, Electrical Engineering and Electrical Mechanics / Compiler: Nina Antonenko. – Zaporizhzhia: National University Zaporizhzhia Polytechnic, 2024. – 76 p.
14. https://www.multitran.com

Appendix A
Table of the derivatives

In the table $u = u(x)$ is a differentiable function

1	$(C)' = 0$, $C = const$	11	$(tgu)' = \dfrac{1}{cos^2 u} \cdot u'$
2	$(x)' = 1$	12	$(ctgu)' = -\dfrac{1}{sin^2 u} \cdot u'$
3	$(u^n)' = n \cdot u^{n-1} \cdot u'$	13	$(\arcsin u)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u'$
4	$(\sqrt{u})' = \dfrac{1}{2\sqrt{u}} \cdot u'$	14	$(\arccos u)' = -\dfrac{1}{\sqrt{1-u^2}} \cdot u'$
5	$(a^u)' = a^u \cdot ln\, a \cdot u', a = const$	15	$(arctgu)' = \dfrac{1}{1+u^2} \cdot u'$
6	$(e^u)' = e^u \cdot u'$	16	$(arcctgu)' = -\dfrac{1}{1+u^2} \cdot u'$
7	$(log_a u)' = \dfrac{1}{u \cdot ln\, a} \cdot u'$	17	$(sh\, u)' = ch\, u \cdot u'$
8	$(\ln u)' = \dfrac{1}{u} \cdot u'$	18	$(ch\, u)' = sh\, u \cdot u'$
9	$(sin\, u)' = cos\, u \cdot u'$	19	$(th\, u)' = \dfrac{1}{ch^2 u} \cdot u'$
10	$(cos\, u)' = -sin\, u \cdot u'$	20	$(cth\, u)' = -\dfrac{1}{sh^2 u} \cdot u'$

Appendix B
Table of the basic indefinite integrals

In the table $u = u(x)$ is a differentiable function

1	$\int du = u + C$	10	$\int ctgu\ du = ln\lvert sin\,u\rvert + C$
2	$\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1} + C$, $\alpha \neq -1$	11	$\int \frac{du}{cos^2 u} = tg\,u + C$
3	$\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$	12	$\int \frac{du}{sin^2 u} = -\,ctg\,u + C$
4	$\int \frac{du}{u} = ln\lvert u\rvert + C$	13	$\int \frac{du}{sin\,u} = ln\left\lvert tg\,\frac{u}{2}\right\rvert + C$
5	$\int a^u du = \frac{a^u}{ln\,a} + C$	14	$\int \frac{du}{cos\,u} = ln\left\lvert tg\left(\frac{u}{2} + \frac{\pi}{4}\right)\right\rvert + C$
6	$\int e^u du = e^u + C$	15	$\int sh\,u\,du = ch\,u + C$
7	$\int \sin u\,du = -\cos u + C$	16	$\int ch\,u\,du = sh\,u + C$
8	$\int cos\,u\ du = sin\,u + C$	17	$\int \frac{du}{ch^2 u} = th\,u + C$
9	$\int tgu\ du = -\,ln\lvert cos\,u\rvert + C$	18	$\int \frac{du}{sh^2 u} = -\,cth\,u + C$

19	$\int \frac{du}{u^2+a^2}=\frac{1}{a}arctg\frac{u}{a}+C$
20	$\int \frac{du}{u^2-a^2}=\frac{1}{2a}ln\left\|\frac{u-a}{u+a}\right\|+C$
21	$\int \frac{du}{\sqrt{a^2-u^2}}=arcsin\frac{u}{a}+C$
22	$\int \frac{du}{\sqrt{u^2\pm a^2}}=ln\left\|u+\sqrt{u^2\pm a^2}\right\|+C$
23	$\int \sqrt{a^2-u^2}du=\frac{1}{2}u\sqrt{a^2-u^2}+\frac{1}{2}a^2\,arcsin\frac{u}{a}+C$
24	$\int \sqrt{u^2\pm a^2}du=\frac{1}{2}u\sqrt{u^2\pm a^2}\pm\frac{1}{2}a^2ln\left\|u+\sqrt{u^2\pm a^2}\right\|+C$

Appendix C
Properties of the Laplace transformation

Property	The calculation formula
Linearity	$\sum\limits_{k=1}^{n} \lambda_k f_k(t) \leftrightarrow \sum\limits_{k=1}^{n} \lambda_k F_k(p)$
Delay theorem	$f(t-\tau)\cdot\eta(t-\tau) \leftrightarrow e^{-\tau p}F(p)$
Shifting theorem	$e^{at} f(t) \leftrightarrow F(p-a)$
Similarity theorem	$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{p}{a}\right)$
Differentiation of original	$f'(t) \leftrightarrow pF(p) - f(0)$, $f^{(n)}(t) \leftrightarrow p^n F(p) - \sum\limits_{k=1}^{n} p^{k-1} f^{(k-1)}(0)$
Differentiation of image	$t^n f(t) \leftrightarrow (-1)^n \frac{d^n}{dp^n} F(p)$
Integration of original	$\int\limits_0^t f(\tau)d\tau \leftrightarrow \frac{F(p)}{p}$
Integration of image	$\frac{f(t)}{t} \leftrightarrow \int\limits_p^{\infty} F(p)dp$
Convolution theorem	$f_1(t) * f_2(t) = \int\limits_0^t f_1(t) f_2(t-\tau)d\tau \leftrightarrow F_1(p)\cdot F_2(p)$

Appendix D
Table of Laplace transforms

Original function $f(t)$	Laplace transform $F(p)$	Original function $f(t)$	Laplace transform $F(p)$
$\eta(t)$	$\dfrac{1}{p}$	sh ωt	$\dfrac{\omega}{p^2-\omega^2}$
$t^n, n \in N$	$\dfrac{n!}{p^{n+1}}$	$e^{\alpha t}\cdot\cos\omega t$	$\dfrac{p-\alpha}{(p-\alpha)^2+\omega^2}$
$t^\beta, \beta > -1$	$\dfrac{\Gamma(\beta+1)}{p^{\beta+1}}$	$e^{\alpha t}\cdot\sin\omega t$	$\dfrac{\omega}{(p-\alpha)^2+\omega^2}$
$e^{\alpha t}$	$\dfrac{1}{p-\alpha}$	$e^{\alpha t}\cdot\mathrm{ch}\,\omega t$	$\dfrac{p-\alpha}{(p-\alpha)^2-\omega^2}$
$t^n e^{\alpha t},\ n \in N$	$\dfrac{n!}{(p-\alpha)^{n+1}}$	$e^{\alpha t}\cdot\mathrm{sh}\,\omega t$	$\dfrac{\omega}{(p-\alpha)^2-\omega^2}$
$t^\beta e^{\alpha t},\ n \in N$	$\dfrac{\Gamma(\beta+1)!}{(p-\alpha)^{\beta+1}}$	$t\cdot\cos\omega t$	$\dfrac{p^2-\omega^2}{\left(p^2+\omega^2\right)^2}$
$\sin\omega t$	$\dfrac{\omega}{p^2+\omega^2}$	$t\cdot\sin\omega t$	$\dfrac{2p\omega}{\left(p^2+\omega^2\right)^2}$
$\cos\omega t$	$\dfrac{p}{p^2+\omega^2}$	$t\cdot\mathrm{ch}\,\omega t$	$\dfrac{p^2+\omega^2}{\left(p^2-\omega^2\right)^2}$
ch ωt	$\dfrac{p}{p^2-\omega^2}$	$t\cdot\mathrm{sh}\,\omega t$	$\dfrac{2p\omega}{\left(p^2-\omega^2\right)^2}$