ДОПОВІДІ

ЧАСТИНА І

BOUNDARIES OF THE ASSORTATIVITY INDEX FOR SCALE-FREE NETWORKS

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Abstract — Assortativity coefficient is a measure of the tendency of nodes in networks to connect with similar in some way, or dissimilar ones. The assortativity coefficients for most of the real-world networks are close to zero and far from the theoretical bounds ±1. Thus, one might think that this property is not very characteristic of complex networks. But the point is, that assortativity properties should not be studied apart of the distribution of nodes by degrees. Most complex networks are scale-free, so the nodes follow an asymptotically power-law distribution. Fixing the distribution of the degree of nodes forms an essential constraint both on the network structure and on the boundaries of the assortativity coefficient. The problem of estimating the bounds of assortativity coefficients for scale-free networks were solved. It was found, that these boundaries are narrowing with increasing the network size.

Keywords— Assortativity Boundaries, Scale-Free Networks, Power Distribution, Barabasi-Albert Model.

It is widely known, that most of the complex real-world networks are scale-free (SF), i.e. the degree of their nodes is distributed by some power law family of distribution (such as Yule-Simon, Benford, zeta etc.), at least asymptotically [1]-[2].

Despite the importance of the nodes degree distribution, it is not an exhaustive characteristic of networks. While degree distribution is a first order graph metric, the *assortativity* [3], that is, the tendency of nodes to link with similar or dissimilar ones, is a second order one. Assortativity is also an important property of networks. It is known [1], [3]-[4], that social networks are positively assortative, while biological, technical and cognitive ones are disassortative.

Despite the large number of studies of scaling and assortativity, these characteristics are most often studied separately that is considered as an essential disadvantage of the current state of the complex networks theory. While the structure of SF-networks with extreme assortativity is widely known [6]-[8], the bounds of assortativity coefficient for the SF-networks with known power exponent are unknown. Estimating this bounds forms the goal of current research.

An *assortativity coefficient* is defined as correlation coefficient of the nodes by their degrees [1] and in general case is in the range $-1 \le r \le 1$. More convenient form of this coefficient is given in [4]:

$$r = \frac{N_3 - S_2^2 / S_1}{S_3 - S_2^2 / S_1}, \tag{1}$$

where

$$S_k = \sum_{i=1}^{n} d_i^k, \quad N_3 = d^T A d = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} d_i d_j, \tag{2}$$

A is adjacency matrix of the network, d_i is node degrees.

Without loss of generality, we can assume that the nodes are ordered by a decreasing degree, so index number i of the node d_i is its range.

As it follows from (1)-(2), network assortativity coefficient (r) depends on degree distribution d_i, so networks mixing cannot be studied apart from scaling. If the distribution of nodes by degree follows to a power law, then the range distribution of nodes asymptotically also follows to a power-family distribution:

$$d_i \approx c \cdot i^{-\beta}, \quad \frac{1}{2} \le \beta < 1. \tag{3}$$

Let's fix the network size and the values of nodes degree. Then, the structure of SF-network with extreme assortativity can be found by greedy algorithms of simple structure [6]-[8]. Links are distributed starting from the richest node. Network is extremely disassortative if on each step current node links with as poorest one as possible. Likewise, while creating an extremely assortative network, current node links with as richest one as possible.

The structure of SF-network with extreme assortativity is shown on Fig.1. As one can see, an extremely disassortative network is tends to be *bipartile* [8],

while in opposite case (extreme assortative mixing) network is close to be a set of almost isolated clusters, all but the largest of which are almost regular.

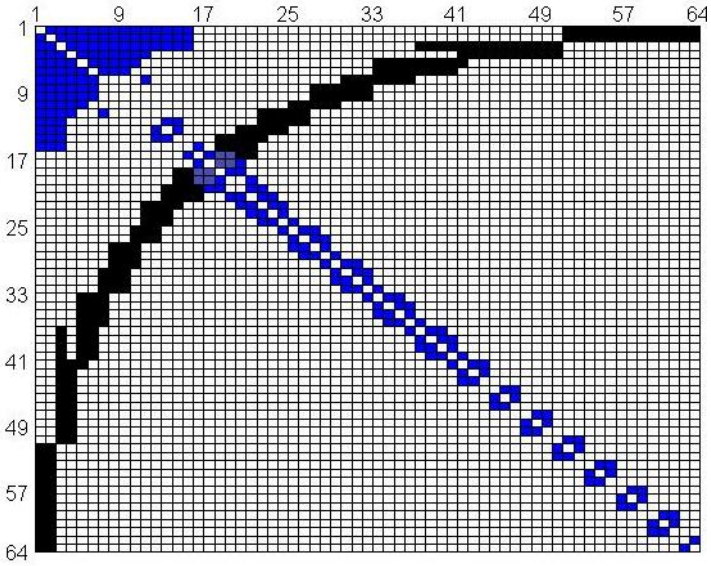

Fig.1 *An adjacency matrix of extremely assortative (blue) and disassortative (black) BA-network of size n=64*

The terms S_1, S_2, S_3 in (1) does not depends on mixing properties (i.e. on adjacency matrix A), but only on network size n and scaling factor β. For SF-model having degree distribution (3) this terms are:

$$S_1 = \frac{c \cdot n}{1-\beta} = 2L, \quad S_2 = O(L^2 \cdot n^{2\beta-2}), \quad S_3 = O(L^3 \cdot n^{3\beta-3}) \qquad (4)$$

where *L* is total number of links.

In disassortative case according to generative algorithm the minimum value of N_3 (2) is

$$N_3^{\min} \approx \frac{2c^3}{2\beta-1} n^{-\beta}. \qquad (5)$$

By substituting (4) and (5) into (1), we get an asymptotic estimate of the lower bound of assortativity coefficient:

$$r_{\min} = -O(n^{-(1-\beta)}). \tag{6}$$

In the extremely assortative case the contribution to N_3 of clusters of "arrow body" is much less than that of "arrow-head" which is equal to

$$N_{31} = \sum_{i=1}^{c+1} d_i \sum_{j=1}^{d_i+1} d_j \approx \frac{c}{1-\beta} \sum_{i=1}^{c+1} (d_i)^{2-\beta} \approx \frac{c^{4-3\beta+\beta^2}}{(1-\beta)^3}. \tag{7}$$

In accordance with (4) and (7), the upper bound of assortavitity coefficient (1) can be estimated as

$$r_{\max} = O((cn^{\beta})^{1-3\beta+\beta^2}). \tag{8}$$

It can be noted that the obtained estimates of the boundaries of the assortativity index can also be used to normalize the "self-similarity index" [9], which is closely related to the assortativity one.

Obtaining more accurate estimates for the model of elastic SF-networks is treated as a direction for further research.

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