EXPLICIT ESTIMATES OF THE PROBABILITIES OF NASH-EQUILIBRIUM STRATEGIES IN LUPI GAME

Shergin V.L.[1], Chala L.E.[1], Udovenko S.G.[2]
[1]Artificial Intelligence dept. NURE, Kharkov, Ukraine,
[2]Informatics and Computer Engineering dept. KNUE, Kharkov, Ukraine

Abstract — LUPI is the game in which players choose positive integers. Winner is the player who picks the lowest unique number. LUPI is the simplest extreme congestion game, in which even one other player's choice of the same number reduces one's payoff to zero. This primitive lottery is used as a model of some auctions, patent races etc. Poisson-Nash equilibrium strategies for this game are known, but the desired exact probabilities are an implicit function of the game parameters. Finding explicit estimates of these probabilities, as well as an estimate of the threshold value, forms the goal of the current research.

Keywords — LUPI game, Poisson-Nash equilibrium, logint function

In the Least Unique Positive Integer (LUPI) game players, choose integers from 1 to *K* simultaneously, and the lowest unique number wins a payoff of 1, while all others earn 0 [1]. In the simplest form of this lottery, the number of players is considered to be a random variable having a Poisson distribution with mean n. The Poisson assumption proves to be easier to work with than a fixed n because of the environmental equivalence (EE) property [2]. LUPI game used as the model of asymptotic case for Lowest Unique Bid Auctions (LUBA) [3].

Due to the symmetry of the game, all players use the same mixed strategy $\mathbf{p} = (p_1, p_2, ..., p_K)$, where p_k is the desired probability of choosing k. Let $\Pr(X(k) = i)$ is the probability that the number of players who pick k in equilibrium is exactly i. According to the above assumption, this

probability follows a Poisson distribution. Hence, the the expected payoffs v_k when choosing the number k are equal to probabilities $\pi(k)$ of k being the winning number:

$$v_k = \pi(k) = \Pr(X(k) = 0) \cdot \prod_{i=1}^{k-1} \Pr(X(i) \neq 1) = e^{-np_k} \prod_{i=1}^{k-1} \left(1 - np_i e^{-np_i}\right)$$

.

(1)

It was shown [1], that the Nash equilibrium $\mathbf{p}$ has full support, i.e. $p_k > 0$ over all $1 \le k \le K$. According to the basic condition of the Nash equilibrium, the expected payoffs must be equal to each other: $v_1 = v_2 = ... = v_K = v$.

A detailed study of the properties of the Nash equilibrium for the LUPI game was provided by Östling et.al [1]. In particular, they shown that the sequence p_k is strictly decreasing. Moreover, this sequence may have super-exponential cutoff k^* [3], such that $p_k = O(\exp(-2^k))$ for $k > k^*$. If so, then probabilities for $k > k^*$ being the winning number are negligible and such numbers can be discarded despite of full support.

Estimation this cutoff value $k^*(n, K)$ as function on number of players (n) and the maximum number allowed (K) is the first interest of current paper.

According to (1), the desired values p_k can be found sequentially and are fully determined by the initial value p_1. Therefore, these probabilities can be easily found by any simple numerical method as a solution to the normalization condition ($\sum p_k = 1$) w.r.t p_1. However, for large values of K, this approach is associated with both a large amount of calculations and the accumulation of a computational error. This fact forms the main interest of current research: estimating the desired probabilities $p_k(n, K)$ as explicit functions, i.e. without loops and without numerical solution of nonlinear equation.

If we denote

$$z_k = \exp(np_k) \quad \Leftrightarrow \quad p_k = \tfrac{1}{n}\ln(z_k), \tag{2}$$

then the sequence (1) can be rewritten as recurrence relation

$$z_{k+1} = z_k - \ln(z_k). \tag{3}$$

It should be noted that the exact solution of the recurrence (3) cannot be written in an explicit form. In [3], the original difference equation is approximated by a differential one ($\frac{dz}{dk} = -\ln(z)$), the solution of which has the form

$$k = c - \mathrm{li}(z), \tag{4}$$

where $\mathrm{li}(z)$ is a logarithmic integral function (also called as "logint"), c is an integration constant. As a result, the estimate of cutoff value has the form $k^* = c + \mathrm{li}(n+1)$.

Despite the rather high accuracy of such approximation, this approach has significant limitations: firstly, the uncertainty of the constant c, and secondly, the impossibility of obtaining an inverse relationship (i.e., $z(k)$) in an explicit form.

We propose to estimate the cutoff as the point at which the curvature of the curve $z(k)$ is maximum. Hence, the second (central) difference of this recurrent sequence should be maximum:

$$\Delta^{(2)} z_k = (z_{k+1} - z_k) - (z_k - z_{k-1}) = \ln z_{k-1} - \ln(z_{k-1} - \ln z_{k-1}) \tag{5}$$

The maximum of function (5) is reached at $z_{k-1} = e$, so $z_k = e - 1 \approx 1.718$. Therefore, cutoff is the solution of $z_{k^*} = e - 1$.

Then we found, that sequence (3) can be approximated much more simply than (4), namely as

$$k^* - k = a \cdot z_k + 2\sqrt{z_k} - 3, \quad a \approx 0.116465577755347. \tag{6}$$

In fact, (6) is an approximation of the logint function $\mathrm{li}(y)$ based on the interval $k^* - k \in [e-1; 10000]$.

Using (6), it is easy both to find the cutoff k^* and to approximate explicitly all values of the array z.

According to (3), the normalization condition can be transformed into a telescopic series:

$$n = n\sum_{k=1}^{K} p_k = \sum_{k=1}^{K} \ln(z_k) = \sum_{k=1}^{K} (z_k - z_{k+1}) = z_1 - z_{K+1}. \tag{7}$$

Suppose that $k^* < K$. Then $z_{K+1} = 1$, so, $z_1 = n+1$, and as per (6):

$$k^* = a \cdot z_1 + 2\sqrt{z_1} - 2. \tag{8}$$

Since $z_{\min} = 1$, and given that k is an integer, the maximum allowable value of k, for which estimate (6) is applicable, is

$$k_{\max} = \left\lceil k^* + 1 - a \right\rceil. \tag{9}$$

Thus, if the value of $k_{\max}$ (9) does not exceed K, then all elements of z_k for $k > k_{\max}$ can be set to unity. Moreover, the remaining elements can be expressed as an explicit function of k:

$$z_k = \left(\frac{-1 + \sqrt{1 + a(3 + k^* - k)}}{a} \right)^2, \quad k = 1, \ldots, k_{\max}. \tag{10}$$

In the opposite case ($k_{\max} > K$), value of z_1 can be evaluated as:

$$z_1 = \left(\frac{n + b^2}{2b} \right)^2, \quad \text{where} \quad b = (K - a \cdot n)/2. \qquad (11)$$

The value of k^* is still evaluated by (8), $k_{\max}$ is set to K, hence all elements of the array *z* are estimated by (10).

In both cases, after calculating the auxiliary array *z*, the probabilities of Poisson-Nash equilibrium strategies can be obtained from (2):

$$p_k = \frac{\ln z_k}{n}. \qquad (12)$$

According to numeric simulations, the proposed algorithm for estimating the probabilities of equilibrium provides high accuracy.

References

1. Östling, Robert & Wang, Joseph & Chou, Eileen & Camerer, Colin. (2007). "Testing Game Theory in the Field: Swedish LUPI Lottery Games." Stockholm School of Economics, Working Paper Series in Economics and Finance. 3. doi:10.1257/mic.3.3.1.
2. Myerson, Roger B. (1998). "Population Uncertainty and Poisson Games." *International Journal of Game Theory*, 27 (3): 375-92.
3. Pigolotti, Simone & Bernhardsson, Sebastian & Juul, Jeppe & Galster, Gorm & Vivo, Pierpaolo. (2012). "Equilibrium Strategy and Population-Size Effects in Lowest Unique Bid Auctions." *Physical review letters*. 108. doi:10.1103/PhysRevLett.108.088701