

The Angular Deformation of the Ring with Reference to the Centrifugal Forces

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Abstract—In the paper the boundary plane elasticity problem for the ring is solved using Muskhelishvili complex potentials method. The ring rotates uniformly under the influence of tangential forces, and the centrifugal forces of inertia are taken into account. The mutual rotation angle of the boundary points is obtained by analyzing the displacement field.

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1. INTRODUCTION AND PRELIMINARIES

The deformations of machine components can be used as signals for monitoring different dynamic modes. For example, the mutual angle of the rotating points on the outer and inner rims of the gear-wheel of engine increases with the transmitted power growth. Using this angle one can create feedback that adjusts the power transmitted. Solution of corresponding plane elasticity problem can be obtained using Muskhelishvili complex potentials method in a closed form. Therefore, the displacement field allows estimating the mutual rotation angle of the border points in a simple form, suitable for engineering applications.

Domain in a form of a ring with inner and outer radii R_1 , R_2 is considered (see figure). Its center is superposed with the beginning of the coordinate system Oxy . Later we will also use polar coordinates (r, θ) . The physical size of the domain in a direction perpendicular to the plane of the figure, is h . M is the external moment, which rotates the domain with a constant angular speed ω . Suppose that tangential stresses T_j ($j = 1, 2$), distributed uniformly around the circumference, are applied on the inner and outer parts of the boundary. With the help of the angle $d\theta$ the contour element of length $d\ell = R_j d\theta$ with outward normal n is selected. The corresponding area element of the lateral surface equals $dS = h d\ell = h R_j d\theta$. The force applied on the area element is $dF = T_j dS = T_j h R_j d\theta$. Moment of this force is $dM = R_j dF = R_j^2 T_j h d\theta$. We integrate the entire circumference: $M = 2\pi R_j^2 T_j h$. Let $M_0 = \frac{M}{h}$ be moment per length unit, measured along the perpendicular to the plane of the figure. Then, $M_0 = 2\pi R_j^2 T_j$. Stress T_j is directed to the left, when viewed along the outer normal on both external and inside contour part. Therefore, the main point is zero and the angular speed is constant if moments M_0 , applied on the inner and outer parts of contour are balanced. Consequently, the elastic equilibrium solution exists under the condition $R_1^2 T_1 = R_2^2 T_2$.

The equilibrium is seen in the rotating coordinate system. We must take into account the centrifugal forces of inertia. Let the element dS of area rotate in a circle of radius r ($R_1 < r < R_2$). The volume of element is $dV = h dS$, the mass of element is $dm = \rho dV$ (ρ is material density). Centripetal acceleration is $w = \omega^2 r$, centrifugal force of inertia is $df = w dm = \rho \omega^2 r dV$, and centrifugal force of inertia per unit

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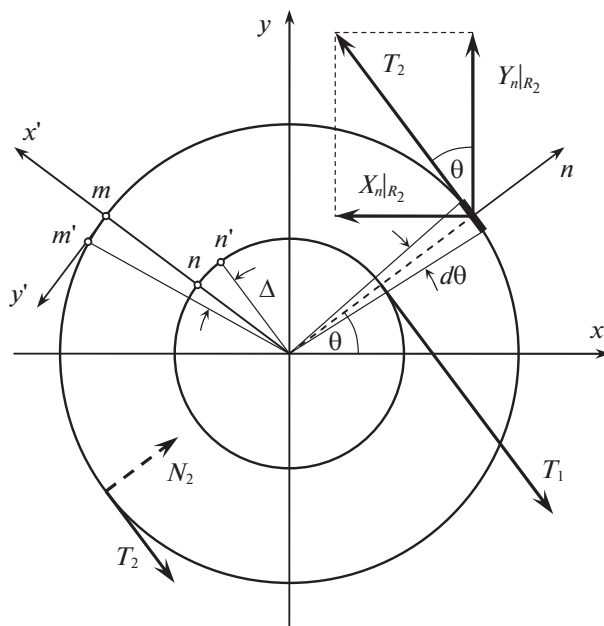


Fig. 1.

of volume is $\frac{df}{dV} = \rho\omega^2 r$. The components of this force along the axis are $X = \rho\omega^2 x$, $Y = \rho\omega^2 y$. Then the equilibrium equation in stresses [1, § 20, (1), p. 70] take the form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \rho\omega^2 x = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \rho\omega^2 y = 0. \quad (1)$$

Let X_n, Y_n be stresses components, applied on the area, oriented perpendicularly to the normal n (i.e. on the length of the contour element). They are $X_n|_{r=R_j} = -T_j \sin \theta$, $Y_n|_{r=R_j} = T_j \cos \theta$ in the global coordinate system according to the figure. Components $X_x, X_y = Y_x, Y_y$ and components X_n, Y_n are correlated with the formulas [1, § 25, (4), p. 89]. Then the boundary conditions take the form

$$\begin{cases} (X_x \cos \theta + X_y \sin \theta)|_{r=R_j} = -T_j \sin \theta, \\ (X_y \cos \theta + Y_y \sin \theta)|_{r=R_j} = T_j \cos \theta, \end{cases} \quad j = 1, 2. \quad (2)$$

Considering the behavior of the material is physically linear, accept Hooke's law in the form of

$$\begin{cases} X_x = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x}, \\ Y_y = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y}, \\ X_y = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right). \end{cases} \quad (3)$$

Here the functions $u = u(x, y)$, $v = v(x, y)$ are the components of the displacement field, and constant parameters λ, μ are Lamé coefficients. They describe the properties of homogeneous isotropic elastic material.

Now formulation of the boundary problem is as follows. We need to find a pair of functions u, v defined in the ring. They generate by (3) X_x, X_y, Y_y stress fields satisfying the inhomogeneous equation (1) inside the ring and the conditions (2) at the boundary of the ring.

2. REDUCTION OF THE PROBLEM TO THE HOMOGENEOUS EQUATIONS

The complex potentials method is directly suitable for solving the problems of elastic equilibrium only in the absence of the volume distributed forces (in this case — the centrifugal forces of inertia). Mathematically, this means that the differential equations of the boundary problem must be homogeneous. In

order to reduce the equations (1) to a homogeneous we look for a solution in the form

$$\begin{cases} u(x, y) = u^{(0)}(x, y) + u^{(1)}(x, y), \\ v(x, y) = v^{(0)}(x, y) + v^{(1)}(x, y). \end{cases} \quad (4)$$

Here the functions $u^{(0)}, v^{(0)}$ can be chosen freely to a certain extent (if only they were twice differentiable). As the law (3) is linear and homogeneous, the expansion (4) implies a search for the corresponding components $X_x^{(0)}, X_y^{(0)}, Y_y^{(0)}$ and $X_x^{(1)}, X_y^{(1)}, Y_y^{(1)}$ separately.

We assume the function $u^{(0)}, v^{(0)}$ in the form $u^{(0)} = axr^2, v^{(0)} = ayr^2$. Substituting these expressions in (3), and the results — in the first of equations (1) we get: $8ax(\lambda + 2\mu) + \rho\omega^2x = 0$ at any y . Therefore, the functions $u^{(0)}, v^{(0)}$ satisfy the first of the equations (1) for arbitrary x, y , if we put $a = -\rho\omega^2/[8(\lambda + 2\mu)]$. Similarly, we obtain the same value of a analyzing the second equation (1). Consequently, the pair of functions $u^{(0)}, v^{(0)}$ removes inhomogeneity of equation (1) with value a . Then for functions $u^{(1)}, v^{(1)}, X_x^{(1)}, X_y^{(1)}, Y_y^{(1)}$ we obtain the homogeneous equations, similar to (1), but devoid of terms, describing the centrifugal force of inertia.

The boundary problem for the new five unknown functions has changed boundary conditions. As equations (3) are linear and homogeneous, then stresses are decayed on the respective components: $X_x = X_x^{(0)} + X_x^{(1)}, X_y = X_y^{(0)} + X_y^{(1)}, Y_y = Y_y^{(0)} + Y_y^{(1)}$ when providing the replacement (4). Then the boundary conditions (2) take the form

$$\begin{cases} \left((X_x^{(0)} + X_x^{(1)}) \cos \theta + (X_y^{(0)} + X_y^{(1)}) \sin \theta \right) \Big|_{r=R_j} = -T_j \sin \theta, \\ \left((X_y^{(0)} + X_y^{(1)}) \cos \theta + (Y_y^{(0)} + Y_y^{(1)}) \sin \theta \right) \Big|_{r=R_j} = T_j \cos \theta, \end{cases} \quad j = 1, 2.$$

We find the stresses $X_x^{(0)}, X_y^{(0)}, Y_y^{(0)}$, and then — their values at the points $r = R_j$, using the function $u^{(0)}, v^{(0)}$ in law (3). As a result:

$$\begin{cases} \left(X_x^{(1)} \cos \theta + X_y^{(1)} \sin \theta \right) \Big|_{r=R_j} = -T_j \sin \theta - N_j \cos \theta, \\ \left(X_y^{(1)} \cos \theta + Y_y^{(1)} \sin \theta \right) \Big|_{r=R_j} = T_j \cos \theta - N_j \sin \theta, \end{cases} \quad j = 1, 2. \quad (5)$$

Here N_j denotes $N_j = 2aR_j^2(2\lambda + 3\mu), j = 1, 2$.

Thus, terms describing the inertial force applied in the radial direction disappear in equilibrium equations (1) when providing the change of variables (4). Instead, fictitious normal components N_j applied in the same direction arise in the boundary conditions (shown with the dashed arrow). Note that these components are directed in the opposite side to the outer normal n , i.e. inside the domain, and therefore they compress the domain. This is indicated by a minus sign if terms $N_j \cos \theta, N_j \sin \theta$.

Now formulation of the boundary problem with respect to the functions $u^{(1)}, v^{(1)}, X_x^{(1)}, X_y^{(1)}, Y_y^{(1)}$ is the following. We need to find a pair of functions $u^{(1)}, v^{(1)}$, defined in the ring, which generate $X_x^{(1)}, X_y^{(1)}, Y_y^{(1)}$ stresses by (3) satisfying the homogeneous equation relevant (1) inside the ring and the conditions (5) at the boundary of the ring.

3. SOLUTION OF THE BOUNDARY PROBLEM

The obtained formulation of the boundary problem contains a homogeneous differential equations, and allows us to apply the complex potentials method directly. The boundary conditions (5) according to [1, § 56, (2), p. 191] written in the Kolosov–Muskhelishvili form are as follows:

$$\left(\Phi(z) + \overline{\Phi(z)} - e^{2i\theta} [\bar{z} \cdot \Phi'(z) + \Psi(z)] \right) \Big|_{|z|=R_j} = -N_j - iT_j, \quad j = 1, 2. \quad (6)$$

Here, $\Phi(z) = \frac{d\varphi}{dz}, \Psi(z) = \frac{d\psi}{dz}$; φ, ψ are Muskhelishvili potentials.

We look for the solution of the boundary problem in the form

$$\Phi(z) = \sum_{k=-\infty}^{+\infty} a_k z^k, \quad \Psi(z) = \sum_{k=-\infty}^{+\infty} a'_k z^k. \quad (7)$$

Substitution of the formula (7) in (6) when $z = R_j e^{i\theta}$ gives

$$\sum_{(k)} R_j^k \left(a_k (1 - k) e^{ik\theta} + \overline{a_k} e^{-ik\theta} \right) + \sum_{(k)} a'_k R_j^k e^{i(k+2)\theta} = -N_j - iT_j.$$

We use the method of undetermined coefficients with the functions $e^{ik\theta}$. As the right side does not depend on θ , then only coefficients with the numbers $k = 0$ (first sum) and $k = -2$ (second sum) differ from zero in the left side. Then

$$a_0 + \overline{a_0} - \frac{a'_{-2}}{R_j^2} = -N_j - iT_j, \quad j = 1, 2.$$

We can assume that the coefficient a_0 is real without loss of generality [1, § 34]. Let $a'_{-2} = \alpha + i\beta$. We get

$$2a_0 - \frac{\alpha + i\beta}{R_j^2} = -N_j - iT_j, \quad j = 1, 2.$$

Separating the real and imaginary parts, we get four equations which have the following solutions:

$$a_0 = \frac{R_1^2 N_1 - R_2^2 N_2}{2(R_2^2 - R_1^2)}, \quad \alpha = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} (N_1 - N_2), \quad \beta = R_1^2 T_1, \quad \beta = R_2^2 T_2.$$

Here the last two relations do not contradict each other, since they are consistent with balance condition of the external moments. So, we can put $\beta = R_1^2 T_1 = R_2^2 T_2 = \frac{M_0}{2\pi}$.

Then Muskhelishvili potentials are

$$\varphi(z) = \int \Phi dz = a_0 z, \quad \psi(z) = \int \Psi dz = -\frac{\alpha + i\beta}{z}.$$

These potentials give rise to the displacement field [1, § 32, (1), p. 110]:

$$2\mu \left(u^{(1)} + iv^{(1)} \right) = \kappa \varphi - z \overline{\varphi'} - \overline{\psi} = (\kappa - 1) a_0 z + \frac{\alpha - i\beta}{\overline{z}}.$$

In the case of plane strain $\kappa = 3 - 4\sigma$ [1, § 32, (2), p. 110], where the Poisson's ratio $\sigma = \frac{\lambda}{2(\lambda + \mu)}$ [1, § 19, (15), p. 69]. To find $u^{(1)}, v^{(1)}$ it is necessary to select the real and imaginary parts. We find a final solution of the initial boundary problem taking into account the expansion of (4):

$$\begin{cases} u = axr^2 + \frac{a_0}{\lambda + \mu} \cdot x + \frac{1}{2\mu} \cdot \frac{\alpha x + \beta y}{r^2}, \\ v = ay r^2 + \frac{a_0}{\lambda + \mu} \cdot y + \frac{1}{2\mu} \cdot \frac{\alpha y - \beta x}{r^2}. \end{cases} \quad (8)$$

4. RESULTS AND DISCUSSION

The following can be proved by direct checking. Stresses X_x, X_y, Y_y , satisfy both non-homogeneous equation (1), and the boundary conditions (2). They arise when (8) is substituted into the (3). Furthermore, it is easy to convert stresses to the axes $Ox'y'$, rotated relatively to the axes Oxy by angle θ , using the formulas [1, § 8, (7), p. 33]. In terms of the polar coordinates:

$$\begin{cases} X'_{x'} = 2ar^2(2\lambda + 3\mu) + 2a_0 - \frac{\alpha}{r^2}, \\ Y'_{y'} = 2ar^2(2\lambda + \mu) + 2a_0 + \frac{\alpha}{r^2}, \\ X'_{y'} = Y'_{x'} = \frac{\beta}{r^2}. \end{cases}$$

This result does not depend on the angle θ . It corresponds to the cylindrical symmetry of the stress-strain state. Using value $r = R_{1,2}$, we get

$$\begin{cases} X'_{x'}|_{r=R_{1,2}} = 0, \\ Y'_{x'}|_{r=R_{1,2}} = T_{1,2}. \end{cases}$$

This means that only tangential stresses are applied on the contour element. We had in mind very this situation, when we was formulate the boundary conditions (2).

Let v_r and v_θ denote the projection of point displacement on the axis Ox' , Oy' . According to [1, § 39, (2), p. 133] we have $v_r + iv_\theta = (u + iv)e^{-i\theta}$, $v_\theta = v \cos \theta - u \sin \theta$. Conversion of (8) to polar coordinates is $v_\theta = -\frac{M_0}{4\pi\mu r}$. Let the position of the two points n , m of the circles be described by the same angle θ . Let these points move into position n' , m' as a result of deformation (see figure). Angular distances of this points are $\Delta\theta_{1,2} = \frac{v_\theta|_{r=R_{1,2}}}{R_{1,2}}$. Searched mutual angle of rotation is equal to

$$\Delta = \Delta\theta_2 - \Delta\theta_1 = \frac{M_0}{4\pi\mu} \cdot \frac{R_2^2 - R_1^2}{R_1^2 R_2^2}.$$

As we can see, the final result does not depend on the angular speed. This means that the centrifugal inertial forces do not influence the mutual angle of rotation of the boundary points on the ring. Apparently, this is to be expected, since the centrifugal forces of inertia are applied in the radial directions, i.e. along the direction, perpendicular to the angular displacement. Note that this result coincided with the similar results obtained previously for the static case [2].

So, in this paper, the boundary problem of the plane theory of elasticity for domain in the form of a ring is solved using Muskhelishvili complex potentials method. The ring rotates uniformly, and inertial forces are taken into account. The solution is obtained in a closed form and can be useful for engineering applications. Mutual rotation angle of the boundary points is evaluated. This angle is able to play the role of stress indicator. It is shown that the centrifugal inertial forces have no effect on the mutual rotation angle. Agreement with the results known for the static case is received.

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