

**MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National University «Zaporizhzhia Polytechnic»**

LECTURES ON ELEMENTS OF THEORY OF PROBABILITY

for students majoring in
141 Power Engineering, Electrical Engineering
and Electrical Mechanics

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CONTENTS

1 Elements of Theory of Probability	4
1.1 Counting Rules	4
1.2 Sample Space and Events	6
1.3 Definition of Probability	8
1.4 Additive and Multiplicative Rules	11
1.5 Total Probability. Bayes' Formula	16
1.6. Bernoulli's Experiment. Bernoulli's Formula	19
1.7 Local and Integral Laplace Theorems	21
1.8 Poisson's Formula	22
2 Random Variables and Their Probability Distribution	24
2.1 Discrete Random Variables and Their Probability Distributions	24
2.2 Continuous Random Variables and Their Probability Distributions	32
References	42
Appendix A	43

1 ELEMENTS OF THEORY OF PROBABILITY

1.1 Counting Rules

The product of the first n positive integers is called n **factorial** and is denoted by $n!$.

By definition, $0!=1$, $n!=1\cdot 2\cdot 3\cdot \dots\cdot n$.

Addition principle (AP). If some object A can be chosen in m ways and another object B can be chosen in n ways then the choice «either A or B » can be made in $m+n$ ways.

Example 1.1. A basket contains 5 apples and 8 oranges. How many ways are there to select one fruit?

Solution. By (AP), the number of selections of one fruit is $5+8=13$. •

Multiplication principle (MP). If the first object A can be chosen in m ways and after each such choice the second object B can be chosen in n ways then the choice of the pair A and B can be made in $m\cdot n$ ways.

Example 1.2. Oleg wants to visit Ternopil, Chernivtsi and Lviv. There are two recommended roads from Ternopil to Chernivtsi and three ones from Chernivtsi to Lviv. How many ways are there to visit all these cities?

Solution. By (MP), the total number of ways from Ternopil to Lviv is $2\cdot 3=6$. •

Example 1.3. A survey consists of three multiple choice questions. The first question has 3 possible answers, the second question has 4 possible answers and the third one has 3 possible answers. What is the total number of different ways in which this survey could be completed?

Solution. By (MP), the total number of different ways in which this survey could be completed is $3\cdot 4\cdot 3=36$. •

A permutation of a group of objects is any arrangement of these objects in a definite order.

The number of permutations of n objects is given by

$$P_n = n!$$

Example 1.4. How many different six-letter words (with or without meaning) can be made from the letters of the word OBJECT?

Solution. The number of different six-letter words is equal to the number of the permutation of 6 letters. We have $P_6 = 6! = 720$. •

Example 1.5. How many different six-letter words (with or without meaning) can be made from the letters of the word OBJECT beginning with E?

Solution.

E					
---	--	--	--	--	--

The first position is filled by the letter E. The number of ways of filling the other 5 positions is $P_5 = 5! = 120$. •

The number of permutations of k objects selected from n distinct objects ($k \leq n$), where repetitions are not allowed, is given by

$$A_n^k = \frac{n!}{(n-k)!}.$$

In permutation, the order that we arrange the objects in is important.

Example 1.6. How many distinct 3-digit numbers can be formed using the digits 2, 4, 6, 8 and 9, if the digits can't be repeated?

Solution. The order of the digits is important. The number of distinct 3-digit numbers formed from 5 given numbers is

$$A_5^3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{2! \cdot 3 \cdot 4 \cdot 5}{2!} = 60. \bullet$$

A combination of k objects that can be selected from n objects ($k \leq n$) is a selection which does not take into account the arrangement of objects. It means that the order of the objects is not important.

The number of combinations of k objects selected from n objects, where repetitions are not allowed, is given by

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

Example 1.7. A committee of 4 persons is chosen from 7 men and 6 women. How many different committees can be chosen if:

- a) there are no restrictions;
- b) there must be 2 of each sex;
- c) at least of one of each sex is needed.

Solution.

a) For no restrictions there are $6+7=13$ people for selection and we want to choose any 4 of them. The number of choices equals

$$C_{13}^4 = \frac{13!}{4!(13-4)!} = \frac{13!}{4!9!} = \frac{9! \cdot 10 \cdot 11 \cdot 12 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 9!} = 715.$$

b) 2 men can be chosen from 7 men in C_7^2 ways and 2 women can be chosen from 6 women in C_6^2 ways. By (MP), the number of choices is

$$C_7^2 \cdot C_6^2 = \frac{7!}{2!(7-2)!} \cdot \frac{6!}{2!(6-2)!} = \frac{5! \cdot 6 \cdot 7}{2! \cdot 5!} \cdot \frac{4! \cdot 5 \cdot 6}{2! \cdot 4!} = 21 \cdot 15 = 315.$$

c) «The committee consists of at least of one of each sex» means «committee consists of 1 man and 3 women or 2 men and 2 women or 3 men and 1 woman».

By (MP) and (AP), the total number of choices equals

$$C_7^1 \cdot C_6^3 + C_7^2 \cdot C_6^2 + C_7^3 \cdot C_6^1 = 665. \bullet$$

1.2 Sample Space and Events

Experiment is any process of observation or procedure that can be repeated an infinite number of times and has a well-defined set of possible outcomes.

Tossing a die, flipping a coin, target shooting are examples of experiments.

A sample space S is a set of all possible non-overlapping outcomes for a random experiment.

Event is a subset of a sample space. Events are denoted by capital letters $A, B, C, \dots$

Example 1.8 Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Some events:

$A = \{\text{an odd number occurs}\}$ is $A = \{1, 3, 5\}$;

$B = \{\text{the number greater than 5 occurs}\}$ is $B = \{6\}$. •

Example 1.9. The experiment consist of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. The sample space is $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$, where H represents head and T represents tail. •

A random event is the event which may or may not occur during an experiment.

A certain (sure) event is the event which will surely occur when the experiment is conducted.

An impossible event or **the null event** is the event which will never occur during an experiment. It is denoted by $\emptyset$.

The union of two events A and B , denoted by $A + B$ or $A \cup B$, is the event C in which at least one of two events A and B occurs.

The intersection of two events A and B , denoted by symbol AB or $A \cap B$, is the event C in which both events A and B occur simultaneously.

Two events A and B are called **mutually exclusive** or **disjoint** if they can not occur simultaneous. It means that $AB = \emptyset$.

The events $A_1, A_2, \dots, A_n$ are **mutually exclusive** if the occurrence of any one of them implies the non-occurrence of another $n-1$ events. It means that $A_i \cap A_j = \emptyset, i \neq j$.

A collection of events $\{A_1, A_2, \dots, A_n\}$ of a sample space S is called **a partition of S** if $A_1, A_2, \dots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \dots \cup A_n = S$.

A compliment of an event A with respect to S , denoted by $\bar{A}$, is the subset of all outcomes of S which are not included in A .

Fig. 1.1 shows Venn diagrams which illustrate the relationships between events and the corresponding sample space.

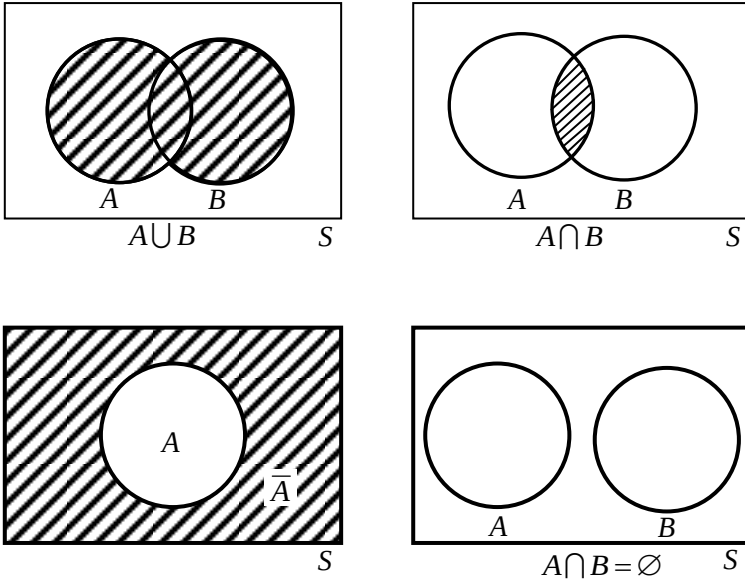


Figure 1.1

Example 1.10. A die is rolled. Find the events:

$A = \{\text{an odd number occurs}\}$; $B = \{\text{an even number occurs}\}$; $C = \{\text{the number greater than 5 occurs}\}$; $D = \{\text{the number greater than 7 occurs}\}$; $A \cup B$; $A \cap B$; $B \cap C$; $\bar{A}$; $\bar{C}$.

Solution. $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$, $C = \{6\}$, $D = \{\emptyset\}$,
 $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \emptyset$, $B \cap C = \{6\}$, $\bar{A} = \{2, 4, 6\}$,
 $\bar{C} = \{1, 2, 3, 4, 5\}$. •

1.3 Definition of Probability

Classical definition of probability. The probability of the event A in the classical sense is

$$P(A) = \frac{m}{n},$$

where n is the total number of equally possible outcomes in the sample space S of the experiment, m is the number of outcomes favorable to A .

Properties of probability:

a) $0 < P(\text{random event}) < 1$;

b) $P(\emptyset) = 0$;

c) $P(\text{certain event}) = 1$.

Example 1.11. An urn contains 2 red, 1 green, 4 blue, and 3 yellow balls. One ball is chosen at random.

a) What is the probability that the red ball is chosen?

b) What is the probability that the yellow or the green ball is chosen?

c) What is the probability that the selected ball is not green?

Solution. The number of ways to choose one ball is $n = 2 + 1 + 4 + 3 = 10$. This is the number of all possible outcomes.

a) The number of ways to choose one red ball is $m = 2$. This is the number of outcomes favorable to the required event.

According to the classical definition of probability we have:

$$P(\text{red}) = \frac{2}{10} = 0.2.$$

b) The number of ways to choose the yellow or the green ball is $m = 1 + 3 = 4$.

$$P(\text{yellow or green}) = \frac{4}{10} = 0.4.$$

c) The number of ways to choose the ball that isn't green is $m = 10 - 1 = 9$, $P(\text{not green}) = \frac{9}{10} = 0.9$. •

Example 1.12. Three cards are drawn from a standard 52-card deck. Find the probability of getting 3 jacks.

Solution. The number of ways of getting any 3 cards is $n = C_{52}^3 = \frac{52!}{3!49!} = 22100$. The number of getting 3 jacks from 4 is

$$m = C_4^3 = \frac{4!}{3!1!} = 4. \quad P(3 \text{ jacks}) = \frac{C_4^3}{C_{52}^3} = \frac{4}{22100} \approx 0.0002. \bullet$$

Axiomatic definition of probability. Let S be a sample space of an experiment. A probability is a real-valued function that assigns a number $P(A)$ to each event A in the sample space S so that the following axioms hold:

a) $P(A) \geq 0$.

b) $0 \leq P(A) \leq 1$.

c) If $A_1, A_2, \dots$ are mutually exclusive events ($A_i \cap A_j = \emptyset$, for $i \neq j$), then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

Frequency definition of probability. Suppose an experiment can be repeated any number of times, so that we can produce a series of independent trials under identical conditions. In each observation the event A either occurs or does not occur. The relative frequency of the event A (in this given series of independent and identical trials) is defined by

$$W(A) = \frac{n(A)}{n},$$

where n is the total number of observations in this series, $n(A)$ is the number of times A occurs. It has been empirically observed that the relative frequency becomes stable if $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \frac{n(A)}{n} = P(A)$.

Geometrical definition of probability. Let mes be a measure (length, area, volume) of a domain. The probability that randomly dropped point is placed in the domain g which contains in the domain G is defined by the formula:

$$P = \frac{mes\ g}{mes\ G}.$$

Example 1.13. A point is randomly dropped inside the circle with center at the origin of the Cartesian coordinate system. What is the probability that it is placed in the square inscribed in the circle?

Solution. Let A be the event that a point is placed in the square inscribed in the circle. If R is the radius of the circle and a is the side of

the square inscribed in the circle then $R = \frac{a\sqrt{2}}{2} \Rightarrow a = \sqrt{2}R$, $S_{circle} = \pi R^2$,

$$S_{square} = a^2 = 2R^2.$$

$$P(A) = \frac{S_{square}}{S_{circle}} = \frac{2R^2}{\pi R^2} = \frac{2}{\pi} \approx 0.6366 \cdot$$

1.4 Additive and Multiplicative Rules

Let A and B be two events. The probability that A occurs given that B has occurred is called the **conditional probability** and is denoted by $P(A|B)$ or $P_B(A)$.

$P(A|B)$ is read “the probability of A , given B ”.

The conditional probability of A , given B is equal to

$$P(A|B) = \frac{P(AB)}{P(B)}, \text{ provided } P(B) > 0.$$

Example 1.14. We toss two dice, and let A be the event that the sum of the face values of two dice is 10, and B be the event that the face value of the first one is 5. Calculate $P(A|B)$.

Solution. The sample space is

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The events A and B are $A = \{(4,6), (5,5), (6,4)\}$, $B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$. Now, $AB = \{(5,5)\}$.

$$P(B) = \frac{6}{36} = \frac{1}{6}, \quad P(AB) = \frac{1}{36}, \quad P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6} \cdot$$

Theorem 1.1. If A and B are mutually exclusive events, then

$$P(A+B) = P(A) + P(B).$$

Proof. Let A and B be mutually exclusive events in some experiment. If n is the total number of equally possible outcomes in the experiment, m_1 is the number of outcomes favorable to A , m_2 is the number of outcomes favorable to B , then $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$. The number of outcomes favorable to the event $A+B$ is equal to $m_1 + m_2$. We have

$$P(A+B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

Theorem 1.2. If $A_1, A_2, \dots, A_n$ are mutually exclusive events, then

$$P(A_1 + A_2 + \dots + A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

Theorem 1.3. If $A_1, A_2, \dots, A_n$ is a partition of a sample space S , then

$$P(A_1 + A_2 + \dots + A_n) = P(A_1) + P(A_2) + \dots + P(A_n) = 1.$$

Proof. Let $A_1, A_2, \dots, A_n$ be a partition of a sample space S . The occurrence of the event $A_1 + A_2 + \dots + A_n$ is the certain event. It means that

$$P(A_1 + A_2 + \dots + A_n) = 1.$$

Since the events A_i ($i = \overline{1, n}$) are mutually exclusive then

$$P(A_1 + A_2 + \dots + A_n) = P(A_1) + P(A_2) + \dots + P(A_n) = 1.$$

Theorem 1.4. If A and $\bar{A}$ are complementary events, then

$$P(A) + P(\bar{A}) = 1.$$

Example 1.15. There are 7 black and 3 white balls in the urn. We select 3 balls at random.

a) What is the probability that at most one black ball is selected?

b) What is the probability that at least one black ball is selected?

Solution. The number of ways of selecting 3 balls out of 10 is

$$n = C_{10}^3 = 120.$$

a) Let A be the event that at most one black ball is selected. The event A means none, or one black ball is selected.

Let B be the event that none of the black balls is selected and let C be the event that exactly one black ball is selected. We have $A = B + C$.

None black ball can be selected from 7 black balls in C_7^0 ways and three white balls can be selected from 3 white balls in C_3^3 ways. By (MP), $m = C_7^0 \cdot C_3^3$.

$$P(A) = \frac{m}{n} = \frac{C_7^0 \cdot C_3^3}{C_{10}^3} = \frac{1}{120}.$$

One black ball can be selected from 7 black balls in C_7^1 ways and two white balls can be selected from 3 white balls in C_3^2 ways. By (MP), $m = C_7^1 \cdot C_3^2$.

$$P(B) = \frac{m}{n} = \frac{C_7^1 \cdot C_3^2}{C_{10}^3} = \frac{7 \cdot 3}{120} = \frac{7}{40}.$$

The events B and C are mutually exclusive events. By theorem 1.1,

$$P(A) = P(B + C) = P(B) + P(C) = \frac{1}{120} + \frac{7}{40} = \frac{22}{120} = \frac{11}{60}.$$

b) Let D be the event at least one black ball is selected. «At least one black ball is selected» means «1 black ball or 2 black balls or 3 black balls are selected». The events D and B (none of the black balls is selected) are the complementary events.

$$\text{By theorem 1.4, } P(D) = 1 - P(C) = 1 - \frac{1}{120} = \frac{119}{120}.$$

If the occurrence or non-occurrence of the event A does not affect on the occurrence of the event B , then A and B are called **independent** events. Otherwise, A and B are dependent.

Two events A and B are independent if and only if

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B).$$

Theorem 1.5. If the events A and B are any two events, then

$$P(AB) = P(A)P(B|A) = P(B)P(A|B).$$

The proof of this theorem follows from the definition of the conditional probability.

Theorem 1.6. If the events A and B are independent, then

$$P(AB) = P(A)P(B).$$

Theorem 1.7. If in an experiment the events $A_1, A_2, \dots, A_k$ can occur, then

$$P(A_1 \cdot A_2 \cdot \dots \cdot A_k) = P(A_1)P(A_2 | A_1)P(A_3 | A_1 A_2) \dots (A_k | A_1 A_2 \cdot \dots \cdot A_{k-1}).$$

Theorem 1.8. If the events $A_1, A_2, \dots, A_k$ are independent then

$$P(A_1 \cdot A_2 \cdot \dots \cdot A_k) = P(A_1)P(A_2)P(A_3) \cdot \dots \cdot P(A_k).$$

Example 1.16. A basket contains 30 apples and five of them are wormy. If two apples are randomly picked in sequence, what is the probability that both are worm-free apples?

Solution. Let A be the event that the first apple is worm-free and let B be the event that the second apple is worm-free. Then AB is the event that the first apple is worm-free and the second apple is worm-free.

The events A and B are dependent. We have $P(A) = \frac{25}{30} = \frac{5}{6}$,

$$P(B|A) = \frac{24}{29}.$$

By theorem 1.5, we have

$$P(AB) = P(A)P(B|A) = \frac{5}{6} \cdot \frac{24}{29} = \frac{20}{29}.$$

Hence the probability that both apples are worm-free is $\frac{20}{29}$. •

Example 1.17. A pen-box contains 2 black pens, 5 blue pens and 3 red pens. Three pens are chosen at random from the pen-box. Find the probability of obtaining black, blue and red pens in that order if

a) the pen isn't put back after each draw;

b) the pen is put back after each draw.

Solution. Let

A be the event that the first pen is black,

B be the event that the second pen is blue,

C be the event that the third pen is red.

Then ABC is the event that the first pen is black, the second pen is blue, the third pen is red.

a) If the pen isn't put back after each draw, then the number of the pens is changed after each draw. Therefore, we have 3 dependent events. By theorem 1.7, we have

$$P(A) = \frac{2}{10} = \frac{1}{5}, \quad P(B|A) = \frac{5}{9}, \quad P(C|AB) = \frac{3}{8}.$$

$$P(ABC) = P(A)P(B|A)P(C|AB) = \frac{1}{5} \cdot \frac{5}{9} \cdot \frac{3}{8} = \frac{3}{72} = \frac{1}{24} = 0.0417.$$

b) If the pen is put back after each draw, then the number of the pens isn't changed after each draw. Therefore, we have 3 independent events. By theorem 1.8, we have

$$P(ABC) = P(A)P(B)P(C) = \frac{2}{10} \cdot \frac{5}{10} \cdot \frac{3}{10} = \frac{30}{1000} = 0.03. \bullet$$

Theorem 1.9. If A and B aren't mutually exclusive events, then

$$P(A+B) = P(A) + P(B) - P(AB).$$

Proof. A and B aren't mutually exclusive events. The event $A+B$ is the sum of such mutually exclusive events: $A\bar{B}$, $\bar{A}B$ and AB . According to theorem 1.1 we have:

$$P(A+B) = P(A\bar{B} + \bar{A}B + AB) = P(A\bar{B}) + P(\bar{A}B) + P(AB).$$

The event A occurs if one of the events $A\bar{B}$ or AB occurs. Thus,

$$P(A) = P(A\bar{B}) + P(AB) \Rightarrow P(A\bar{B}) = P(A) - P(AB).$$

Analogously, $P(B) = P(\bar{A}B) + P(AB) \Rightarrow P(\bar{A}B) = P(B) - P(AB).$

We get

$$\begin{aligned} P(A+B) &= P(A) - P(AB) + P(B) - P(AB) + P(AB) = \\ &= P(A) + P(B) - P(AB). \end{aligned}$$

Example 1.18. In a high school graduating class of 100 students, 54 studied mathematics, 69 studied history, and 35 studied both mathematics and history. If one of these students is selected at random, find the probability that

a) the student took mathematics or history;

b) the student did not take either of these subjects.

Solution. Let A be the event that student took math and let B be the event that student took history.

a) Events A and B aren't mutually exclusive, $P(A) = \frac{54}{100} = 0.54$,

$P(B) = \frac{69}{100} = 0.69$, $P(AB) = \frac{35}{100} = 0.35$. According to theorem 1.9,

$$P(A+B) = P(A) + P(B) - P(AB) = 0.54 + 0.69 - 0.35 = 0.88.$$

b) Let C be the event that student did not take either of these subjects.

The events “the student took mathematics or history” and “student did not take either of these subjects” are complementary. By theorem 1.4,

$$P(C) = 1 - P(A+B) = 1 - 0.88 = 0.12. \bullet$$

1.5 Total Probability. Bayes' Formula

Let $H_1, H_2, \dots, H_n$ be a set of mutually exclusive and exhaustive events, then this set is called **a complete group of hypotheses**. For a complete group of hypotheses, the following equality holds: $P\left(\sum_{i=1}^n H_i\right) = 1$.

Theorem 1.10 (Theorem of total probability). Let an event A occurs only if one hypothesis of a complete group $H_1, H_2, \dots, H_n$ has occurred, then a formula of total probability holds:

$$P(A) = \sum_{i=1}^n P(H_i)P(A|H_i),$$

where $P(A)$ is a probability of the event A ;

$P(H_i)$ is a probability of the hypothesis H_i ;

$P(A|H_i)$ is a conditional probability of the event A given that H_i has occurred.

Proof. The occurrence of the event A means the occurrence any of mutually exclusive events H_1A , H_2A , ..., H_nA . Since $A = H_1A + H_2A + \dots + H_nA$, then by theorem 1.1 we get

$$P(A) = P(H_1A + H_2A + \dots + H_nA) = P(H_1A) + P(H_2A) + \dots + P(H_nA).$$

Using theorem 1.5, we obtain

$$P(A) = P(H_1)P(A|H_1) + P(H_2)P(A|H_2) + \dots + P(H_n)P(A|H_n).$$

Example 1.19. In a certain plant, three machines (I, II, and III) make 30%, 45%, and 25% of the details. It is known that 2%, 3%, and 5% of the details made by each machine, respectively, are defective. A detail is randomly selected. What is the probability that it is defective?

Solution. Let H_i be a hypothesis that the detail is made by i -th machine, $i=1,2,3$. According to the condition of the problem we have

$$P(H_1) = 0.3, \quad P(H_2) = 0.45, \quad P(H_3) = 0.25.$$

Let A be the event that the selected detail is defective.

$P(A|H_1) = 0.02$ is the conditional probability that the detail is defective, given that it made by machine I;

$P(A|H_2) = 0.03$ is the conditional probability that the detail is defective, given that it made by machine II,

$P(A|H_3) = 0.05$ is the conditional probability that the detail is defective, given that it made by machine III.

Applying the theorem of total probability, we get

$$\begin{aligned} P(A) &= P(H_1)P(A|H_1) + P(H_2)P(A|H_2) + P(H_3)P(A|H_3) = \\ &= 0.3 \cdot 0.02 + 0.45 \cdot 0.03 + 0.25 \cdot 0.05 = 0.032. \bullet \end{aligned}$$

Theorem 1.11 (Bayes' formula). Let an event A occurs only if one hypothesis of a complete group $H_1, H_2, \dots, H_n$ has occurred, then Bayes' formula holds

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{P(A)}, \quad i=1,2,\dots,n,$$

where $P(H_i|A)$ is the probability of realization of the hypothesis H_i given that A has already occurred, $P(A)$ is the total probability.

Proof. Using definition of conditional probability and theorem 1.5, we get

$$P(H_i | A) = \frac{P(H_i A)}{P(A)} = \frac{P(H_i)P(A | H_i)}{P(A)},$$

where $P(A)$ can be found using theorem of total probability.

Example 1.20. Consider the information in the previous example. If a randomly selected detail is defective, what is the probability that it was made by machine III?

Solution. Using Bayes' formula, the probability that a randomly selected detail was made by machine III and, given that it is defective, is

$$P(H_3 | A) = \frac{P(H_3)P(A | H_3)}{P(A)} = \frac{0.25 \cdot 0.05}{0.032} \approx 0.3906 . \bullet$$

Example 1.21. A store has light bulbs from two suppliers, 1 and 2. The chance of supplier 1 delivering defective bulbs is 5%, whereas supplier 2 has a defective rate of 2%. Suppose 70% of the current supply of light bulbs came from supplier 1. One bulb is randomly chosen the current supply.

a) Find the probability that the chosen bulb is non-defective.

b) The chosen bulb is non-defective. Find the probability that it came from supplier 2.

Solution.

a) Let H_1 be a hypothesis that the bulb came from supplier 1, and let H_2 be a hypothesis that the bulb came from supplier 2. According to the condition of problem we have

$$P(H_1) = 0.7, \quad P(H_2) = 0.3.$$

Let A be an event that the chosen bulb is non-defective.

$P(A | H_1) = 1 - 0.05 = 0.95$ is the conditional probability that the chosen bulb is non-defective, given that it came from supplier 1,

$P(A | H_2) = 1 - 0.02 = 0.98$ is the conditional probability that the chosen bulb is non-defective, given that it came from supplier 2.

$$P(A) = P(H_1)P(A | H_1) + P(H_2)P(A | H_2) = 0.7 \cdot 0.95 + 0.3 \cdot 0.98 = 0.959 .$$

b) By Bayes' formula, the probability that the chosen bulb came from supplier 2, given that it is non-defective equals

$$P(H_2 | A) = \frac{P(H_2)P(A|H_2)}{P(A)} = \frac{0.3 \cdot 0.98}{0.959} \approx 0.3066 . \bullet$$

1.6 Bernoulli's Experiment. Bernoulli's Formula

Bernoulli's experiment is one that possesses the following properties:

- a)** the experiment consists of n repeated trials;
- b)** each trial results in an outcome that may be classified as a **success** or a **failure**;
- c)** the probability of a success, denoted by p , remains constant from trial to trial, the probability of a failure is $q = 1 - p$;
- d)** repeated trials are independent.

The probability of k successes in n Bernoulli's trials is

$$P_n(k) = C_n^k \cdot p^k \cdot q^{n-k} .$$

Example 1.22. A coin is tossed 10 times. Find the probability that

- a)** exactly 2 heads occur; **c)** at most 2 heads occur;
- b)** from 2 to 4 heads occur; **d)** at least 2 heads occur.

Solution. $n = 10$, $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$.

$$\mathbf{a)} \quad P_{10}(2) = C_{10}^2 \cdot p^2 \cdot q^{10-2} = \frac{10!}{2!8!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 = \frac{45}{1024} \approx 0.0439 .$$

b) The event «from 2 to 4 heads occur» means «2, 3 or 4 heads occur». Therefore,

$$\begin{aligned} P_{10}(2 \leq k \leq 4) &= P_{10}(2) + P_{10}(3) + P_{10}(4) = \\ &= C_{10}^2 \cdot p^2 \cdot q^8 + C_{10}^3 \cdot p^3 \cdot q^7 + C_{10}^4 \cdot p^4 \cdot q^6 = \\ &= \frac{45}{1024} + \frac{120}{1024} + \frac{210}{1024} = \frac{375}{1024} \approx 0.3662 . \end{aligned}$$

c) The event «at most 2 heads occur» means that 0, 1, 2 heads occur.

$$P_{10}(0) = C_{10}^0 \cdot p^0 \cdot q^{10-0} = \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} = \frac{1}{1024};$$

$$P_{10}(1) = C_{10}^1 \cdot p^1 \cdot q^{10-1} = \frac{10!}{1!9!} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9 = \frac{10}{1024};$$

$$P_{10}(2) = C_{10}^2 \cdot p^2 \cdot q^{10-2} = \frac{45}{1024};$$

$$P_{10}(k \leq 2) = P_{10}(0) + P_{10}(1) + P_{10}(2) = \frac{1}{1024} + \frac{10}{1024} + \frac{45}{1024} = \frac{56}{1024} \approx 0.0547.$$

d) the event «at least 2 heads occur» means that 2, 3, ..., 10 heads occur. The event «at least 2 heads occur» is compliment of the event «less than 2 heads occur». Therefore,

$$\begin{aligned} P_{10}(k \geq 2) &= 1 - P_{10}(k < 2) = 1 - P_{10}(0) - P_{10}(1) = \\ &= 1 - \frac{1}{1024} - \frac{10}{1024} = \frac{1013}{1024} \approx 0.9893. \bullet \end{aligned}$$

Example 1.23. Observations have shown that in City X in April it rains on an average 12 days. A five-day period is chosen. What is the probability that it rains on: **a)** exactly three days; **b)** less than two day; **c)** at least four days.

Solution. There are 30 days in April. So, the probability of raining (on a single day) is $p = \frac{12}{30} = \frac{2}{5}$, the probability of not raining is

$q = 1 - p = 1 - \frac{2}{5} = \frac{3}{5}$. The conditions of the problem satisfies Bernoulli's

experiment conditions with $n = 5$, $p = \frac{2}{5}$, $q = \frac{3}{5}$.

$$\mathbf{a)} \quad P_5(3) = C_5^3 \cdot p^3 \cdot q^2 = \frac{5!}{3!2!} \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^2 = \frac{144}{625} = 0.2304.$$

b) The event «it rains on less than two days» means «it rains on 0 or 1 day».

$$P_5(0) = C_5^0 \cdot p^0 \cdot q^{5-0} = \left(\frac{2}{5}\right)^0 \left(\frac{3}{5}\right)^5 = \frac{243}{3125} = 0.07776;$$

$$P_5(1) = C_5^1 \cdot p^1 \cdot q^{5-1} = \frac{5!}{1!4!} \left(\frac{2}{5}\right)^1 \left(\frac{3}{5}\right)^4 = \frac{162}{625} = 0.2592 ;$$

$$P_5(k < 2) = P_5(0) + P_5(1) = 0.07776 + 0.2592 = 0.337 .$$

c) The event «it rains on at least four days» means «it rains on 4 or 5 days».

$$P_5(4) = C_5^4 \cdot p^4 \cdot q^{5-4} = \frac{5!}{4!1!} \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^1 = \frac{48}{625} = 0.0768 ;$$

$$P_5(5) = C_5^5 \cdot p^5 \cdot q^{5-5} = \left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right)^0 = \frac{32}{625} = 0.0512 ;$$

$$P_5(k \geq 4) = P_5(4) + P_5(5) = 0.0768 + 0.0512 = 0.128 . \bullet$$

1.7 Local and Integral Laplace Theorems

It is difficult to use Bernoulli's formula for large n . In this case, the local Laplace theorem is applied in the Bernoulli trials.

Theorem 1.12 (Local Laplace theorem). The probability of k successes in n Bernoulli's trials with probability of success p ($0 < p < 1$) is approximately equal to

$$P_n(k) \approx \frac{1}{\sqrt{npq}} \cdot \varphi(x) ,$$

where $q = 1 - p$, $x = \frac{k - np}{\sqrt{npq}}$, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, the function $\varphi(x)$ is the

Gauss function (its values can be found using special tables (Table A.1)).

Function $\varphi(x)$ has such properties:

1) $\varphi(-x) = \varphi(x)$; 2) for $x \geq 4$ $\varphi(x) = 0$.

Example 1.24. It is known that 20% of engines made by the plant are defective. 900 engines are selected at random and inspected. Find the probability that exactly 210 engines will be defective.

Solution. $n = 900$, $k = 210$, $p = 0.2$, $q = 0.8$.

According to the local Laplace theorem, we have

$$x = \frac{k - np}{\sqrt{npq}} = \frac{210 - 900 \cdot 0.2}{\sqrt{900 \cdot 0.2 \cdot 0.8}} = 2.5, \quad \varphi(2.5) = 0.0175 ,$$

$$P_{400}(80) \approx \frac{1}{\sqrt{900 \cdot 0.2 \cdot 0.8}} \cdot \varphi(0) = \frac{1}{12} \cdot 0.0175 = 0.0015 . \bullet$$

Theorem 1.13 (Integral Laplace theorem). The probability that the number of successes is between k_1 and k_2 in n Bernoulli's trials with probability of success p ($0 < p < 1$) is approximately equal to

$$P_n(k_1, k_2) \approx \Phi(x_2) - \Phi(x_1),$$

where $x_1 = \frac{k_1 - np}{\sqrt{npq}}$, $x_2 = \frac{k_2 - np}{\sqrt{npq}}$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$, the function $\Phi(x)$ is the Laplace function (its values can be found using special tables (Table A.2)).

The function $\Phi(x)$ has such properties:

1) $\Phi(-x) = -\Phi(x)$; 2) for $x > 5$ $\Phi(x) = 0.5$.

Example 1.25. The factory produces 70% of the first quality items. What is the probability that in an inspection lot of 1000 items the number of the first quality items will be between 652 and 760?

Solution. According to the statement of the problem, we have $n = 1000$, $p = 0.7$, $q = 0.3$, $k_1 = 652$, $k_2 = 760$.

$$x_1 = \frac{k_1 - np}{\sqrt{npq}} = \frac{652 - 1000 \cdot 0.7}{\sqrt{1000 \cdot 0.7 \cdot 0.3}} \approx -3.31,$$

$$x_2 = \frac{k_2 - np}{\sqrt{npq}} = \frac{760 - 1000 \cdot 0.7}{\sqrt{1000 \cdot 0.7 \cdot 0.3}} \approx 4.14.$$

Using table A.2, we find $\Phi(4.14) = 0.49998$, $\Phi(-3.31) = -\Phi(3.31) = -0.49950$. According to the Laplace integral theorem, we get

$$\begin{aligned} P_{1000}(652, 760) &\approx \Phi(x_2) - \Phi(x_1) = \Phi(4.14) - \Phi(-3.31) = \\ &= 0.49998 + 0.49950 = 0.99948 . \bullet \end{aligned}$$

1.8 Poisson's Formula

Consider Bernoulli's experiment with a small probability p of a success in each trial and a large number n of trials.

Theorem 1.14 (Poisson's theorem). Let $p \rightarrow 0$ and $n \rightarrow \infty$, so that $p \cdot n \rightarrow \lambda$ ($\lambda > 0$). Then the probability of k successes in n Bernoulli's trials is

$$P_n(k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}, \quad k = 1, 2, \dots$$

The last formula is used when the following conditions are fulfilled: $n \gg 1$, $p \ll 1$ and $\lambda = np < 10$.

Example 1.26. If the probability that an individual suffers an adverse reaction from a particular drug is known to be 0.002, determine the probability that out of 3000 individuals

a) exactly two will suffer an adverse reaction;

b) more than two individuals will suffer an adverse reaction.

Solution. We have $n = 3000$, $p = 0.002$, $q = 0.999$, $\lambda = p \cdot n = 6 < 10$. Because n is large and p is small, we can use the Poisson formula.

a) The probability that exactly three individuals will suffer an adverse reaction is

$$P_{3000}(2) = \frac{6^2 \cdot e^{-6}}{2!} = 0.0446.$$

That is, there is approximately an 4.46% chance that exactly two individuals of 3000 will suffer an adverse reaction.

b) The probability that more than two individuals will suffer an adverse reaction is

$$\begin{aligned} P_{3000}(k > 2) &= 1 - P_{3000}(0) - P_{3000}(1) - P_{3000}(2) = \\ &= 1 - \frac{6^0 \cdot e^{-6}}{0!} - \frac{6^1 \cdot e^{-6}}{1!} - \frac{6^2 \cdot e^{-6}}{2!} = 0.938. \end{aligned}$$

Similarly, there is approximately a 93,8% chance that more than two individuals will have an adverse reaction. •

2 RANDOM VARIABLES AND THEIR PROBABILITY DISTRIBUTION

A random variable is a variable whose value is determined by the outcome of a random experiment.

The random variables are denoted by the capital letters X , Y , Z , ..., and any particular real values of the random variables are denoted by the corresponding lowercase letters x , y , z , ...

The probability that X takes on the value x $P(X = x)$ is denoted by $p(x)$.

A random variable X is said to be **discrete (D.v.)** if it can assume only a finite or countably infinite number of distinct values.

The number of students in Moodle, the number of red pens in a pen-box, the number of heads when flipping four coins, the number of defective items of 10 ones are examples of D.v.

A continuous random variable X (C.v.) is one whose set of assumed values is uncountable (arises from measurement).

Height of students in class, weight of students in class, time it takes to get to university, the temperature of a cup of coffee are examples of C.v.

2.1 Discrete Random Variables and Their Probability Distributions

A discrete probability distribution is a table (or a formula) listing all possible values that a discrete variable X can take on, together with the associated probabilities:

X	x_1	x_2	...	x_n
p	p_1	p_2	...	p_n

where $x_1 < x_2 < \dots < x_n$ and $p_1 + p_2 + \dots + p_n = 1$.

A piecewise linear graph based on this table is called a **distribution polygon**.

A distribution function (cumulative distribution function) $F(x)$ of a discrete random variable X is defined by the formula

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i).$$

Every distribution function has such properties:

a) $0 \leq F(x) \leq 1$;

b) $\lim_{x \rightarrow -\infty} F(x) = 0$;

c) $\lim_{x \rightarrow \infty} F(x) = 1$;

d) $F(x)$ is a nondecreasing function;

e) $\lim_{x \rightarrow x_0+0} F(x) = F(x_0)$.

Example 2.1. When tossing three fair coins, let the random variable X denote the number of tails in the outcome.

a) Find the probability distribution of X .

b) Find the distribution function of X .

Solution. The sample space has form:
 $S = \{HTT, HTH, HHT, HHH, THH, TTH, THT, TTT\}$.

D.v. X can assume values $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$.

a) $P(X=0) = \frac{1}{8}$, $P(X=1) = \frac{3}{8}$, $P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$.

The probability distribution of X has the form:

X	0	1	2	3
p	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b) According to the definition of the distribution function, we have

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{8}, & 0 < x \leq 1, \\ \frac{1}{8} + \frac{3}{8}, & 1 < x \leq 2, \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8}, & 2 < x \leq 3, \\ 1, & x > 3. \end{cases} = \begin{cases} 0, & x \leq 0, \\ \frac{1}{8}, & 0 < x \leq 1, \\ \frac{1}{2}, & 1 < x \leq 2, \\ \frac{7}{8}, & 2 < x \leq 3, \\ 1, & x > 3. \end{cases}$$

Fig. 2.1. shows the graph of $F(x)$. •

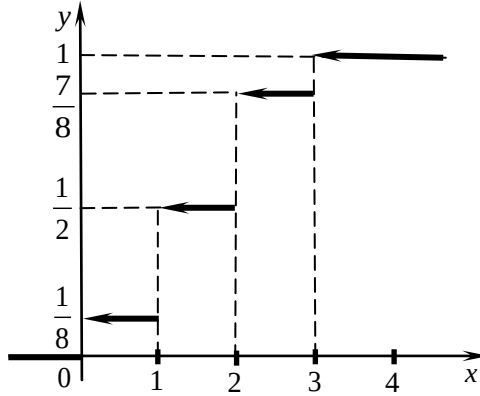


Figure 2.1

Notice that the distribution function of D.v. is a step function and is defined for all real numbers. This is true for all discrete random variables.

Numerical characteristics of D.v.

The expected value of D.v. X , denoted by $E(X)$ or μ , is defined by

$$E(X) = \sum_{i=1}^n x_i p_i = x_1 p_1 + x_2 p_2 + \dots + x_n p_n.$$

The expected value of a random variable X has such properties:

- 1) $E(C) = C$ ($C = \text{const}$);
- 2) $E(CX) = C \cdot E(X)$;
- 3) $E(X + Y) = E(X) + E(Y)$, where X and Y are random variables;
- 4) $E(X \cdot Y) = E(X) \cdot E(Y)$, where X and Y are independent random variables.

The variance of D.v. X , denoted by $V(X)$ or σ^2 , is defined by

$$V(X) = E([X - E(X)]^2).$$

$$V(X) = E([X - E(X)]^2) = E(X^2) - [E(X)]^2. \quad (2.1)$$

$$V(X) = E(X^2) - [E(X)]^2 = \sum_{i=1}^n x_i^2 \cdot p_i - (E(X))^2. \quad (2.2)$$

Let proof (2.1).

$$\begin{aligned} V(X) &= E([X - E(X)]^2) = E(X^2 - 2X \cdot E(X) + [E(X)]^2) = \\ &= E(X^2) - E(2X \cdot E(X)) + [E(X)]^2 = \\ &= E(X^2) - 2E(X)E(X) + [E(X)]^2 = E(X^2) - [E(X)]^2. \end{aligned}$$

Notice that the variance can be thought as the average squared distance between the values of X and the expected value $E(X)$.

Properties of the variance.

$$1) V(C) = 0.$$

Proof. $V(C) = E([C - E(C)]^2) = E([C - C]^2) = E(0) = 0.$

$$2) V(CX) = C^2 \cdot V(X).$$

Proof. $V(CX) = E([CX - E(CX)]^2) = E(C^2[X - E(X)]^2) =$
 $= C^2 E([X - E(X)]^2) = C^2 V(X).$

3) $V(X \pm Y) = V(X) + V(Y)$, where X and Y are independent random variables.

Proof. $V(X + Y) = E([X + Y]^2) - (E[X + Y])^2 = E(X^2 + 2XY + Y^2) -$
 $-(E(X) + E(Y))^2 = E(X^2) + 2E(X)E(Y) + E(Y^2) - E^2(X) - 2E(X)E(Y) -$
 $- E^2(Y) = \{E(X^2) - E^2(X)\} + \{E(Y^2) - E^2(Y)\} = V(X) + V(Y).$

$$V(X - Y) = V(X) + V(-Y) = V(X) + (-1)^2 V(Y) = V(X) + V(Y).$$

The standard deviation of D.v. X is the square root of the variance and is given by

$$\sigma = \sqrt{V(X)}.$$

The standard deviation is a number which describes the spread of the distribution. Small standard deviation means small spread, large standard deviation means large spread.

The most probable value of D.v. X is called the **mode** M_0 .

The **median** M_e of D.v. X is the value of X for which
 $P(X < M_e) = P(X \geq M_e) = 0.5.$

Example 2.2. When tossing three fair coins, let the random variable X denote the number of tails in the outcome. Find the expected value, variance and standard deviation of X .

Solution.

The probability distribution of X has the form (see Ex. 2.1):

X	0	1	2	3
p	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$1) E(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5.$$

$$2) E(X^2) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3;$$

$$V(X) = E(X^2) - [E(X)]^2 = 3 - 2.25 = 0.75.$$

$$3) \sigma = \sqrt{V(X)} = \sqrt{0.75} \approx 0.87.$$

Discrete probability distributions

1. Binomial distribution. A random variable X possesses a **binomial distribution** if the following conditions are satisfied:

- a) the experiment consists of n repeated trials;
- b) each trial results in an outcome that may be classified as a success or a failure (hence the name «binomial»);
- c) the probability of a success, denoted by p , remains constant from trial to trial;
- d) repeated trials are independent.

The number of successes X in n trials of a binomial experiment is called a **binomial random variable**. Binomial random variable can get the values $0, 1, 2, \dots, n$. The probability distribution of the binomial random variable X is given by the formula:

$$P(X = x) = C_n^x \cdot p^x \cdot q^{n-x},$$

where n is the number of trials; $x = 0, 1, 2, \dots, n$; p is the probability of success in a single trial; $q = 1 - p$ is the probability of failure in a single trial.

In a binomial distribution, only 2 parameters, namely n and p , are needed to determine the probability.

The expected value and the variance are given by the formulas

$$E(X) = n \cdot p, \quad V(X) = n \cdot p \cdot q. \quad (2.3)$$

Example 2.3. A man fires at a target five times; the probability of his hitting each time is independent of other tries and is $\frac{2}{5}$. Random variable

X is the number of hitting the target.

a) Write the probability distribution for this random variable and find its expected value, variance and standard deviation.

b) What is the probability that he will hit the target one time?

c) What is the probability that he will hit the target at least once?

d) What is the probability that he will hit the target at most two times?

Solution. A random variable X is the number of hitting the target. X takes 6 values $x_1 = 0, x_2 = 1, \dots, x_6 = 5$. It has binomial distribution with parameters: $n = 5, p = \frac{2}{5}$. Find $q : q = 1 - p = \frac{3}{5}$.

$$P(X = 0) = C_5^0 \cdot \left(\frac{2}{5}\right)^0 \cdot \left(\frac{3}{5}\right)^5 = \frac{243}{3125}; \quad P(X = 1) = C_5^1 \cdot \left(\frac{2}{5}\right)^1 \cdot \left(\frac{3}{5}\right)^4 = \frac{162}{625};$$

$$P(X = 2) = C_5^2 \cdot \left(\frac{2}{5}\right)^2 \cdot \left(\frac{3}{5}\right)^3 = \frac{216}{625}; \quad P(X = 3) = C_5^3 \cdot \left(\frac{2}{5}\right)^3 \cdot \left(\frac{3}{5}\right)^2 = \frac{144}{625};$$

$$P(X = 4) = C_5^4 \cdot \left(\frac{2}{5}\right)^4 \cdot \left(\frac{3}{5}\right)^1 = \frac{48}{625}; \quad P(X = 5) = C_5^5 \cdot \left(\frac{2}{5}\right)^5 \cdot \left(\frac{3}{5}\right)^0 = \frac{32}{3125}.$$

The probability distribution is:

X	0	1	2	3	4	5
P	$\frac{243}{3125}$	$\frac{162}{625}$	$\frac{216}{625}$	$\frac{144}{625}$	$\frac{48}{625}$	$\frac{32}{3125}$

Let's find the expected value, the variance and the standard deviation of X :

$$E(X) = 0 \cdot \frac{243}{3125} + 1 \cdot \frac{162}{625} + 2 \cdot \frac{216}{625} + 3 \cdot \frac{144}{625} + 4 \cdot \frac{48}{625} + 5 \cdot \frac{32}{3125} = 2,$$

$$\begin{aligned}
 V(X) &= 0^2 \cdot \frac{243}{3125} + 1^2 \cdot \frac{162}{625} + 2^2 \cdot \frac{216}{625} + 3^2 \cdot \frac{144}{625} + \\
 &\quad + 4^2 \cdot \frac{48}{625} + 5^2 \cdot \frac{32}{3125} - 2^2 = 1.2, \\
 \sigma(X) &= \sqrt{1.2} \approx 1.0954.
 \end{aligned}$$

The expected value and the variance can be found using formulas (2.3):

$$E(X) = n \cdot p = 5 \cdot \frac{2}{5} = 2, \quad V(X) = n \cdot p \cdot q = 5 \cdot \frac{2}{5} \cdot \frac{3}{5} = 1.2.$$

$$b) P(X = 1) = \frac{162}{625}.$$

$$c) P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{243}{3125} = \frac{2882}{3125} \approx 0.9222.$$

$$\begin{aligned}
 d) P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) = \frac{243}{3125} + \frac{162}{625} + \\
 &+ \frac{216}{625} = \frac{2133}{3125} \approx 0.6826. \bullet
 \end{aligned}$$

2. Poisson probability distribution. Let X be a binomial random variable with parameters n and p and $p \rightarrow 0$, $n \rightarrow \infty$, $\lambda = np$, then the probability distribution of the random variable X is called the **Poisson probability distribution**, and is given by the formula

$$P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}.$$

The Poisson probability distribution is characterized by the single parameter λ , which represents the mean of a Poisson probability distribution.

The expected value and the variance are given by the formulas:

$$E(X) = \lambda, \quad V(X) = \lambda.$$

Rare events are modeled by the Poisson distribution. For example, the Poisson probability distribution has been used in the study of telephone systems, the number of misprints in a book, radioactivity counts per unit time, the number of plankton (microscopic plant or animal organisms that

float in bodies of water) per aliquot of seawater, or count of bacterial colonies per petri plate in a microbiological study.

3. Geometric distribution. Let the random variable X be the number of independent trials up to and including the first successful outcome with a probability of success p and a probability of failure q . This random variable X has a geometric distribution. This probability distribution is represented by the probability function

$$P(X = x) = q^{x-1} \cdot p.$$

The expected value and the variance are given by the formulas:

$$E(X) = 1/p, \quad V(X) = q/p^2.$$

Example 2.4. A shooter shoots at the target until the first hit. Probability of hitting the target is $p = 0.57$. What is the probability that he will hit the target the first time with the fifth shot?

Solution. Let X be the number of shots until the first hit of the target including it. We have

$$p = 0.57, \quad q = 1 - p = 0.43, \quad P(X = 5) = 0.43^{5-1} \cdot 0.57 = 0.0195. \bullet$$

4. Hypergeometric distribution. Suppose that an urn contains N balls M of which are white ($M < N$). Suppose n balls are taken from the urn at random and without replacement, and let the random variable X be the number of white balls obtained. Then X has the hypergeometric distribution. This probability distribution is represented by the probability function

$$P(X = x) = \frac{C_M^x \cdot C_{N-M}^{n-x}}{C_N^n}.$$

Example 2.5. There are 7 laptops in the store, 2 of which are defective. A firm makes a random purchase of 3 of these laptops. Find the probability distribution for the number of defective laptops purchased by the firm and its expected value, variance and standard deviation.

Solution. Let X be a random variable whose values x are the numbers of the defective laptops purchased by the firm. Then x can take on the values 0, 1, and 2. We have:

$$P(X=0)=\frac{C_2^0 \cdot C_5^3}{C_7^3}=\frac{2}{7}; \quad P(X=1)=\frac{C_2^1 \cdot C_5^2}{C_7^3}=\frac{4}{7}; \quad P(X=2)=\frac{C_2^2 \cdot C_5^1}{C_7^3}=\frac{1}{7}.$$

The probability distribution of X is:

X	0	1	2
p	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

Let's find the expected value, the variance and the standard deviation of X :

$$E(X)=0 \cdot \frac{2}{7} + 1 \cdot \frac{4}{7} + 2 \cdot \frac{1}{7} = \frac{6}{7};$$

$$V(X)=0^2 \cdot \frac{2}{7} + 1^2 \cdot \frac{4}{7} + 2^2 \cdot \frac{1}{7} - \left(\frac{6}{7}\right)^2 = \frac{20}{49};$$

$$\sigma(X)=\sqrt{\frac{20}{49}} \approx 0.6389.$$

2.2 Continuous Random Variables and Their Probability Distributions

A random variable is called **continuous random variable** (C.v.) if it can take on any value from a certain interval (finite or infinite). The distribution function of C.v. $F(x)=P(X < x)$ is continuous function.

The main characteristic of C.v. is a **probability density function** $f(x)$. **The probability density function of C.v.** $f(x)$ is called a derivative of its distribution function $f(x)=F'(x)$.

Properties of the probability density distribution of C.v.

1) $f(x) \geq 0, \forall x \in R$.

Proof. The distribution function $F(x)$ is non-decreasing function therefore $F'(x)=f(x) \geq 0$.

$$2) P(a < X < b) = \int_a^b f(x) dx.$$

$$\mathbf{Proof.} \quad P(a < X < b) = F(b) - F(a) = \int_a^b F'(x) dx = \int_a^b f(x) dx.$$

$$3) F(x) = \int_{-\infty}^x f(x)dx.$$

$$4) \int_{-\infty}^{\infty} f(x)dx = 1.$$

$$\text{Notice that for C.v. } X \quad P(X = a) = \int_a^a f(x)dx = 0.$$

For C.v. X and constants a and b ($a < b$),

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b).$$

Numerical characteristics of C.v.

Let X be C.v. with probability density function $f(x)$.

The expected value of C.v. X is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

The variance of C.v. X is given by

$$V(X) = \int_{-\infty}^{\infty} [X - E(X)]^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - [E(X)]^2.$$

The standard deviation of a C.v. X is given by

$$\sigma(X) = \sqrt{D(X)}.$$

The mode M_0 of C.v. X is the value of X for which the probability density function has a local maximum.

The median M_e of C.v. X is the value of X for which $P(X < M_e) = P(X \geq M_e) = 0.5$.

Example 2.6. The probability density function of a C.v. X is given by

$$f(x) = \begin{cases} 0, & x \leq 4, \\ a(4-x)^2, & 4 < x \leq 5, \\ 0, & x > 5. \end{cases}$$

a) Find the value of a ;

- b)** compute $P(4.2 < X < 4.3)$;
c) find the distribution function $F(x)$;
d) find $E(X)$, $V(X)$, $\sigma(X)$;
e) sketch $f(x)$ and $F(x)$.

Solution.

a) Since $\int_{-\infty}^{\infty} f(x)dx = 1$, we have

$$\int_4^5 a(4-x)^2 dx = 1 \Rightarrow a \int_4^5 (4-x)^2 dx = 1,$$

$$\int_4^5 (4-x)^2 dx = -\frac{(4-x)^3}{3} \Big|_4^5 = \frac{1}{3},$$

$$\frac{1}{3}a = 1, \quad a = 3,$$

$$f(x) = \begin{cases} 0, & x \leq 4, \\ 3(4-x)^2, & 4 < x \leq 5, \\ 0, & x > 5. \end{cases}$$

b) $P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f(x)dx,$

$$\begin{aligned} P(4.2 < X < 4.3) &= \int_{4.2}^{4.3} 3(4-x)^2 dx = -(4-x)^3 \Big|_{4.2}^{4.3} = \\ &= -((4-4.3)^3 - (4-4.2)^3) = 0.019. \end{aligned}$$

c) If $x \leq 4$, then $F(x) = \int_{-\infty}^x f(x)dx = \int_{-\infty}^x 0dx = 0.$

If $4 < x \leq 5$, then

$$F(x) = \int_{-\infty}^x f(x)dx = \int_{-\infty}^4 0dx + \int_4^x 3(4-x)^2 dx = -(4-x)^3 \Big|_4^x = (x-4)^3.$$

If $x > 5$, then

$$F(x) = \int_{-\infty}^x f(x)dx = \int_{-\infty}^4 0dx + \int_4^5 3(4-x)^2 dx + \int_5^x 0dx = -(4-x)^3 \Big|_4^5 = 1.$$

The distribution function has the form:

$$F(x) = \begin{cases} 0, & x \leq 4, \\ (x-4)^3, & 4 < x \leq 5, \\ 1, & x > 5. \end{cases}$$

$$\begin{aligned} \text{d) } E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_4^5 3x(4-x)^2 dx = \int_4^5 (48x - 24x^2 + 3x^3)dx = \\ &= \left(24x^2 - 8x^3 + \frac{3x^4}{4} \right) \Big|_4^5 = 24 \cdot 5^2 - 8 \cdot 5^3 + \frac{3 \cdot 5^4}{4} - \left(24 \cdot 4^2 - 8 \cdot 4^3 + \frac{3 \cdot 4^4}{4} \right) = \\ &= 68.75 - 64 = 4.75; \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_4^5 3x^2(4-x)^2 dx = \int_4^5 (48x^2 - 24x^3 + 3x^4)dx = \\ &= \left(16x^3 - 6x^4 + \frac{3x^5}{5} \right) \Big|_4^5 = 16 \cdot 5^3 - 6 \cdot 5^4 + \frac{3 \cdot 5^5}{5} - \left(16 \cdot 4^3 - 6 \cdot 4^4 + \frac{3 \cdot 4^5}{5} \right) = \\ &= 125 - 102.4 = 22.6; \end{aligned}$$

$$V(X) = E(X^2) - [E(X)]^2 = 22.6 - 4.75^2 = 0.0375;$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{0.0375} \approx 0.1936.$$

e) Fig. 2.2 shows $f(x)$ (Fig. 2.2a) and $F(x)$ (Fig. 2.2b).

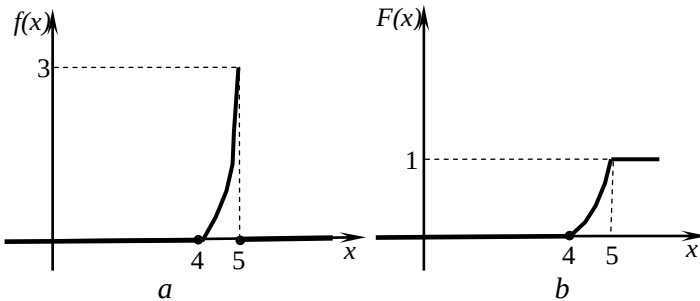


Figure 2.2

Probability distributions of C. v.

1. Uniform probability distribution. A random variable X is said to have a uniform probability distribution on $(a; b)$, if its probability density function is constant over an interval $[a, b]$.

The probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a; b], \\ 0, & x \notin [a; b]. \end{cases}$$

Distribution function $F(x)$ has the form:

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

Fig. 2.3 shows $f(x)$ (Fig. 2.3a) and $F(x)$ (Fig. 2.3b).

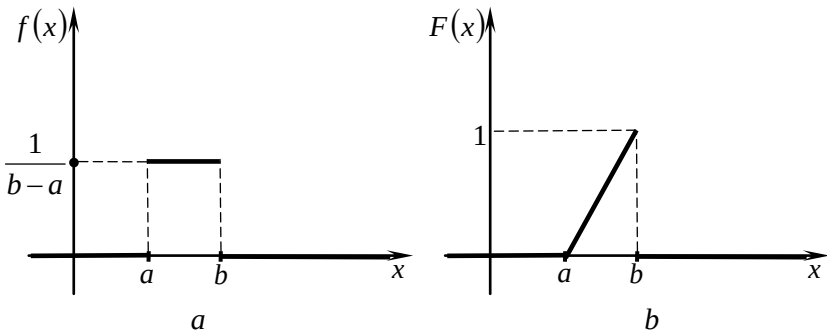


Figure 2.3

Numerical characteristics of X :

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}, \quad \sigma(X) = \frac{b-a}{2\sqrt{3}}.$$

For a r.v. the probability X to get the value from the interval $(\alpha; \beta)$ ($(\alpha; \beta) \subset [a; b]$) is calculated by the formula $P(\alpha < X < \beta) = \frac{\beta - \alpha}{b - a}$.

Example 2.6. The tram arrives at the University stop every 20 minutes. Olena arrives at a random point in time to take the tram. The distribution of waiting time in minutes is uniform on $(0, 20)$. The mean is $E(X) = \frac{0+20}{2} = 10$ minutes, the variance is $V(X) = (20-0)^2 / 12 \approx 33.333$ and the standard deviation is $\sigma(X) = (20-0) / (2\sqrt{3}) \approx 5.774$ minutes. The probability that the waiting time is more than 15 minutes is $P(X > 15) = (20-15) / (20-0) = \frac{5}{20} = 0.25$. •

2. Exponential distribution. The C.v. X has an exponential distribution with parameter λ if its density function is given by

$$f(x) = \begin{cases} 0, & x < 0, \\ \lambda e^{-\lambda x}, & x \geq 0, \end{cases}$$

where $\lambda > 0$.

The distribution function is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Fig. 2.4 shows $f(x)$ (Fig. 2.4a) and $F(x)$ (Fig. 2.4b).

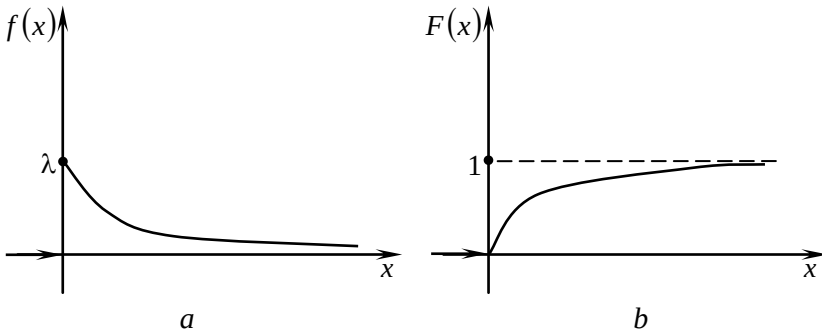


Figure 2.4

Numerical characteristics of X : $E(X)=1/\lambda$, $V(X)=1/\lambda^2$, $\sigma(X)=1/\lambda$.

The probability that the r.v. X will get value from interval $(a;b)$ is calculated by the formula: $P(a < X < b)=e^{-\lambda a}-e^{-\lambda b}$.

Exponential random variables are often used to model the lifetimes of electronic components such as fuses, for survival analysis, and for reliability analysis, among others. The exponential distribution is also used in developing models of insurance risks.

Example 2.7. The time, in hours, during which an electrical generator is operational is a random variable that follows the exponential distribution with mean 100. What is the probability that a generator of this type will be operational

a) less than 20 hours? **b)** between 30 and 100 hours? **c)** more than 120 hours?

Solution. Let X denote the random variable corresponding to the time (in hours) during which the generator is operational.

$E(X)=\frac{1}{\lambda}=100 \Rightarrow \lambda=\frac{1}{100}$. The density function of X is given by

$$f(x)=\begin{cases} \frac{1}{100}e^{-\frac{x}{100}}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

We will use the formula: $P(a < X < b)=e^{-\lambda a}-e^{-\lambda b}$.

a) $P(X < 20)=P(0 < X < 20)=e^{-\frac{0}{100}}-e^{-\frac{20}{100}} \approx 0.1813$. There is about a 18.13% chance that a generator of this type will be operational less than 20 hours.

b) $P(30 < X < 100)=e^{-\frac{30}{100}}-e^{-\frac{100}{100}} \approx 0.3729$. Hence, there is about a 37.29% chance that a generator of this type will be operational between 30 and 100 hours.

$$c) \quad P(X > 120) = 1 - P(X < 120) = 1 - \left(e^{-\frac{0}{100}} - e^{-\frac{120}{100}} \right) \approx 0.3012.$$

The chance that the generator will be operational more than 120 hours is about 30.12%.

3. Normal distribution (Gaussian distribution). A random variable X is said to have a normal probability distribution with parameters a and σ^2 , if it has a probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}},$$

where $\sigma > 0$, $x \in (-\infty; +\infty)$.

If $\mu = 0$ and $\sigma = 1$, we call it **standard normal random variable**.

It can be proved that a is the mean (μ) and σ is the standard deviation of normal distribution.

Let prove that a is equal to the mean μ :

$$\begin{aligned} \mu = E(x) &= \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \left| \begin{array}{l} z = \frac{x-a}{\sigma} \\ x = \sigma z + a \\ dx = \sigma dz \end{array} \right| \\ &= \frac{\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + a) e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma z e^{-\frac{z^2}{2}} dz + \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \\ &= 0 + \frac{a}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = a. \end{aligned}$$

(The second term is the Poisson integral $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$).

For any normal random variable with mean μ and variance σ^2 , the notation $X \sim N(\mu, \sigma^2)$ is used.

When a random variable X has a standard normal probability distribution, the notation $X \sim N(0,1)$ is used.

Probabilities for a standard normal probability distribution are given in Table A.2. The graph of $f(x)$, called the normal curve, is the bell-shaped curve (Fig. 2.5).

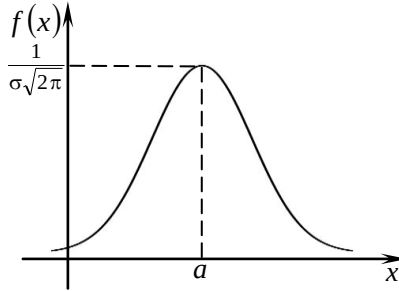


Figure 2.5

It has the following properties:

- 1) The curve is symmetric about a vertical line $x = a$.
- 2) The curve has maximum at $x = a$, $f(a) = \frac{1}{\sigma\sqrt{2\pi}}$.
- 3) If $x \rightarrow \pm\infty$ then the normal curve approaches to the x -axis asymptotically.
- 4) The curve has its points of inflection at $x = a \pm \sigma$. It is concave downward if $x \in (a - \sigma; a + \sigma)$, and is concave upward if $x \in (-\infty; a - \sigma) \cup (a + \sigma; +\infty)$.

Distribution function $F(x)$ has the form:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-a)^2}{2\sigma^2}} dt.$$

Numerical characteristics of normal distribution: $\mu = E(X) = a$, $V(X) = \sigma^2$, $\sigma(X) = \sigma$.

The probability that the random variable X assumes a value between $x = \alpha$ and $x = \beta$ can be calculated by the formula

$$P(\alpha \leq X \leq \beta) = \Phi(x_2) - \Phi(x_1), \quad (2.4)$$

where $x_1 = \frac{\alpha - a}{\sigma}$, $x_2 = \frac{\beta - a}{\sigma}$, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$. The function $\Phi(x)$ can be found using special tables.

Theorem 2.1. Let X be a normal random variable. The probability of a falling X into an interval symmetric with respect to the mean is determined by the formula:

$$P(|X - a| < \delta) = 2 \cdot \Phi(\delta / \sigma).$$

Proof. $|X - a| < \delta \Rightarrow a - \delta < X < a + \delta$. By the formula (2.4):

$$\begin{aligned} P(|X - a| < \delta) &= P(a - \delta < X < a + \delta) = \Phi\left(\frac{(a + \delta) - a}{\sigma}\right) - \Phi\left(\frac{(a - \delta) - a}{\sigma}\right) = \\ &= \Phi\left(\frac{\delta}{\sigma}\right) - \Phi\left(-\frac{\delta}{\sigma}\right) = \left\{ \Phi\left(-\frac{\delta}{\sigma}\right) = -\Phi\left(\frac{\delta}{\sigma}\right) \right\} = 2 \cdot \Phi(\delta / \sigma). \end{aligned}$$

Theorem 2.2. If $\delta = 3\sigma$ then $P(|X - a| < 3\sigma) = 2 \cdot \Phi(3) = 0.9973$.

Proof. Setting $\delta = \sigma t$ in $P(|X - a| < \delta) = 2 \cdot \Phi(\delta / \sigma)$, we get

$$P(|X - a| < \sigma t) = 2 \cdot \Phi(t).$$

If $t = 3$, then $\sigma t = 3\sigma$, then

$$P(|X - a| < 3\sigma) = 2 \cdot \Phi(3) = 2 \cdot 0.49865 = 0.9973.$$

Example 2.8. An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 774 and 830 hours.

Solution.

$$a = 800, \sigma = 40, x_1 = \frac{\alpha - a}{\sigma} = \frac{774 - 800}{40} = -0.65,$$

$$x_2 = \frac{\beta - a}{\sigma} = \frac{830 - 800}{40} = 0.75,$$

$$\begin{aligned} P(774 \leq X \leq 830) &= \Phi(0.75) - \Phi(-0.65) = \\ &= \Phi(0.75) + \Phi(0.65) = 0.2734 + 0.2422 = 0.5156. \bullet \end{aligned}$$

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Appendix A

Table A.1 – Values of the Function $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

x	0	1	2	3	4	5	6	7	8	9
0.0	0.3989	3989	3989	3988	3986	3984	3982	3980	3977	3973
0.1	3970	3965	3961	3956	3951	3945	3939	3932	3925	3918
0.2	3910	3902	3894	3885	3876	3867	3857	3847	3836	3825
0.3	3814	3802	3790	3778	3765	3752	3739	3726	3712	3697
0.4	3683	3668	3652	3637	3621	3605	3589	3572	3555	3538
0.5	3521	3503	3485	3467	3448	3429	3410	3391	3372	3352
0.6	3332	3312	3292	3271	3251	3230	3209	3187	3166	3144
0.7	3123	3101	3079	3056	3034	3011	2989	2966	2943	2920
0.8	2897	2874	2850	2827	2803	2780	2756	2732	2709	2685
0.9	2661	2637	2613	2589	2565	2541	2516	2492	2468	2444
1.0	0.2420	2396	2371	2347	2323	2299	2275	2251	2227	2203
1.1	2179	2155	2131	2107	2083	2059	2036	2012	1989	1965
1.2	1942	1919	1895	1872	1849	1826	1804	1781	1758	1736
1.3	1714	1691	1669	1647	1626	1604	1582	1561	1539	1518
1.4	1497	1476	1456	1435	1415	1394	1374	1354	1334	1315
1.5	1295	1276	1257	1238	1219	1200	1182	1163	1145	1127
1.6	1109	1092	1074	1057	1040	1023	1006	0989	0973	0957
1.7	0940	0925	0909	0893	0878	0863	0848	0833	0818	0804
1.8	0790	0775	0761	0748	0734	0721	0707	0694	0681	0669
1.9	0656	0644	0632	0620	0608	0596	0584	0573	0562	0551
2.0	0,0540	0529	0519	0508	0498	0488	0478	0468	0459	0449
2.1	0440	0431	0422	0413	0404	0396	0387	0379	0371	0363
2.2	0355	0347	0339	0332	0325	0317	0310	0303	0297	0290
2.3	0283	0277	0270	0264	0258	0252	0246	0241	0235	0229
2.4	0224	0219	0213	0208	0203	0198	0194	0189	0184	0180
2.5	0175	0171	0167	0163	0158	0154	0151	0147	0143	0139
2.6	0136	0132	0129	0126	0122	0119	0116	0113	0110	0107
2.7	0104	0101	0099	0096	0093	0091	0088	0086	0084	0081

Table A.1 (continued).

x	0	1	2	3	4	5	6	7	8	9
2.8	0079	0077	0075	0073	0071	0069	0067	0065	0063	0061
2.9	0060	0058	0056	0055	0053	0051	0050	0048	0047	0046
3.0	0.0044	0043	0042	0040	0039	0038	0037	0036	0035	0034
3.1	0033	0032	0031	0030	0029	0028	0027	0026	0025	0025
3.2	0024	0023	0022	0022	0021	0020	0020	0019	0018	0018
3.3	0017	0017	0016	0016	0015	0015	0014	0014	0013	0013
3.4	0012	0012	0012	0011	0011	0010	0010	0010	0009	0009
3.5	0009	0008	0008	0008	0008	0007	0007	0007	0007	0006
3.6	0006	0006	0006	0005	0005	0005	0005	0005	0005	0004
3.7	0004	0004	0004	0004	0004	0004	0003	0003	0003	0003
3.8	0003	0003	0003	0003	0003	0002	0002	0002	0002	0002
3.9	0002	0002	0002	0002	0002	0002	0002	0002	0001	0001

Table A.2 – Values of the Function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz$.

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.0000	0.14	0.0557	0.28	0.1103	0.42	0.1628
0.01	0.0040	0.15	0.0596	0.29	0.1141	0.43	0.1664
0.02	0.0080	0.16	0.0636	0.30	0.1179	0.44	0.1700
0.03	0.0120	0.17	0.0675	0.31	0.1217	0.45	0.1736
0.04	0.0160	0.18	0.0714	0.32	0.1255	0.46	0.1772
0.05	0.0199	0.19	0.0753	0.33	0.1293	0.47	0.1808
0.06	0.0239	0.20	0.0793	0.34	0.1331	0.48	0.1844
0.07	0.0279	0.21	0.0832	0.35	0.1368	0.49	0.1879
0.08	0.0319	0.22	0.0871	0.36	0.1406	0.50	0.1915
0.09	0.0359	0.23	0.0910	0.37	0.1443	0.51	0.1950
0.10	0.0398	0.24	0.0948	0.38	0.1480	0.52	0.1985
0.11	0.0438	0.25	0.0987	0.39	0.1517	0.53	0.2019
0.12	0.0478	0.26	0.1026	0.40	0.1554	0.54	0.2054
0.13	0.0517	0.27	0.1064	0.41	0.1591	0.55	0.2088

Table A.2 (continued).

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.56	0.2123	0.86	0.3051	1.16	0.3770	1.46	0.4279
0.57	0.2157	0.87	0.3078	1.17	0.3790	1.47	0.4292
0.58	0.2190	0.88	0.3106	1.18	0.3810	1.48	0.4306
0.59	0.2224	0.89	0.3133	1.19	0.3830	1.49	0.4319
0.60	0.2257	0.90	0.3159	1.20	0.3849	1.50	0.4332
0.61	0.2291	0.91	0.3186	1.21	0.3869	1.51	0.4345
0.62	0.2324	0.92	0.3212	1.22	0.3883	1.52	0.4357
0.63	0.2357	0.93	0.3238	1.23	0.3907	1.53	0.4370
0.64	0.2389	0.94	0.3264	1.24	0.3925	1.54	0.4382
0.65	0.2422	0.95	0.3289	1.25	0.3944	1.55	0.4394
0.66	0.2454	0.96	0.3315	1.26	0.3962	1.56	0.4406
0.67	0.2486	0.97	0.3340	1.27	0.3980	1.57	0.4418
0.68	0.2517	0.98	0.3365	1.28	0.3997	1.58	0.4429
0.69	0.2549	0.99	0.3389	1.29	0.4015	1.59	0.4441
0.70	0.2580	1.00	0.3413	1.30	0.4032	1.60	0.4452
0.71	0.2611	1.01	0.3438	1.31	0.4049	1.61	0.4463
0.72	0.2642	1.02	0.3461	1.32	0.4066	1.62	0.4474
0.73	0.2673	1.03	0.3485	1.33	0.4082	1.63	0.4484
0.74	0.2703	1.04	0.3508	1.34	0.4099	1.64	0.4495
0.75	0.2734	1.05	0.3531	1.35	0.4115	1.65	0.4505
0.76	0.2764	1.06	0.3554	1.36	0.4131	1.66	0.4515
0.77	0.2794	1.07	0.3577	1.37	0.4147	1.67	0.4525
0.78	0.2823	1.08	0.3599	1.38	0.4162	1.68	0.4535
0.79	0.2852	1.09	0.3621	1.39	0.4177	1.69	0.4545
0.80	0.2881	1.10	0.3643	1.40	0.4192	1.70	0.4554
0.81	0.2910	1.11	0.3665	1.41	0.4207	1.71	0.4564
0.82	0.2939	1.12	0.3686	1.42	0.4222	1.72	0.4573
0.83	0.2967	1.13	0.3708	1.43	0.4236	1.73	0.4582
0.84	0.2995	1.14	0.3729	1.44	0.4251	1.74	0.4591
0.85	0.3023	1.15	0.3749	1.45	0.4265	1.75	0.4599

Table A.2 (continued).

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
1.76	0.4608	1.97	0.4756	2.36	0.4909	2.78	0.4973
1.77	0.4616	1.98	0.4761	2.38	0.4913	2.80	0.4974
1.78	0.4625	1.99	0.4767	2.40	0.4918	2.82	0.4976
1.79	0.4633	2.00	0.4772	2.42	0.4922	2.84	0.4977
1.80	0.4641	2.02	0.4783	2.44	0.4927	2.86	0.4979
1.81	0.4649	2.04	0.4793	2.46	0.4931	2.88	0.4980
1.82	0.4656	2.06	0.4803	2.48	0.4934	2.90	0.4981
1.83	0.4664	2.08	0.4812	2.50	0.4938	2.92	0.4982
1.84	0.4671	2.10	0.4821	2.52	0.4941	2.94	0.4984
1.85	0.4678	2.12	0.4830	2.54	0.4945	2.96	0.4985
1.86	0.4686	2.14	0.4838	2.56	0.4948	2.98	0.4986
1.87	0.4693	2.16	0.4846	2.58	0.4951	3.00	0.49865
1.88	0.499	2.18	0.4854	2.60	0.4953	3.20	0.49931
1.89	0.4706	2.20	0.4861	2.62	0.4956	3.40	0.49966
1.90	0.4713	2.22	0.4868	2.64	0.4959	3.60	0.499841
1.91	0.4719	2.24	0.4875	2.66	0.4961	3.80	0.499928
1.92	0.4726	2.26	0.4881	2.68	0.4963	4.00	0.499968
1.93	0.4732	2.28	0.4887	2.70	0.4965	4.50	0.499997
1.94	0.4738	2.30	0.4893	2.72	0.4967	5.00	0.499997
1.95	0.4744	2.32	0.4898	2.74	0.4969		
1.96	0.4750	2.34	0.4904	2.76	0.4971		