

ON ONE BORDER PROBLEM OF RING DOMAIN DEFORMATION

В даній роботі методом комплексних потенціалів Мусхелішвілі побудовано розв'язок граничної задачі теорії пружності для області в формі кільця з кусково-сталими граничними умовами на контурі. Розв'язок одержано в аналітичному вигляді і зведено до форми, придатної для чисельного моделювання. Встановлено, що в околі контура має місце деформація області, близька до зсуву (на ділянках границі з ненульовою граничною умовою) або до радіального стиску (на ділянках границі з нульовою граничною умовою).

Ключові слова: комплексні потенціали Мусхелішвілі, гранична задача теорії пружності, безрозмірні параметри, деформація зсуву, радіальне стискання.

In this paper, the method of Muskhelishvili's complex potentials is used to solve the boundary value problem of elasticity theory for a domain in the form of a ring with piecewise constant boundary conditions on the contour. The solution is obtained in an analytical form and it is put to a form suitable for numerical simulation. It is established that in the neighborhood of the contour there is deformation of the region close to the shift (on the sections of the boundary with a nonzero boundary condition) or to radial compression (on the parts of the boundary with the zero boundary condition).

Keywords: Muskhelishvili's complex potentials, boundary value problem of elasticity theory, dimensionless parameters, shift deformation, radial contraction.

1. Introduction.

A system of partial differential equations for a domain in the form of a ring, known [1] as the first fundamental boundary problem of the elasticity theory, is solved using the method of Muskhelishvili's complex potentials. The ring models the power element of engine, transmitting rotational motion, therefore the boundary conditions are formulated taking into account the tangential load. Such a problem has already been considered [2]. However, the load in these works was thought to be uniformly distributed along the contour. It is physically realistic to assume that the load is uniformly distributed only within the angles corresponding to the ring bindings and that it is equal to zero outside these limits. The aim of the paper is to find and analyze the solution of the corresponding boundary value problem.

2. Basic notation.

We denote the inner and outer radii of the ring as R_1 , R_2 . The position of the current point on the contour will be characterized by the angle θ , where $0 \leq \theta \leq 2\pi$. Suppose that N

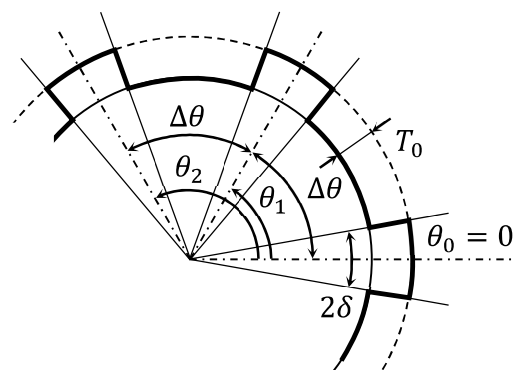


Fig. 1 - Determination of the geometric parameters of the problem.

bindings are placed along the contour with the angular size 2δ each (Fig. 1), where the middle points of bindings are described by the angles $\theta_m = m \Delta\theta$, $m = 0, N-1$, $\Delta\theta = \frac{2\pi}{N}$. To ensure that the bindings do not overlap in the angle, we must put $2\delta \cdot N < 2\pi$, from which $N\delta < \pi$. Let C denote the fraction that the binding angular size forms from the period of the bindings:

$$C = \frac{2\delta}{\Delta\theta} = \frac{N\delta}{\pi}.$$

Then the binding overlapping in the angle does not occur at values $0 < C < 1$.

Fig. 1 also shows the diagram of the tangential mechanical stress equal to $T_0 = T_{01}$ for the inner part of the contour and $T_0 = T_{02}$ for the outer one.

It was shown in [2] that accounting for centrifugal inertia forces is reduced only to correction of the normal component of the external stress, and this does not change the mutual angle of points rotation on the outer and inner parts of the contour. Therefore, in the present paper, the normal component is not taken into account at all. Thus, volumetrically distributed forces are absent, the problem can be considered in a static formulation, and the method of complex potentials can be applied directly. In this case, the distribution of the external stress for angles can be taken in the form

$$T(\theta) = T_0 \cdot t(\theta). \quad (1)$$

Here the function

$$t(\theta) = \sum_{m=0}^N \{ \sigma(\theta - \theta_m^-) - \sigma(\theta - \theta_m^+) \}, \quad (2)$$

where σ is the Heaviside function, and the points of discontinuity $\theta_m^\pm = \theta_m \pm \delta$. The function $t(\theta)$ plays the role of a «comb»: it identically is equal to one within the angular intervals corresponding to the bindings, and is equal to zero outside these intervals.

The sum (2) contains the $(N+1)$ summand. Among them the summands with the numbers $m = 0$ and $m = N$ describe the halves of the same binding: its parts $\theta \in [0; \delta]$ and $\theta \in [2\pi - \delta; 2\pi]$, respectively. Let the thickness of the ring be h . The angle $d\theta$ is supported by a contour arc of length $d\ell = R d\theta$ (the index of the radius is omitted, since these calculations are the same for the outer and inner parts of the contour). Area element $dS = h d\ell = Rh d\theta$. Elementary force $dF = T(\theta) dS = T(\theta) \cdot Rh d\theta$. Elementary moment

$$dM = R dF = T(\theta) \cdot R^2 h d\theta.$$

Taking into account (1), (2) integration of the last expression leads to the result

$$M = T_0 R^2 h \cdot (N \cdot 2\delta) = 2\pi C T_0 R^2 h.$$

Let $M_0 = \frac{M}{h}$ – be the external torque per length unit in the direction of the perpendicular to the ring plane. Then $M_0 = 2\pi C T_0 R^2$. It is this moment that is applied separately to the contour inner part, and separately to the external one (in the static case, these moments must be balanced). Then

$$T_{01} = \frac{M_0}{2\pi C R_1^2}, \quad T_{02} = \frac{M_0}{2\pi C R_2^2}.$$

To specify the form of the function (1), it remains to substitute these values in (1) instead of T_0 .

Let $\gamma = \frac{R_1}{R_2}$ denote the fraction that the inner radius forms from the outer one. Then

$$T_{02} = \gamma^2 T_{01}.$$

It is clear that the fulfillment of this relation ensures that the total external torque vanishes.

3. Boundary conditions. In terms of complex potentials Φ, Ψ the boundary condition [1, § 41, (23)] in the Kolosov-Muskhelishvili form in the absence of the normal component ($N = 0$) has the form ($j = 1, 2$):

$$\{ \Phi + \bar{\Phi} - e^{2i\theta} [\bar{z} \cdot \Phi' + \Psi] \} \big|_{|z|=R_j} = -iT. \quad (3)$$

Here the tangent component of $T = T_{01} \cdot t(\theta)$ at $j = 1$ or $T = T_{02} \cdot t(\theta)$ at $j = 2$. The condition (3) must be fulfilled identically with respect to θ angle.

4. The solution of the boundary value problem. We seek for the solution of the boundary value problem in the form

$$\Phi(z) = \sum_{-\infty}^{+\infty} a_k z^k, \quad \Psi(z) = \sum_{-\infty}^{+\infty} a'_k z^k. \quad (4)$$

The substitution of these expressions in (3) and the subsequent use of the method of indefinite coefficients for the functions $e^{ik\theta}$ allow us to find the coefficients a_k, a'_k , and consequently – the solution of the boundary value problem. In addition, for contour points we must assume

that $z = R_j e^{i\theta}$, $j = 1, 2$. Such approach requires the expansion of the right side of (3) in the complex Fourier series.

The function (2) can be expanded in a Fourier series with respect to the angle θ on the interval $0 \leq \theta \leq 2\pi$. Obviously, the coefficients of the sines are zero, and therefore:

$$t(\theta) = c_0 + \sum_{k=1}^{+\infty} c_k \cos k\theta. \quad (5)$$

The constant component (integral mean) c_0 is equal to the ratio of the area under the graph $t(\theta)$ to the width of the expansion interval:

$$c_0 = \frac{2\delta \cdot N}{2\pi} = C.$$

Coefficients for cosines:

$$c_k = \frac{1}{\pi} \int_0^{2\pi} t(\theta) \cos k\theta d\theta.$$

Restricting ourselves to integration along the intervals where $t(\theta) \neq 0$, we have:

$$\begin{aligned} \pi c_k &= \int_0^{\delta} \cos k\theta d\theta + \int_{2\pi-\delta}^{2\pi} \cos k\theta d\theta + \\ &+ \sum_{m=1}^{N-1} \int_{\theta_m-\delta}^{\theta_m+\delta} \cos k\theta d\theta = \frac{2 \sin k\delta}{k} \sum_{m=0}^{N-1} \cos k\theta_m. \end{aligned}$$

We have $k\theta_m = k \cdot \frac{2\pi m}{N} = mx$, where $x = \frac{2k\pi}{N}$ is temporarily designated. As it is known,

$$\sum_{m=0}^{N-1} \cos mx = \begin{cases} N, & x = 2\pi p; \\ 0, & x \neq 2\pi p. \end{cases}$$

The condition $x = 2\pi p$ is $k = Np$, then

$$c_k = \begin{cases} \frac{2N \sin k\delta}{k\pi}, & k = N, 2N, \dots; \\ 0, & \text{at other } k. \end{cases}$$

Now the Fourier expansion (5) is formed. We transform it into a complex form:

$$t(\theta) = \sum_{k=-\infty}^{+\infty} A_k e^{ik\theta}. \quad (6)$$

We have:

$$\begin{aligned} t(\theta) &= A_0 + (A_1 e^{i\theta} + A_{-1} e^{-i\theta}) + \\ &+ (A_2 e^{2i\theta} + A_{-2} e^{-2i\theta}) + \dots = \\ &= A_0 + (A_1 + A_{-1}) \cos \theta + i(A_1 - A_{-1}) \sin \theta + \\ &+ (A_2 + A_{-2}) \cos 2\theta + i(A_2 - A_{-2}) \sin 2\theta + \dots \end{aligned}$$

Comparing with (5), we obtain: $A_0 = c_0 = C$, and also

$$\begin{cases} A_k + A_{-k} = c_k; \\ i(A_k - A_{-k}) = 0. \end{cases}$$

From here it follows that

$$A_k = A_{-k} = \frac{c_k}{2} = \begin{cases} \frac{N \sin k\delta}{k\pi}, & k = N, 2N, \dots; \\ 0, & \text{at other } k. \end{cases}$$

To establish the correspondence with the notation [1, § 59, (2)], we put:

$$A'_k = -iT_{01}A_k, \quad A''_k = -iT_{02}A_k.$$

Now (3) coincides with [1, § 59, (2)], and we can use the technique [1, § 59].

We substitute (4) in (3)

$$\begin{aligned} &\sum_{k=-\infty}^{k=+\infty} R_j^k [(1-k)a_k e^{ik\theta} + \overline{a_k} e^{-ik\theta}] - \\ &- \sum_{k=-\infty}^{k=+\infty} R_j^k a'_k e^{i(k+2)\theta} = -iT. \end{aligned}$$

Here the right side is

$$-iT = \begin{cases} \sum_{k=-\infty}^{+\infty} A'_k e^{ik\theta}, & j = 1; \\ \sum_{k=-\infty}^{+\infty} A''_k e^{ik\theta}, & j = 2. \end{cases}$$

Let us compare free terms. In addition, in the left side the summands with the number $k = 0$ should be taken from the first sum, and from the second one – the summands with the number $k = -2$. Separately assuming $j = 1$ (points on the inner circle) and $j = 2$ (points on the outer circle), we obtain a pair of equations:

$$\begin{cases} a_0 + \overline{a_0} - \frac{\overline{a'_{-2}}}{R_1^2} = A'_0 = -iT_{01}A_0; \\ a_0 + \overline{a_0} - \frac{\overline{a'_{-2}}}{R_2^2} = A''_0 = -iT_{02}A_0. \end{cases}$$

Eliminating $\overline{a'_{-2}}$ from these equations, we obtain: $a_0 + \overline{a_0} = 0$. Here it is taken into account that

$$-iT_{01}A_0 \cdot R_1^2 = -iT_{02}A_0 \cdot R_2^2 = -i \cdot \frac{M_0}{2\pi}.$$

In solving the first fundamental boundary value problem, the potential $\Phi(z)$ is defined only to a purely imaginary additive constant [1, § 34], therefore without generality restriction we may assume that a_0 is a real number. Then $\overline{a_0} = a_0$, and $a_0 = 0$. Now we find: $a'_{-2} = i \cdot \frac{M_0}{2\pi}$.

Let us compare the factors at $e^{ik\theta}$. This function is contained in $\pm k$ summands in the first sum, and $(k-2)$ summands in the second one. Such a comparison should be carried out twice: for $j = 1$ for points of the inner circle, and for $j = 2$ for points of the outer circle. Therefore, one comparison generates a couple of equations:

$$\begin{pmatrix} R_1^k(1-k) & R_1^{-k} & -R_1^{k-2} \\ R_2^k(1-k) & R_2^{-k} & -R_2^{k-2} \end{pmatrix} \times \begin{pmatrix} a_k \\ \overline{a_{-k}} \\ a'_{k-2} \end{pmatrix} = \begin{pmatrix} A'_k \\ A''_k \end{pmatrix}. \quad (7)$$

Excluding a'_{k-2} from here, we obtain:

$$\begin{aligned} (1-k)(R_2^2 - R_1^2)a_k + (R_2^{2-2k} - R_1^{2-2k})\overline{a_{-k}} = \\ = R_2^{2-k}A''_k - R_1^{2-k}A'_k. \end{aligned} \quad (8)$$

Let us compare factors at $e^{-ik\theta}$. From the resulting pair of equations (for $j = 1, 2$) by eliminating the a'_{-k-2} coefficient, we find:

$$\begin{aligned} (1+k)(R_2^2 - R_1^2)a_{-k} + (R_2^{2+2k} - R_1^{2+2k})\overline{a_k} = \\ = R_2^{2+k}A''_{-k} - R_1^{2+k}A'_{-k}. \end{aligned}$$

Conjugating this equation, we have:

$$\begin{aligned} (1+k)(R_2^2 - R_1^2)\overline{a_{-k}} + (R_2^{2+2k} - R_1^{2+2k})a_k = \\ = R_2^{2+k}\overline{A''_{-k}} - R_1^{2+k}\overline{A'_{-k}}. \end{aligned} \quad (9)$$

Here it is taken into account that $A'_{-k} = A'_k$, $A''_{-k} = A''_k$. Now the simultaneous solution of equations (8), (9) allows us to find a pair of coefficients a_k, a'_{-k} at once. It is clear then that

the solving of the corresponding system of linear algebraic equations is sufficient only for natural k . The matrix of this system has the form:

$$\begin{pmatrix} (1-k)(R_2^2 - R_1^2) & R_2^{2-2k} - R_1^{2-2k} \\ R_2^{2+2k} - R_1^{2+2k} & (1+k)(R_2^2 - R_1^2) \end{pmatrix}.$$

If $k = 1$, then this matrix degenerates, because it takes the form:

$$\begin{pmatrix} 0 & 0 \\ R_2^4 - R_1^4 & 2(R_2^2 - R_1^2) \end{pmatrix}.$$

Therefore, the coefficients $a_1, \overline{a'_{-1}}$ must be found in a special way.

For $k = 1$, the first equation of this system takes the form:

$$0 \cdot a_1 + 0 \cdot \overline{a_{-1}} = R_2A'_1 - R_1A'_1.$$

On mechanical grounds, it is clear that the ring must have more than one binding, $N \geq 2$ (for $N = 1$, the equilibrium conditions for a solid cannot be satisfied). Then the Fourier coefficient $c_1 = 0$, from which $A_{\pm 1} = 0$, $A'_{\pm 1} = A''_{\pm 1} = 0$, and the first equation of the compatibility system does not contradict. The remaining equation can be solved together with the condition of single-valuedness of displacements [1, § 35]; its solution is trivial: $a_1 = \overline{a_{-1}} = 0$. The remaining pairs of coefficients $a_k, \overline{a'_{-k}}$ are solutions of non-degenerate systems composed of the equations (8), (9) for $k \geq 2$: $a_k = \frac{P_k}{Q_k}$. Here is denoted:

$$P_k = (1+k)(R_2^2 - R_1^2)B_k - (R_2^{2-2k} - R_1^{2-2k})\overline{B_{-k}},$$

$$\begin{aligned} Q_k &= (1-k^2)(R_2^2 - R_1^2)^2 - \\ &- (R_2^{2+2k} - R_1^{2+2k})(R_2^{2-2k} - R_1^{2-2k}), \\ B_k &= A''_k R_2^{2-k} - A'_k R_1^{2-k}. \end{aligned}$$

Now the coefficients a'_k can be found from any equation of the system (7). If the numbering is shifted by two units, we obtain:

$$a'_k = \frac{\overline{a_{-k-2}}R_1^{-k-2} - (1+k)a_{k+2}R_1^{k+2} - A'_{k+2}}{R_1^k}.$$

Among other things for $k = -2$ we have:

$$a'_{-2} = \frac{-A'_0}{R_1^{-2}} = \frac{iT_{01}A_0}{R_1^{-2}} = i \cdot \frac{M_0}{2\pi},$$

which has already been obtained above.

In addition, it should be taken into consideration that the coefficients $a_{\pm 1} = 0$ were calculated in a special way. Hence we obtain $a'_{-1} = a'_{-3} = 0$.

From now on, all the coefficients a_k, a'_k are found, the complex potentials Φ, Ψ are defined, and the solution of the boundary problem is constructed.

5. Use of the solution of the boundary value problem for finding physical fields. The series (4) converge, and therefore admit a term-by-term integration: $\varphi = \int \Phi dz$, $\psi = \int \Psi dz$. The analytic functions φ, ψ are also potentials. Now the components u, v displacement fields according to [1, § 32, (1)] are determined from equation

$$2\mu(u + iv) = \kappa\varphi - z\overline{\varphi'} - \psi. \quad (10)$$

Here μ is the material shear modulus; the coefficient $\kappa = 3 - 4\nu$ for the state of plane strain or $\kappa = \frac{3-\nu}{1+\nu}$ for a plane stress state, ν – Poisson's ratio. The transformation of the rotation of the coordinate system according to [1, § 39, (2)] also makes it possible to obtain the radial v_r and angular v_θ components of the displacement field:

$$\begin{cases} v_r = u \cos \theta + v \sin \theta; \\ v_\theta = -u \sin \theta + v \cos \theta. \end{cases}$$

The components $X_x, Y_y, X_y = Y_x$ of the stress field are determined by the relations:

$$X_x + Y_y = 2 \left[\Phi(z) + \overline{\Phi(z)} \right], \quad (11)$$

$$Y_y - X_x + i(X_y + Y_x) = 2 [\bar{z}\Phi'(z) + \Psi(z)]. \quad (12)$$

The transformation of the rotation of the coordinate system to the axes $x'Oy'$ also makes it possible to obtain stresses in the rotated axes:

$$\begin{aligned} X'_{x'} + Y'_{y'} &= X_x + Y_y, \\ Y'_{y'} - X'_{x'} + i(X'_{y'} + Y'_{x'}) &= \\ &= (Y_y - X_x + i(X_y + Y_x)) e^{2i\theta}. \end{aligned}$$

6. Reduction to a form suitable for numerical simulation. The obtained solution (10), (11), (12) of the problem must be

reduced to a dimensionless form, since in numerical simulation this will allow analyzing the basic characteristics of fields without using the values of physical parameters.

The position of the current point of the domain will be characterized by the number

$$z = Re^{i\theta} = \rho R_2 e^{i\theta} = \frac{\rho}{\gamma} R_1 e^{i\theta}.$$

Here $\rho = \frac{R}{R_2}$ denotes the fraction that the actual radius R forms from the largest possible radius R_2 . Then $R_1 < \rho R_2 < R_2$, $\gamma < \rho < 1$.

Thus the value ρ describes the dimensionless modulus of the current value z . In this case, a natural unit $\ell = \frac{R_1}{\gamma} = R_2$ of length measurement arises when determining the position of the current point for a given boundary value problem. In fact, the module for moving away the current point from the center of circles can be defined in ℓ units, since $|z| = \rho\ell$, and the variable ρ is dimensionless.

For termwise integration, for example, of the first of the series (4), the typical term of the expansion takes the form

$$\frac{a_k}{k+1} \cdot z^{k+1} = \frac{a_k}{k+1} \cdot \left(\frac{\rho}{\gamma} \right)^{k+1} R_1^{k+1} e^{i(k+1)\theta}.$$

Obviously, the dimension of this term is concentrated only in the factor $a_k R_1^{k+1}$. Taking into account (10) it should be equal to Pascal multiplied by a meter. In this case, the meter as an integral part of the dimension of the φ, ψ potentials linear combination can not arise due to the use of only the geometric characteristics of the region (in our case, the radii R_1, R_2). In fact, the cause of the displacement field is the external load. Assuming physical linearity, the components of the displacement field must be proportional to this load. Therefore, the physical characteristic of the load, and not just the geometric characteristics of the region, should be the source of the dimension of the potentials linear combination. Taking into consideration all said above it makes sense to introduce the following changes of variables:

$$a_k = \frac{iM_0}{2\pi C R_1} \cdot \frac{1}{R_1^{k+1}} \cdot \alpha_k,$$

$$a'_k = \frac{iM_0}{2\pi CR_1} \cdot \frac{1}{R_1^{k+1}} \cdot \alpha'_k.$$

Now α_k, α'_k are dimensionless parameters that replace the coefficients a_k, a'_k . Using the results obtained above, we find:

$$\alpha_k = \begin{cases} A_k \cdot f_k(\gamma), & k = \pm 2, \pm 3, \dots; \\ 0, & k = 0, \pm 1. \end{cases}$$

Here is denoted

$$f_k(\gamma) = (1 - \gamma^k) \times \frac{(1+k)(\gamma^{2k} - \gamma^{2k+2}) - \gamma^{3k} + \gamma^{k+2}}{(1-k^2)(\gamma^{k-1} - \gamma^{k+1})^2 - (1 - \gamma^{2k+2})(\gamma^{2k-2} - 1)}.$$

We also obtain

$$\alpha'_k = \begin{cases} 0, & k = -3; -1; \\ C, & k = -2. \end{cases}$$

For other values of k we have:

$$\alpha'_k = \alpha_{k+2} + \alpha_{-k-2} + A_{k+2}.$$

When dividing (10) by $\frac{M_0}{2\pi CR_1}$, it turns out to be expedient to change the variables

$$U = u \cdot \frac{4\pi\mu CR_1}{M_0}, \quad V = v \cdot \frac{4\pi\mu CR_1}{M_0}. \quad (13)$$

Consequently, value

$$L = \frac{M_0}{4\pi\mu CR_1};$$

it plays the role of a natural unit of length measurement when calculating the components of the displacement field for a given boundary value problem. In fact, the u, v components of the displacement field can be defined in L units, since $u = LU, v = LV$, and the variables U and V are dimensionless.

Using the substitutions (13), the expression (10) takes the form

$$U + iV = i \sum_{k=-\infty}^{+\infty} \frac{\rho^{k+1}}{\gamma^{k+1}} e^{i\theta} \left(\frac{\alpha_k - \alpha'_k}{k+1} \cdot e^{ik\theta} + \alpha_k e^{-ik\theta} \right).$$

For an arbitrary pair $(\rho; \theta)$, the value of this sum can be obtained approximately by the accumulation method: it is necessary to add pairs of summands with numbers $k =$

$\pm 1, \pm 2, \dots$ to the initial term arising at $k = 0$. It should be remembered that some coefficients α_k, α'_k are calculated in a non-typical way.

Then, the dimensionless components U, V of the displacement of a point with coordinates $(\rho; \theta)$ arise when the real and imaginary parts are separated in the accumulated sum.

Finally, we define the natural unit for measuring the stresses for a given problem. Bearing in mind (11), from the point of view of dimensions it is sufficient to consider the following sum:

$$a_k z^k + \overline{a_k z^k} = -\frac{M_0}{\pi CR_1^2} \cdot \alpha_k \left(\frac{\rho}{\gamma} \right)^k \sin k\theta.$$

Taking into account (12), we also obtain

$$\bar{z} \cdot k a_k z^{k-1} = \frac{M_0}{\pi CR_1^2} \cdot \frac{ik\alpha_k}{2} \left(\frac{\rho}{\gamma} \right)^k e^{i(k-2)\theta}.$$

It is clear then that the natural unit for measuring stresses for a given problem is the factor $\sigma_0 = \frac{M_0}{\pi CR_1^2}$. Consequently, the stresses X_x, Y_y, X_y can be expressed in σ_0 units:

$$X_x = \sigma_0 \tilde{X}_x, \quad Y_y = \sigma_0 \tilde{Y}_y, \quad X_y = \sigma_0 \tilde{X}_y;$$

these changes of variables reduce (11), (12) to the dimensionless form.

Numerical simulation of a similar problem for specific values of physical parameters was carried out in [3]. The results of this section of the present article generalize the result obtained in [3].

7. Conclusions. In this paper, a solution of the boundary value problem of elasticity theory for a domain in the form of a ring with piecewise constant boundary conditions on the contour was constructed using the Muskhelishvili's method of complex potentials. The solution is obtained in an analytical form and is reduced to a form suitable for numerical simulation. Analysis of the solution shows that in the neighborhood of the contour there is deformation of the region close to the shift (on the sections of the boundary with a nonzero boundary condition) or to radial contraction (on the sections of the boundary with the zero boundary condition). The results of this paper

can be applied to describe the state of rotating mechanical structures.

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