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Затверджено
на засіданні кафедри ТЗЕ
Протокол №1 від 15.11.22 р.

Затверджено
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TRANSIENTS

6.1 Transients Defined

Transients are the phenomena which occur between two permanent (or steady-state) conditions, as a rule periodic, differing from each other in, say, peak value, phase, wave-form or frequency of the e.m.f., the parameters or configuration of the circuit.

Hence the changing process of any electrical circuit from one of the stationary conditions to another is called the transient. The stationary conditions is called such a rate when currents and voltages can exist infinitely long without changing of their character with given configuration of the circuit and its parameters.

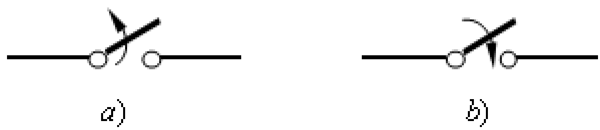


Fig. 6.1

Normally, transient phenomena are brought about in electric circuits by switching operations, that is by closing or opening switches. There may be, however, transients due to other causes, such as faults.

In diagrams the operation performed on a switch is shown by an arrow. Thus, the arrow (Fig. 6.1, *a*) shows that the switch is to be opened, and the one (Fig. 6.1, *b*) that it is to be closed.

Physically, transient phenomena associated with switching operations are changes leading from the energy state existing prior to a switching operation to an energy state which exists after a switching operation. Transients usually last for a few tenths, hundredths or even millionths of a second; it is seldom that they may last for several seconds. Their study is important, as it shows in advance what dangerous rises in voltage or current, many times their steady-state values may happen in the individual sections of a circuit. Transient analysis also throws light on how signals are distorted in wave-form or amplitude as they pass through different circuit elements.

6.2 Step Response of an *RL* Circuit

Any circuit composed of resistance and inductance represents a situation in which there occurs a dissipation of energy as well as storage of energy in a magnetic field. Since we now know how do deal with both voltage and current sources, the step response of the *RL* circuit is found for each source type. Attention is directed first to the step-voltage response.

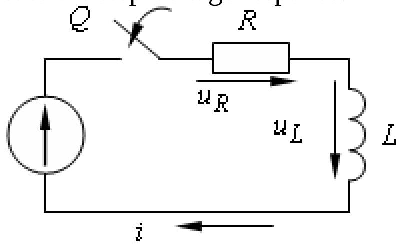


Fig. 6.2

Appearing in Fig. 6.2 is the circuit arrangement of an initially deenergized inductor in series with a resistor. It is desired to fond the complete solution for the current when the switch *Q* is close. The governing differential equation results upon applying Kirchoff's voltage law to the circuit of this figure. Thus

$$E = Ri + L\frac{di}{dt} \tag{6.1}$$

We have got a linear *differential equation* with constant coefficients here. The current i is the solution being sought and E is the applied forcing function which causes the response i to exist.

From mathematics it is known that the general solution to a linear differential equation is the sum of a particular solution of an inhomogeneous equation and the complete solution of a homogeneous equation. The particular solution (or the particular integral) of Eq. (6.1) is

$$i_{ss} = I = \frac{E}{R},$$

where E is the direct current voltage. The homogeneous equation is obtained from the original one by equating its right-hand side to zero:

$$L\frac{di}{dt} + Ri = 0 \tag{6.2}$$

Let's transfer from the differential equation to the algebraic one

$$R + Lp = 0\,. \qquad (6.3)$$

This algebraic expression is termed a *characteristic equation*, as its root determine a form of a free component.

Solving the equation, we will get

$$p = -\frac{R}{L}\,. \qquad (6.4)$$

The solution of the homogeneous equation (or the complementary solution or function) is an exponential function of the form

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t}\,, \qquad (6.5)$$

where A is a constant independent of time (time-invariant) and p is a root of the characteristic equation.

We assume that for all transient time $t = 0$ corresponds to the instant when the switch is thrown from one position to the other. A and p are constants independent of time (time-invariant). Without proof, for our case they are $A = -E/R$ and $p = -R/L$. Therefore, the solution to Eq. (6.1) may be re-written thus

$$i = \frac{E}{R} - \frac{E}{R}\cdot e^{-\frac{R}{L}\cdot t} \qquad (6.6)$$

where $\frac{E}{R}$ is the particular integral of Eq. (6.1) and the next value is the complementary solution to Eq. (6.5).

The particular integral represents the *forced* or *steady-state component* of the response (current or voltage), and the complementary solution is the *force-free,* or *transient component* of the response. The name "free" (or "force-free") is due to the fact that transient component of the response is independent of the excitation or drive.

To tell one component from the other and both from the total quantity, the convention is to supply the symbol of the steady-state component with the subscript **s**, the transient component with the subscript **t**, and the total current with no subscript at all. Thus $i = i_s + i_t$.

Physically, the forced or steady-state component has the same frequency as the excitation or drive. If the drive is a sinusoid of frequency *ω*, the steady-state component will be also a sinusoid of the same frequency *ω* and of the same wave-form. In a circuit containing a d.c. voltage source (see Fig. 6.2), the forced component will be d.c. one.

It should be remembered that capacitance presents an open-circuit to a direct current. Therefore, there will be no forced current in the response of a capacitor. Also, inductance presents a short-circuit to dc circuits, and there is no forced voltage in the response of an inductor.

6.3 Evaluation of initial conditions

Information concerning the initial value of the response, and its derivatives when required, is always derivable from the defining differential equation. Sometimes there is need for appropriate algebraic manipulation of the differential equation before the specific information can be deduced, but the important thing is that this equation can always be made to yield the required information. In the case at hand, for example, knowledge of the value of the response immediately after switching, ;(0+), is obtained by integrating both sides of Eq. (6-1) with respect to time from $0-$ (the time just before switching) to 0^+ (the time just after switching). Thus

$$\int_{0^=}^{0^+} E dt = \int_{0^=}^{0^+} Ri dt + L \int_{0^=}^{0^+} \frac{di}{dt} \qquad (6.7)$$

Now note that the quantity

$$\int_{0^-}^{0^+} E dt = 0$$

because for a constant integrand the net contribution of the integral from 0^- to 0+ is infinitesimally small. A similar reasoning applies for the first term on the right side for a finite value of *i*. Accordingly, Eq. (6.7) may be written as

$$0 = 0 + L \int_{0^-}^{0^+} di \text{ or } 0 = L\left[i(0^+) - i(0^-)\right] \qquad (6.8)$$

It follows then that

$$i(0^+) = i(0^-) \qquad (6.9)$$

Therefore Eq. (6.9) states that the current flowing through the *RL* circuit immediately after the switch is closed is equal to the current existing before switching. This is another way of stating that the current through an inductance cannot change instantaneously.

The forced solution is often called the *steady-state solution* because, after all transient terms have virtually disappeared, it is the only current which remains for the circuit of Fig. 6.2 with the switch closed.

However, note that as time elapses the transient term decays to zero, leaving just the forced solution. Of course, as *i* decays, time is allowed for transferring energy to the inductance. When steady state is virtually reached, the energy stored in the inductor is

$$W = \frac{1}{2} L \left(\frac{E}{R} \right)^2 \tag{6.10}$$

It is because this energy cannot be transferred instantaneously that the need for the transient term arises.

If the changing of the inductive current carried out instantaneously (i.e. by great advance) it would lead to appearing of perpetually large e.m.f. of self-induction

$$e_L = -\frac{d\psi}{dt} = -L\frac{di_L}{dt} \to \infty .$$

The energy of magnetic field would also vary instantaneously and, hence, the circuit supply source should develop perpetually large power

$$p = \frac{dW_м}{dt} = Li_L \frac{di_L}{dt} \to \infty .$$

As the e.m.f. of self-induction and the power source in real circuits have finite values, the transition from one steady-state condition of the circuit with an inductive element to another one is possible only during some period of time.

If the voltage u_{C1} is applied to the terminals of a capacitor, some charge is retained on its facings, and the electrical field energy is reserved here:

$$q_1 = Cu_{C1}, \; W_{э1} = \frac{Cu_{C1}^2}{2} .$$

If voltage is changed to the value u_{C2} the new installed condition will be characterized by the new values of a charge and an electrical field energy

$$q_2 = Cu_{C2},\ W_{э2} = \frac{Cu_{C2}^2}{2}.$$

If this transition was carried out instantaneously, it would lead to appearing of the perpetually large charging (or discharge) current

$$i = \frac{dq}{dt} = C\frac{du_C}{dt} \to \infty$$

The electrical field energy would also change instantaneously, and, hence, the circuit supply source would develop perpetually large power

$$p = \frac{dW_э}{dt} = Cu_C\frac{du_C}{dt} \to \infty .$$

The values of the transient current and voltage components depend on a level of disparity of the energy reserved in magnetic and electric fields to a new steady-state rate. In process of diminution of this disparity the transient currents gradually decrease to zero values.

One can represent a real circuit current during transient as a sum of two components: a new steady-state current and a transient current

$$i = i_{ss} + i_t .$$

The transient time is spotted by a small time interval, at the end of which currents and voltages so come nearer to the steady-state values, that the distinction appears to become practically imperceptible.

Consider the transient solution

$$i_t = -\frac{E}{R}e^{-\frac{R}{L}t} \qquad (6.11)$$

which makes it clear that the decay of the transient term is dependent upon the circuit parameters *R* and *L* and entirely independent of the source function.

This is certainly not unexpected when it is recalled that the transient term is really a description of the manner in which the circuit reacts to external disturbances, whatever their origin or nature.

The only influence that the forcing function has on the transient term is in determining the magnitude of the coefficient of the exponential, but it in no way influences the rate of decay of the transient term. A plot of Eqs. (6.6) and (6.11) appears in Fig. 6.3. Note that at t = 0 the steady-state and the transient solutions have equal and opposite values, thus yielding a zero value for the total current as called for by the boundary condition.

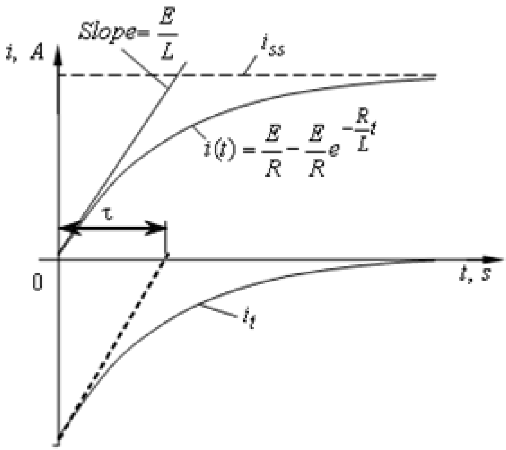


Fig. 6.3

Note also that as the transient term decays to zero, the total current reaches its forced value. In the interest of establishing a convenient measure of the duration of transient, let us look more closely at the exponential function of Eq. (6.11). A glance at the power of the exponential, which must be a numeric, reveals that the quantity *L/R* must in turn bear units of time. Because of this fact and because the quantity is determined in terms of the circuit parameters, it is called the time constant of the circuit and is denoted by τ. Thus $\tau = L/R$. Its physical sense: it is the time during which a free component decreases in $e \approx 2{,}718$ times in comparison with its initial value $i_t(0+)$

$$i_t(\tau) = i_t(0+)e^{-1} = \frac{i_t(0+)}{e} = \frac{A}{e}. \qquad (6.12)$$

The value inverse to the time constant termed as a *damping factor*

$$\alpha = \frac{1}{\tau}.$$

Transient current damps more slowly and, hence, the more a time constant (and the less a damping factor), the longer a new steady-state rate is installed. As we can see from the figure, in spite of the fact that transient process lasts perpetually long, in practice it is accepted to consider that the transient lasts only during the time $t = (3...5)\tau$.

6.4 The Rules of Switching

Under any transient or steady-state conditions two basic points hold: the current through any inductance and the voltage across a capacitance cannot change instantaneously. The case of a current through inductance can be proved by reference to the network of Fig. 6.2. By Kirchhoff's voltage law: $L\frac{di}{dt} + Ri = E$. The current i and the voltage E can take on only finite values. Assume that the current i can change instantaneously. In other words, the current changes by a finite amount Δi during an interval of time $\Delta t \to 0$ tending to zero, such that ${}^{\Delta i}\!/\!_{\Delta t} = \infty$. If ∞ is substituted in Eq. (6.1), the left-hand side of the equation will no longer be equal to its right-hand side, and Kirchhoff's voltage law will not be satisfied. In other words, the assumption that the current through an inductance can change instantaneously runs counter to Kirchhoff's voltage law. While the current through an inductance cannot change instantaneously, the voltage across it can. This does not contradict Kirchhoff's voltage law.

Consider now the simple circuit with a capacitance (see Fig. 6.4).

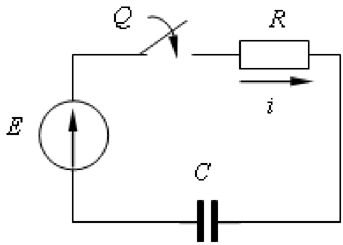


Fig.6.4

Write an equation by Kirchhoff's voltage law for this network

$$Ri + u_C = E \qquad (6.13)$$

where E is the source voltage (which is a finite quantity) and u_C is the voltage across the capacitance. Since

$$i = C\frac{du_C}{dt},$$

it follows that

$$RC\frac{du_C}{dt} + u_C = E \qquad (6.14)$$

If we assume that the voltage u_c can change instantaneously than $\dfrac{\Delta u_c}{\Delta t} \approx \dfrac{du_c}{dt} = \infty$, and the left-hand side of Eq. (6.14) will no longer be equal to its right-hand side. Hence the assumption that the voltage across the capacitance can change instantaneously runs counter to Kirchhoff's voltage law.

However, a current equal to $C\dfrac{du_c}{dt}$ flowing through a capacitor can vary step-wise; this does not contradict Kirchhoff's voltage law.

There are two corollaries to the above two basic points, which may be called switching rules.

Let $i_L(0-)$ be the current through any inductance just prior to switching. It is equal to the current through the same inductance immediately after switching, $i_L(0+)$, or

$$i_L(0-) = i_L(0+) \tag{6.15}$$

Time $t = 0-$ is the instant immediately before switching; $t = 0+$ is the time just after switching. Eq. (6.12) expresses the first rule of switching: *the inductor current immediately prior to and immediately after a switching operation must be the same.*

Let the voltage across a capacitance just before switching be $u_c(0-)$, and that immediately after switching, $u_c(0+)$. Since the voltage across a capacitance cannot change instantaneously, it follows that

$$u_c(0-) = u_c(0+) \tag{6.16}$$

Eq (6.13) expresses the second rule of switching: the capacitor voltage immediately prior to and immediately after a switching operation must be the same.

6.5 Independent and Dependent Initial Conditions

In the literature, under *initial values* or *conditions* are meant the values of current and voltage in the circuit at time $t = 0$. As have already been stated, currents through inductances and voltages across capacitances immediately before and after switching are respectively equal. Other quantities, such as voltages across inductances and resistances, currents through capacitances and resistances, can change instantaneously, and so their values immediately after switching may differ from those before switching. Hence, a distinction should be made between initial conditions before and after switching. Initial conditions before switching are those which exist at time $t = 0-$. Initial conditions after switching are those which exist at time $t = 0+$.

For any circuit the currents and voltages that exist in or across any element at time $t = 0+$ (after switching) can be found by Kirchhoff's laws. In the equations written by these laws the currents through inductances and the voltages across capacitances known from the state of the circuit prior to switching are referred to as *independent initial conditions (i.i.c.)* The remaining currents and voltages at $t = 0+$, found by Kirchhoff's laws are referred to as *dependent initial conditions*.

If by the time a transient is initiated, that is, before switching, all currents through and all voltages across passive elements have become equal to zero, the circuit is said to be in the state of zero initial conditions and the respective values are termed *zero initial conditions*.

If, however, by the time of a transient the same values are other than zero, the circuit is said to be in the state of non-zero initial conditions, and the respective variables are termed *non-zero initial values*.

6.6 Short circuit in a RL circuit

Let's guess, that an inductive coil which has been connected through resistive element R_0 to a source of a direct current e.m.f., at the time instant $t = 0$ becomes to be short circuit. Before a switching at time instant $t = 0-$ the current flows through the circuit

$$i\ (0-) = \frac{E}{R_o + R}.$$

As a result of the switching in the organized contour which includes an inductive coil, thanks to a stored energy of a magnetic field, the current will not disappear instantaneously. The e.m.f. of self-induction which has the same direction, as a current before switching, tends to support the current in the mesh at the expense of energy of a disappearing magnetic field. As the magnetic field energy is gradually diffused, being transmuted into heat in a resistor, the current in a contour decreases to zero. The steady-state value of a current in a short-circuited contour is equal to zero. The transient process taking place in this mesh is free or (transient) and the solution consists only of a free component, i.e. $i = i_t$. As well as in the previous case, the free component of a current does not depend on the energy source, therefore the equation for it, a characteristic equation and its solution, a form of a transient component are the same

$$u_R + u_L = 0\text{, or } Ri_t + L\frac{di_t}{dt} = 0.$$

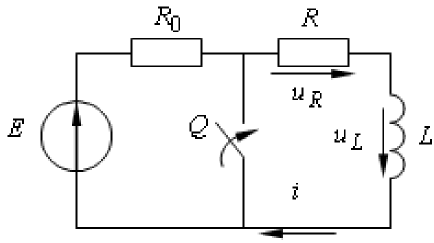


Fig. 6.5

The characteristic equation

$$R + pL = 0\text{, from where } p = -\frac{R}{L}.$$

Then the transient component

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t}.$$

For integration constant definition we use initial conditions and the first law of switching

$$i(0-) = i(0+) = \frac{E}{R_o + R} = A$$

From here

$$i = \frac{E}{R_o + R}e^{-\frac{R}{L}t}.$$

The change of the current in a short-circuited contour and the voltage across the inductor are plotted in Fig. 6.6.

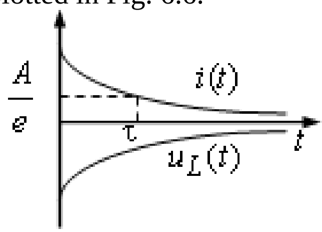


Fig.6.6

The transient process in a short-circuited contour theoretically lasts perpetually. Just as in the previous case, in a short-circuited contour it is possible to consider that the transient is practically over after the time $t=(3...5)\tau$. With that the energy is deposited in the form of heat in the resistor

$$\int_0^\infty Ri^2dt = \int_0^\infty R\left(i(0-)e^{-\frac{R}{L}t}\right)^2 dt = Ri^2(0-)\int_0^\infty e^{-\frac{2R}{L}t}dt =$$

$$= Ri^2(0-)\left(-\frac{L}{2R}e^{-\frac{2R}{L}t}\right)\Bigg|_0^\infty = \frac{Li^2(0-)}{2},$$

i.e. all energy reserved in a magnetic field of the coil before a switching.

6.7 Step Response of an RC Circuit

Appearing in Fig. 6.7 is the circuit arrangement of a resistor in series with a capacitor. The capacitor is assumed to have an initial charge of value q_o owing to the flow of a previously applied current. Let us now find the complete expression for the current after the switch Q is closed.

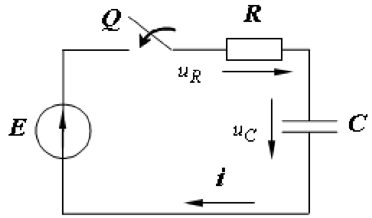


Fig. 6.7

The governing differential equation is obtained upon applying Kirchhoff's voltage law to the closed circuit. Thus,

$$Ri + u_C = Ri + \int idt = E . \qquad (6.17)$$

We search the solution of the given differential equation in an aspect

$$u_C = u_{Css} + u_{Ct},$$

where u_{Css} is a steady-state component and u_{Ct} if a free (transient) component of the capacitive voltage.

As the e.m.f. is constant, all steady-state magnitudes (including capacitive voltage) will be also constant $u_{C_{ss}} = E$.

As the capacitive current is $i = C\frac{du_C}{dt}$, the equation by the KVL is

$$RC\frac{du_C}{dt} + u_C = E$$

Before switching at *t = 0-*, the capacitor has not been charged. After closing a key the capacitive voltage must be the same after switching

$$u_C(0-) = u_C(0+) = 0$$

Let's make a characteristic equation and, on solving it, we will find a form of the free component

$$pRC + 1 = 0 \text{ from where } p = -\frac{1}{RC},$$

$$u_{C_t} = Ae^{pt} = Ae^{-\frac{1}{RC}t}.$$

Then at $t = 0+$ we have got

$$0 = E + A.$$

From here $A = -E$. Hence, on combining the solution for *ss* and free components, we'll get a transient voltage across the capacitive element

$$u_C = E - Ee^{pt} = E - Ee^{-\frac{1}{RC}t} \qquad (6.18)$$

The time constant $\tau = \frac{1}{|p|} = RC$.

Now we can determine the current in this contour

$$i = C\frac{du_C}{dt} = C\frac{d}{dt}\left(E - Ee^{-\frac{1}{RC}t} \right) =$$

$$= C\left(-\frac{1}{RC} \right)\left(-Ee^{-\frac{1}{RC}t} \right) = \frac{E}{R}e^{-\frac{1}{RC}t} \qquad (6.19)$$

And the voltage across the resistor

$$u_R = Ri = R\frac{E}{R}e^{-\frac{1}{RC}t} = Ee^{-\frac{1}{RC}t} \qquad (6.20)$$

A plot of these equations as functions of time is depicted in Fig. 6.8.

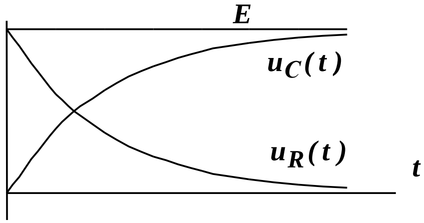


Fig.6.8

Example 6.1. In the circuit of Fig. 6.7 the capacitor has an initial charge of such value that the voltage appearing across its plates is 40 *V*. The active resistance is equal to $0.2\times 10^{6}\ \Omega$, the capacitance is equal to 10 *mF*. Find the expression for the current which flows when the switch is closed. Also, find the total energy dissipated in the resistor during the transient state and compare it with the energy initially stored in the capacitor.

Solution: Taking voltage drops in a clockwise direction as positive. Kirchhoff's voltage law applied to the circuit with *Q* closed yields

$$Ri+\int idt=0 \qquad (6.21)$$

A comparison of Eq. (6.18) with Eq. (6.14) discloses that it ts identical except for the fact that *E* is zero. Accordingly, the complete current response for this ease follows from Eq. (6.16) upon setting *E* to zero. Thus

$$i=-\frac{E}{R}e^{-\frac{1}{RC}t}$$

The time constant is $\tau=RC=0.2\times 10^{6}\times 10\times 10^{-6}=2\ s$. During this time interval, the function $i(t)$ decreases by the factor $e = 2.71$.

The current solution may therefore be written as

$$i(t)=-\frac{40}{0.2\times 10^{6}}e^{-1/2t}=-2\times 10^{4}e^{-1/2t}\ A \qquad (6.22)$$

The negative sign indicates that the actual current flow is counter clockwise rather than clockwise as initially assumed.

The amount of energy dissipated in the resistor is found from

$$W_{dis}=\int_0^{\infty} i^{2}(t)dt=\int_0^{\infty}\left(4\times 10^{-8}\right)0.2\times 10^{6}e^{-t/2}dt=$$

$$=-8\times 10^{-3}\left[e^{-t}\right]\frac{\infty}{0}=8\times 10^{-3}\ J \qquad (6.23)$$

The amount of energy initially stored in the electric field of the capacitor is given by

$$W_{el} = \frac{1}{2}CU^2 = \frac{1}{2}10^{-5} \times 40^2 = 8 \times 10^{-3} \ J$$

A comparison of Eqs. (6.20) and (6.21) shows that all the capacitor energy is dissipated as heat after the switch is closed.

6.8 Short Circuit in RC Circuit

In dealing with a current source the resistance and capacitance parameters are considered to be in parallel as illustrated in Fig. 6.9.

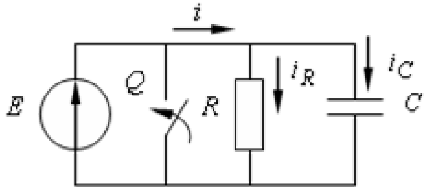


Fig. 6.9

As previously explained, a step-current forcing function is applied to the parallel combination by opening the switch. Then by Kirchhoff's current law we can write

$$i = i_R + i_C \tag{6.24}$$

Moreover, the same potential difference appears across *R* and *C*. Hence

$$R\,i_R = \frac{\int_0^t i_C\,dt}{C} \tag{6.25}$$

There can exist no initial charge on the capacitor because the switch *Q* is initially assumed closed. Inserting Eq. (6.24) into Eq. (6.25) yields

$$R\,i = Ri_C + \int_0^t i_C\,dt \tag{6.26}$$

A study of the last equation can be made to reveal information about the steady-state value of i_C without resorting to the operator form of the governing differential equation and setting *p* to zero.

First note that the left side of the equation is a finite quantity, *IR*. Also note that, as time increases after opening the switch, the contribution from the integral term becomes larger and larger. As time gets very large and approaches infinity, the only way the right side of the equation can remain finite so as to balance the left side is for the current to approach zero.

Our experience with differential equations having the form of Eq. (5.26) tells us that the capacitor carries a transient current given by

$$i_C = i_t = Ae^{-t/RC} \tag{6.27}$$

To evaluate *A*, we need the initial value of i_C. This too can be deduced from Eq. (6.23) upon inserting $t=0+$. In this instance the contribution from the integral term is infinitesimal so that it follows that

$$i_C = I \tag{6.28}$$

Therefore, the complete expression for the capacitor current is

$$i_C = Ie^{-t/RC} \tag{6.29}$$

The time solution for the current through the resistor is found from Eq. (6.21) and (6.24). Thus

$$i_R = I - i_C = I(1 - e^{-t/RC} \tag{6.30}$$

In the configuration of Fig. 6.9 the total source current initially flows entirely through the capacitor and then, as time elapses, it gradually transfers to the resistor. At steady-state all the current *I* flows through *R* and none through *C*.

6.9 Step Response of Second-order System (RLC Circuit)

The behaviour of a circuit which contains two independent energy-storing elements is described by a second-order differential equation.

Let it be desired to find the complete current response in the circuit of Fig. 6.10 after the switch is closed. Assume that the capacitor has an initial charge of q_o. The governing differential equation for the circuit is found upon applying Kirchhoff's voltage law. Thus

$$E = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \tag{6.31}$$

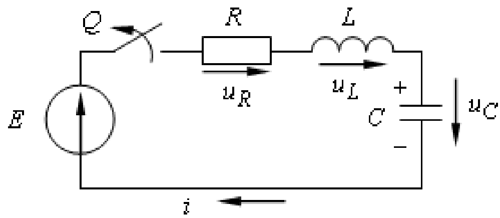


Fig.6.10

Equation (6.31) is a second-order nonhomogeneous linear differential equation. Introducing the initial condition voltage we can then write

$$E = Ri + L\frac{di}{dt} + \frac{q_o}{C} + \frac{1}{C}\int idt \tag{6.32}$$

Rearranging and employing the differential operator, the expression becomes

$$E - \frac{q_o}{C} = \left(R + pL + \frac{1}{pC}\right)i \tag{6.33}$$

or

$$i = \left(\frac{1}{R + pL + 1/pC}\right)\cdot\left(E - \frac{q_o}{C}\right). \tag{6.34}$$

First let us discuss the steady-state solution. One must remember; to find the steady-state solution for dc sources for any branch voltage of any energy storage element, we replace the inductor with a short circuit and the capacitor with an open circuit. Note that in the series *LC* problem only the steady-state capacitor voltage is nonzero and is determined by the remainder circuit. Similarly, for the parallel *LC* problem, only the steady-state inductor current will be nonzero.

The steady-state (or forced) solution is found upon setting *p* equal to zero in the last equation which clearly leads to

$$i_{ss} = \text{forced solution} = 0$$

Since the capacitor represents an open circuit to a constant forcing function, this result is entirely expected.

The first step consists of finding the characteristic equation which here is readily determined by setting the denominator of Eq. (6.34) equal to zero. After rearranging terms, we get

$$Z(p) = p^2 + \frac{R}{L}p + \frac{1}{LC} = 0 \tag{6.35}$$

It is worth noting that this equation also results when the left side of Eq. (6.33) is set equal to zero, thus yielding the homogeneous differential equation in operator form. Moreover, note that each of the three circuit parameters appears and that the highest order of p s is *2*, which is consistent with the presence of two independent energy-storing elements. If p_1 and p_2 denote the roots of this equation, the formal expression for the response can be written as

$$i(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{6.36}$$

Although Eq. (6.33) presents a formal solution of the problem at hand, we are interested in writing this solution in amore useful and significant form. In this connection, then, let us return to Eq. (6.32) and write the specific expressions for the roots. Thus

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{6.37}$$

Depending upon the expression under the radical the transient response can be any one of the following: (I) overdamped if $(R/2L)^2 > 1/LC$; (2) critically damped if $(R/2L)^2 = l/LC$; (3) underdamped if $(R/2L)^2 < 1/LC$.

Now let us investigate each of these three cases in more detail.

The first case**:** *Two unequal real roots*.

The characteristic equation has the form

$$p^2 + ap + b = 0 \tag{6.38}$$

The two roots are given by

$$p_{1,2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}\,, \tag{6.39}$$

which can be written, by factoring out $-a/2$, as

$$p_{1,2} = -\frac{a}{2}\left(1 \pm \sqrt{1 - \frac{4b}{a^2}}\right) \tag{6.40}$$

The two roots will be real and unequal if $4b/a^2 < 1$. Then, from Eq. (6.40) it is clear that if a is positive, both of these roots will be negative. For circuits containing positive resistors, inductors, and capacitors, *R*, *L* and *C* will be positive and also will a. Therefore, for realistic circuits, both roots will be negative and we will designate them as $p_1 = -\alpha_1$; $p_2 = -\alpha_2$; where α_1 and α_2 are positive numbers.

Therefore, transient solution will be of the form

$$i(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \qquad (6.41)$$

where A_1 and A_2 are undetermined constants.

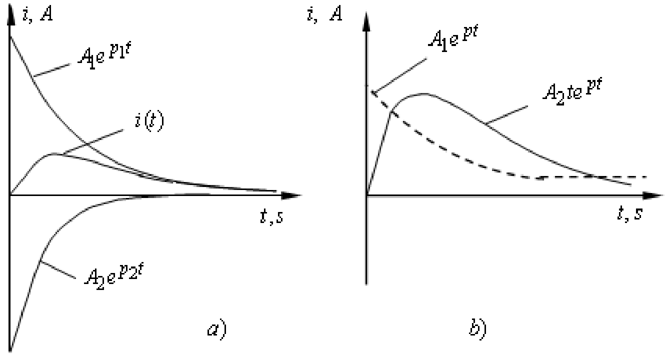


Fig.6.11

Thus, the transient solution consists of the sum of two decaying exponentials, as shown in Fig. 6.11, a. For this case the transient solution is said to be overdamped.

The second case: Two equal real roots. If $4b = a^2$. The radical in Eq. (6.40) will be zero and the roots will be equal: $p_1 = p_2 = -a/2 = -\alpha$.

From mathematics it is known that if among the roots of the characteristic equation there are two which are equal, the respective parts of the solution may be written thus

$$i_t = A_1 e^{pt} + A_2 te^{pt} = (A_1 + tA_2)e^{-\alpha t} \qquad (6.42)$$

This solution cannot be reduced any further and that is contains exactly two undetermined constants. For this case, the transient solution is said to be critically damped, that is shown in Fig. 6.11*b*.

The third case**:** *Complex Roots*. Because the underdamped case is the most interesting as well as the most frequently encountered case, attention is confined to it throughout the remainder of this section. The term damping is an appropriate one to use in our description because, as previously demonstrated, it is characteristic of the resistance parameter to dissipate energy. Its presence serves to prevent an uninterrupted interchange of energy between the two energy-storing elements. Such an uninterrupted interchange would constitute a sustained (or undamped) oscillation.

For the underdamped case, then, the expression for the roots of the characteristic equation becomes

$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,, \qquad (6.43)$$

This result is obtained by factoring out the minus sign under the radical and recalling that $j = \sqrt{-1}$.

The critical value of damping for fixed values of L and C corresponds to that value of R which makes the radical term go to zero. Hence

$$\left(\frac{R_c}{2L}\right)^2 = \frac{1}{LC} \qquad (6.44)$$

where Rc denotes the value of resistance which yields a critically damped transient response. In order to express the roots of the characteristic equation, two figures of merit are introduced and defined.

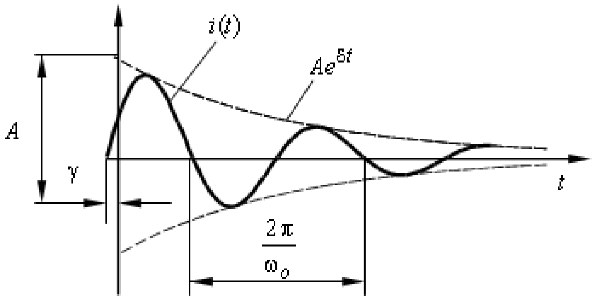


Fig. 6.12

If $4b/a^2$ >1 in Eq. (6.45), we can rewrite this equation as

$$p_{1,2} = -\frac{a}{2}\left(1 \pm j\sqrt{\frac{4b^2}{a} - 1}\right) \qquad (6.45)$$

So that we will denote the roots as

$$p_{1,2} = \delta \pm j\omega_o \qquad (6.46)$$

Complex roots always occur in conjugate pairs. If one root is $p_1 = -\delta + j\omega_o$, the other will be $p_2 = -\delta - j\omega_o$. It is known that the respective term, say the transient current i_t should be written thus

$$i_t = Ae^{-\delta t}\sin(\omega_o t + \gamma \qquad (6.47)$$

This is illustrated in Fig. 6.12 and is called a *damped sine-wave*. It is of angular frequency ω_o and of initial phase (epoch angle) γ. The envelope of the wave is decided by the curve $Ae^{-\delta t}$. The greater δ, the faster the transient decays. The magnitude A and an initial face of damped sinusoid γ depend on the circuit parameters, initial conditions and the energy source. On the other hand, ω_o and δ depend solely on the circuit parameters after switching. The complex frequency ω_o is termed the *angular natural frequency*, and δ is the *decay factor*, or *decrement*. The value $T = 2\pi/\omega_o$ is called the period of a function.

6.10. General Outline of Transient Analysis as Applied to Linear Circuits

Transient analysis involves basically the following steps:

1. At first we represent an electric circuit before switching. Positive directions of currents through inductive elements and voltages across capacitive elements are assumed. Then one must find independent initial conditions (i.e. the values of currents through inductive elements and voltages across capacitances immediately before switching). Independent initial conditions are found by some calculation method for direct current circuits, if energy sources are **dc**; or by the calculation method for sinusoidal alternating current, if energy sources are **ac**.

2. Now we represent an electric circuit after switching. Positive directions are assumed for branch currents and voltages. Positive directions of inductive currents and capacitive voltages must be the same, as in the circuit before switching. One must work out the integro-differential equations for transient, using Kirchhoff's laws.

3. Define the steady-state components of currents and voltages after switching when transient process has been over. The steady-state components are found by some calculation method for direct current circuits, if energy sources are dc; or by the calculation method for sinusoidal alternating current, if energy sources are ac.

4. The characteristic equation is written down and its roots are found. Now we can define the law of change of transient components by the form of the roots.

5. Required values are written down as the sum of steady-state and free components.

6. Then we can calculate the constants of integration which enter into transient components. If the transient component contains more than one constant of integration, it is required to find at first not only the values of currents and voltages, but also their derivatives for the moment of time after switching.

The main difficulty of a classical method just also consists in definition of dependent initial conditions and the constants of integration.

7. The unknown currents and voltages are expressed as functions of time in the form of algebraic sum of the steady-state and transient components (taking into account the calculated constants of integration).

6.11 Determination of Integration Constants in the Classical Method

The classical method shows that the current flowing under transient conditions is described by a differential equation whose solution is the sum of a particular integral (the steady-state component) and a complementary function (the transient component), The constants of integration of the complementary function (the transient current or voltage) are found by solving a system of algebraic equations from the known roots of the characteristic equation and from the known values of the transient currents and its derivatives for $t = 0+$.

Any transient current or voltage may be represented as the sum of exponential terms. The number of such terms is equal to the number of roots of the characteristic equation. For any network using Kirchhoff's laws and the switching rules one can find:

(1) the numerical value of $i_t(0+)$, or the transient current at $t = 0+$;

(2) the numerical value of the first and, if necessary, higher derivatives of the transient current for $t = 0+$.

Let's consider how the constants of integration A_1 and A_2 are found, assuming that $i_t(0+)$ and $i'_t(0+)$ and the roots p_1 and p_2 are known. When a network is described by a first-order equation, then $i_t = Ae^{pt}$. The constant of integration is determined from the transient current $i_t(0+)$: $A = i_t(0+)$.

When a network is described by a second-order equation which has real and unequal roots, then

$$i_t = A_1 e^{p_1 t} + A_2 e^{p_2 t} \qquad (6.48)$$

Differentiating this equation with respect to time gives

$$i'_t = p_1 A_1 e^{p_1 t} + p_2 A_2 e^{p_2 t} \,. \qquad (6.49)$$

Writing Eqs. (6.48) and (6.49) for $t = 0$ (and noting that for $t = 0$, $e^{p_1 t} = e^{p_2 t} = 1$) gives

$$\begin{cases} i_t(0+) = A1 + A2 \\ i_t{}'(0+) = p_1 A1 + p_2 A2 \end{cases} \qquad (6.50)$$

Solving these two equations simultaneously gives us unknown values A_1 and A_2.

If the characteristic equation has conjugate complex roots, the transient current is given by Eq. (6.47), where the angular frequency ω_o and the decay factor δ are known from the solution of the characteristic equation. So, the transient current is given by

$$i_t = Ae^{-\delta t} \sin(\omega_o t + \gamma) \,. \qquad (6.51)$$

Differentiating Eq. (6.51) with respect to time gives

$$i'_t(0+) = -A\delta e^{-\delta t} \sin(\omega_o t + \gamma) + A\omega_o e^{-\delta t} \cos(\omega_o t + \gamma) \qquad (6.52)$$

Writing Eqs. (6.51) and (6.52) for the time $t = 0$, we get

$$i_t(0+) = A \sin\gamma \qquad (6.53)$$

$$i'_t(0+) = -A\delta \sin\gamma + A\omega_o \cos\gamma \,. \qquad (6.54)$$

Thus, we have two equations to determine two unknown values: the constant in integration A and the initial phase γ.

Example 6.2. There were zero initial conditions in the network of Fig. 6.10 before switch closure. The parameters of the network are E = 60 *V*, *R*=10 *Ω*, *L*=0.1 *H*, *C*=10^{-4}*F*. Determine the law according to which the current vary with time.

Solution: (1) Assign positive directions to the current. The positive directions for the voltages will, as always, be assumed coincident with the current.

Before switch closure, the inductive current and the voltage across the capacitor were zero:

$$i(0-) = i(0+) = 0$$

$$u_C(0-) = u_C(0+) = 0$$

Hence, we have zero initial conditions here.

In accordance with the second switching rule, they remain zero immediately after closure of the switch, at $t = 0+$.

After switching the key Q is close. The equation by the second Kirchhoff's law:

$$u_R + u_L + u_C = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt = E \qquad (6.55)$$

where i *and* u_C are zero initial conditions. Then $u_L = E = 60\ V$.

(2) Consider steady-state condition. Remember: to find the steady-state solution for dc sources for any branch voltage of any energy storage element, we replace the inductor with a short circuit and the capacitor with an open circuit. Note that in the series RLC problem only the steady-state capacitor voltage is nonzero and is determined by the remainder circuit. Hence, $i_{ss} = 0$, $u_{Css} = E = 60\ V$.

(3) To find the transient current, represent the current at t = 0+ as the sum of a steady-state and a transient component

$$i(0+) = i_{ss}(0+) + i_t(0+) = 0$$

The characteristic equation of the network

$$Z(p) = R + pL + \frac{1}{pC} = 0$$

or
$$p^2LC + pRC + 1 = 0$$

has two complex conjugate roots

$$p_{1,2} = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = \frac{-10^{-3} \pm j6.24\cdot 10^{-3}}{2\cdot 10^{-5}} = -50 \pm j312\ s^{-1}.$$ Since the characteristic equation has such roots, the transient component will have the form

$$i_t = Ae^{-50t}\sin(312t + \gamma)$$

where A and γ can be found from the transient solution and its first derivative for $t = 0+$.

From (1) above, $u_L = E = L\frac{di}{dt}$,

from where $\frac{di}{dt} = \frac{E}{L} = 600\ A/s$.

The function it for $t = 0+$ is equal to $A\sin\gamma$.

The derivative of the function $Ae^{-\delta t}\sin(\omega_o t+\gamma)$ for t = 0+ is equal to $\frac{di}{dt}(0+)=-A\delta\sin\gamma+A\omega_o\cos$. To determine A and γ for the transient component of the current, write two equations:

$$\begin{cases} A\sin\gamma=0 \\ -A\delta\sin\gamma+A\omega_o\cos\gamma=600 \end{cases}$$

Solving them simultaneously gives $\gamma = 0^o$; A=1.92 A.

Therefore,

$$i(t)=i_t(t)=1.92e^{-50t}\sin 312t\ .$$

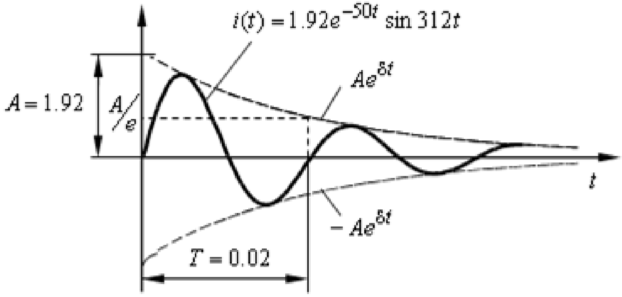


Fig. 6.13

This function is represented by curve in Fig. 6.13. The period of the function is equal to $T={}^{2\pi}\!/_{312}=0.02\,s$, the time constant

$\tau=\frac{1}{|50|}=0.02\,s$. During this time interval, the function $1.92e^{-50t}$ decreases by the factor e = 2.71.

6.12. Complete Response of RL Circuit to Sinusoidal Input

Now we determine the total response of a series *RL* circuit when a second type of input, the sinusoidal function, is applied to the circuit terminals. The circuit is depicted in Fig. 6.14. The voltage source is assumed to be varying in a cosinusoidal fashion. When the switch is closed, the governing differential equation for the circuit becomes

$$e(t)=Em\cos\omega t=Ri+L\frac{di}{dt}\ \text{v} \qquad (6.56)$$

The solution is searched as the sum of two components

$$i = i_{ss} + i_t \,.$$

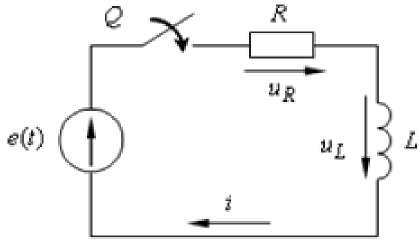


Fig. 6.14

In operator form the response is

$$i = \left[\frac{1}{R + pL}\right] E_m \cos \omega t \tag{6.57}$$

The forced solution is found directly from the Eq. (6.57). Before doing so, recall that a considerable saving of effort results when the cosinusoidal function is replaced by the corresponding exponential function which contains it. Accordingly, we have

$$i_{ss} = \mathrm{Re}\left[\frac{E_m}{Z} e^{(j\omega t - \psi)}\right] = \frac{E_m}{Z}\cos(\omega t - \psi) =$$

$$= I_m \cos(\omega t - \psi)\,, \tag{6.58}$$

where $Z = \sqrt{R^2 + \omega^2 L^2}$; $\psi = \tan^{-1}\dfrac{\omega L}{R}$

It is instructive here to note that the forced solution has the same form as the forcing function but that it differs in two respects: amplitude and phase. The amplitude of the current is modified from that of the source voltage by the factor $1\Big/\sqrt{\left(R^2 + \omega^2 L^2\right)} = 1/Z$. The phase or argument is altered by the angle ψ.

To find the transient component of the solution, we first obtain the characteristic equation. This follows upon setting the denominator of Eq. (6.57) equal to zero. Thus

$$R + pL = 0 \tag{6.59}$$

from which $p = R/L$.

This is a familiar result and leads to an expression for the transient solution that is the same as was found for the case with dc energy source. See Eqs. (6.3) and (6.4). Thus

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t} \tag{6.60}$$

The complete solution is obtained by adding Eq. (6.58) to Eq. (6.60), which yields

$$i = i_{ss} + i_t = I_m \cos(\omega t - \psi) + Ae^{pt} \tag{6.61}$$

The presence of the inductor means that $i(0+) = 0$. Inserting this result into Eq. (6.61) for t = 0+ identifies the value of A as

$$0 = I_m \cos\,(-\psi) + A$$

from here

$$A = -i_{ss}(0+) = -I_m \cos\,\psi$$

The final expression for the response thus becomes

$$i(t) = I_m \cos(\omega t - \psi) - I_m\, e^{pt} \cos\psi \tag{6.62}$$

A graphical representation of Eq (6.62) appears m Fig 6.15. Initially transient component of the solution provides a magnitude of current which is equal and opposite to the instanta-neous value of the steady-state current at the instant of switching. In Fig. 6.15 it is assumed that the switch is closed at the instant when the voltage has its positive maximum value. But corresponding steady-state current is not zero at this instant. In Fig. 6.15 it is shown to have a value 0a. Since the boundary condition demands that the current at this instant be zero, it is necessary for the transient term to have a value equal and opposite. This is depicted as –0a. As time elapses after switching, the transient term decays to zero at a rate determined by the circuit time con-stant, and the total current in Fig. 6.15 then becomes identical to the steady-state current.

The presence of a transient component in the total solution for the current is called for whenever the initial value of current in the *RL* circuit is at variance with the value of the steady state current it the instant of switching. As an illustration, if in the problem just discussed the initial current flowing through L were equal to 0*a* there would be no need for a transient term because the ensuing current would proceed directly into the steady state As a result of the correspondence of the currents before and after switching, then is no force attempting to bring about an abrupt change in the amount of energy stored in the inductor.

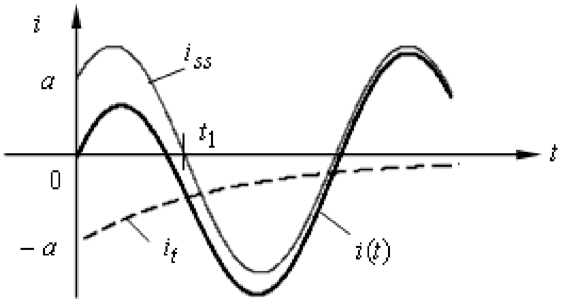


Fig. 6.15

This behaviour can be described in another way. Let t1 in Fig. 6.15 denote the time when the steady-state current is zero. If the switch in Fig. 6.14 is closed at this instant, no violation of the boundary condition occurs; consequently, there is no need for a transient current. In such a case the response proceeds directly into the steady state. Note that although the steady-state current is zero the corresponding value of the applied voltage is not. This is characteristic of the RL circuit for sinusoidal forcing functions.

Example 6.3. A sinusoidal forcing function $e(t) = 141 \sin 377t$ is applied to an initially series *RL* circuit in which $R = 100\ \Omega$ and L = 0.5 H.

(a) If the switch which applied the voltage to the *RL* circuit is closed at the instant when $e(t)$ is passing through zero with a positive slope, determine the initial value of the transient current.

(b) Write the complete expression for the transient solution.

(c) Write the expression for the complete solution of current.

(d) At what instantaneous value of the applied voltage will the closing of the switch result in no transient component of current?

Solution: (a) The general form of the steady-state solution is

$$i_{ss} = \frac{E_m}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \theta) \qquad (6.63)$$

This follows directly from Eq. (6.58) except that the sine function is used in order to make it consistent with the assumed sinusoidal forcing function. Since $\omega = 377$ it follows that

$$\omega L = 377(0.5) = 188.5\ \Omega$$

and

$$\sqrt{R^2 + \omega^2 L^2} = \sqrt{100^2 + 188.5^2} = 213.5\,\Omega$$

Also

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}1.885 = 62^o$$

Therefore, the value of the steady-state current at t = 0+, which is the time immediately following application of the forcing function, is

$$i_{ss}(0+) = \frac{E_m}{\sqrt{R^2+\omega^2L^2}}\sin(-\theta) = \frac{141}{213.5}\sin(-62^o) = -0.584\ A$$

However, in accordance with the boundary condition the initial value of the current through the inductance must be zero. That is

$$i(0) = i_{ss}(0+) + i_t(0+) + 0$$

or 0 = -0.584 + i_t(0+). Hence the initial value of the transient current is i_t = 0.584 A.

(b) The rate of decay of the transient term is determined by the time constant of the circuit, which here is

$$T = \frac{L}{R} = \frac{0.5}{100} = \frac{1}{200}\ s$$

Therefore, the complete expression for the transient term becomes

$$i_t = 0.584e^{-200t}\ \ A$$

(c) Then the equation for the total solution results:

$$i(t) = 0.66\sin(377t - 62^o) + 0.584e^{-200t}\ \ A$$

(d) In order that there be no transient current in the solution, it is necessary to close the switch at that time instant for which if is zero. A glance at Eq. (6.63) shows that this occurs whenever

$$(\omega t - \psi) = 0,\ \pi,\ 2\pi,...$$

Choosing the first value, it follows that $\omega t = \psi = 62^o$.

The corresponding instantaneous value of the forcing function is then e = 141 sin 62º =124.2 V.

Therefore, if the switch is closed when the applied voltage has an instantaneous value of 124.2 volts, there is no need for a transient term and so none exists.

6.13. Response of RLC Circuit to Sinusoidal Inputs

The treatment of this case presents nothing new in the way of concepts about transients; therefore, the development of the solution is kept brief. The circuit configuration is shown in Fig. 6.16.

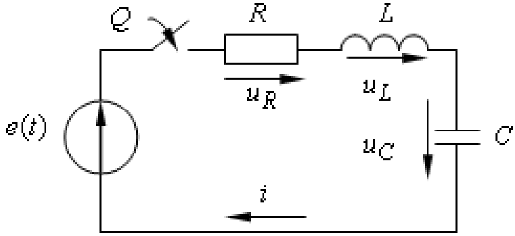


Fig. 6.16

With the switch closed, the differential equation for the circuit is

$$E_m \cos \omega t = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i dt$$

All initial conditions are assumed to be zero.

Proceeding as outlined in the previous section, we can write the expression for the forced solution directly as

$$i_{ss} = \frac{E_m}{Z} e^{j(\omega t - \theta)} = \frac{E_m}{Z} \cos(\omega t - \theta)$$

where now

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \text{ and } \theta = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}$$

Again note that this forced solution has the same form as the source function but differs in its amplitude and argument.

Write down the characteristic equation equalled to zero. Thus

$$R + pL + \frac{1}{pC} = 0 \; or \; p^2 LC + pRC + 1 = 0$$

This characteristic equation has two roots, p_1 and p_2. Accordingly, the transient component takes the form (if we have two real roots)

$$i_t = A_1 e^{p_1 t} \cos \omega t + A_2 e^{p_2 t} \sin \omega t$$

The expression for the total solution then becomes

$$i(t) = i_{ss} + i_t = \frac{E_m}{Z}\cos(\omega t - \psi) + A_1 e^{p_1 t}\cos\omega t + A_2 e^{p_2 t}\sin\omega t$$

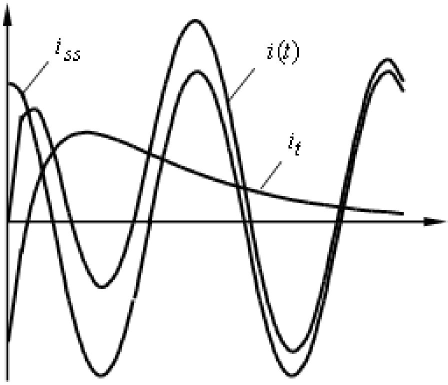


Fig.6.17

Comparing the last equation, the current response of the *RLC* case, with the similar one, the current response for the *RL* case, shows that the responses differ essentially in the character of the transient terms. Because the *RL* configuration involves one energy-storing element and is described by a first-order differential equation there occurs just a simple exponential decay of the transient. The steady-state solution merely rides on this exponential decay, as illustrated in Fig.6.15. For the underdamped *RLC* configuration, however, the transient solution itself involves an exponentially damped oscillation with definite frequency. Here the steady-state term can be looked upon as riding on the damped oscillation, as depicted in Fig. 6.17. It is basically in this respect that the two responses differ.

Example 6.4. In the figure of Fig.6.18, the switch in the third branch is to be closed. Before closure the network is in a steady state; *e(t)* = 127 sin (314t + 40^o) volts. The parameters of the circuit are $R_1 = R_3 = 50\ \Omega$, $R_2 = 10\ \Omega$, $L = 2H$, $C = 150\ \mu F$. Find u_C (0+).

Solution: Before closure of the switch

$$\underline{I}_{1m} = \frac{\underline{E}_m}{R_1 + R_2 + jX_L} = \frac{127e^{j40^o}}{60 + j628} = 0.202e^{-j44}\ A$$

The instantaneous steady-state current

$$i_1 = 0.202\sin(\omega t - 44^o)$$

At the close of the switch ($t = 0$)

$$i_1 = 0.202\sin(-44^o) = -0.14\ A$$

Write down independent initial conditions

$$i_1(0+) = -0.14\ A;\ u_C(0+) = 0$$

Now find the steady-state currents to determine the steady-state voltage across the capacitor after closure of the switch.

The driving-point impedance of the network

$$\underline{Z}_{eq} = R_1 + \frac{(R_2 + j\omega L)(R_3 - \frac{j}{\omega C})}{R_2 + j\omega L + R_3 - \frac{j}{\omega C}} = 104.8e^{-j10^o}\ \Omega$$

Then

$$\underline{I}_{1m} = \frac{\underline{E}_{1m}}{\underline{Z}_{eq}} = \frac{127e^{j40^o}}{104.8e^{-j10^o}} = 1.213e^{j50^o}\ \ A$$

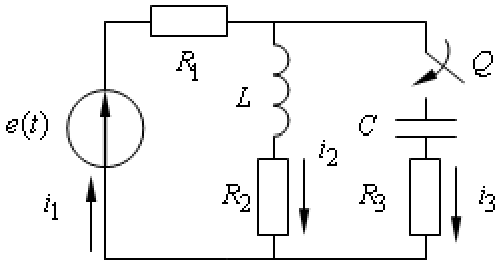


Fig. 6.18

The instantaneous steady-state current after closure of the switch

$$i_{1ss} = 1.213\sin(\omega t + 50^o)\ A$$

$$i_{1ss}(0+) = 1.213\sin 50^o = 0.923\ A$$

The complex impedance of two parallel branches is

$$\underline{Z}_{23} = \frac{(R_2 + j\omega L)(R_3 - \frac{j}{\omega C})}{R_2 + j\omega L + R_3 - \frac{j}{\omega C}} = 56.3e^{-j18^o}\ \Omega$$

The complex voltage across the parallel portion of the network

$$\underline{U}_{23m} = \underline{I}_{1m} \cdot \underline{Z}_{23} = 1.213e^{j50^o} \cdot 56.3e^{-j18^o} = 68.2e^{j32^o}\ V$$

Then we can find the current through the capacitor

$$\underline{I}_{3m} = \frac{\underline{U}_{23}}{\underline{Z}_3} = \frac{68.2e^{j32^o}}{50 - j21.3} = 1.253e^{j54^o}\ A$$

The steady-state voltage across the capacitor after closure of the switch

$$\underline{U}_{Cm} = \underline{I}_{3m} \cdot \left(-\frac{j}{\omega C}\right) = 1.253e^{j54^o} \cdot 21.3e^{-j90^o} = 26.7e^{-j36^o}\ V$$

The instantaneous steady-state voltage across the capacitor and the instantaneous steady-state currents i_2 and i_3 after closure of the switch

$$u_{Css} = 26.7\sin(\omega t - 36^o)\ V$$

$$u_{Css}(0+) = 26.7\sin(-36^o) = -15.57\ V$$

$$i_{2ss} = 0.11\sin(\omega t - 58^o)$$

$$i_{3ss} = 1.253\sin(\omega t + 54^o)\ A$$

And their values at the moment of switching

$$u_{Css}(0+) = 26.7\sin(-36^o) = -15.57\ V$$

$$i_{2ss}(0+) = 0.11\sin(-58^o) = -0$$

$$i_{3ss}(0+) = 1.253\sin 54^o = 1.02$$

Now find i_{2t} (0+). By the first switching rule

$$i_2(0-) = i_2(0+) = -0.14 = i_{2ss}(0+) + i_{2t}(0(\ );$$

From here

$$_{2t}(0+) = i_2(0+) - i_{2ss}(0+) = -0.14 + 0.1 = 0.04\ \ \text{A}$$

The transient voltage across the capacitor after closure of the switch is found by the second switching rule:

$$u_C(0+) = u_{Css}(0+) + u_{Ct}(0+);$$

$$u_{Ct}(0+) = u_C(0+) - u_{Css}(0+) = 0 - (-15.57) = 15.57\ \text{V}$$

To determine $i_{3t}(0+)$, write an equation for the loop formed by the first and third branches:

$$R_1 i_{1t}(0+) + R_3 i_{3t}(0+) + u_{Ct}(0+)$$

Substituting $-0.04 + i_{3t}(0+)$ for $i_{1t}(0+)$ and noting that $u_{C_t}(0+) = 15.57$ volts, we get

$$i_{3t}(0+) = \frac{-15.57 + 2}{50 + 50} = -0.1357 \quad \text{A}$$

From the equation $i_{3_t}(0+) = C\dfrac{du_{Ct}}{dt}$ we can determine

$$\left(\frac{du_{Ct}}{dt}\right)_{t=0+} = \frac{i_{3t}(0+)}{C} = \frac{-0.1357}{150 \cdot 10^{-6}} = -905 \text{ V/s}$$

The characteristic equation

$$p^2 LC(R_1 + R_2) + p\left[C\left(R_1R_2 + R_2R_3 + R_1R_3\right) + L_2\right] + R_1 + R_2 = 0$$

has two roots

$$p_1 = -42 + j15.2\ s^{-1};\ p_2 = -42 - j15.2\ s^{-1}.$$

Therefore, the transient component should be given the form

$$u_{C_{ss}}(0) = Ae^{-\delta t}\sin(\omega_0 + \gamma)$$

where $\delta = 42$ and $\omega_o = 15.2$. A and γ can be found from the transient component and its first derivative for $t = 0+$.

To determine A and γ for u_{Ct} write two equations

$$\begin{cases} A\sin\gamma = 15.57 \\ -\delta A\sin\gamma + \omega_o A\cos\gamma = -905 \end{cases}$$

Solving them simultaneously gives

A = 21.5 and $\gamma = 137^{o};$

Therefore

$$u_{C_{ss}}(0) = 26.7\sin(-36^o) = -15.57\ V$$

The total capacitive current

$$u_C(0+) = u_{Css}(0+) + u_{C_t}(0+) = 26.7\sin(\omega t - 36^o) +$$

$$+ 21.5e^{-42^o}\ \sin\ (15.2t + 137^o)\ V$$

6.14. The Laplace-Transform Method of Finding Circuit Solutions

The mathematical formulation of the Laplace transform permits a uniformity of analysis and of interpretation of results in situations which we find constantly recurring in all areas of electrical engineering and many other branches of engineering and science.

In the treatment which follows, the emphasis is put on the Laplace-transform method. A matter such as obtaining the final solution to a problem after applying the Laplace transform is accomplished through the use of tables. When we accept the use of tables as a valid means of performing the inverse Laplace transformation, there is every reason to use the Laplace transform as a means of solving linear differential equations.

6.15. The Nature of a Mathematical Transform

The Laplace transform is a mathematical tool for transforming functions. In linear analysis it also serves to convert operation in time, such as differentiation and integration, into simpler algebraic functions of an intermediate variable such as multiplication and division.

The first step in the procedure is to find the transform of each number. Finally, the solution in the original system of numbers is obtained by consulting the logarithm table. This last step is known as performing the inverse transformation through the use of tables.

Thus, to solve an integrodifferential equation the same basic steps are involved. (1) The Laplace transform is used to convert the integrodifferential equation into an algebraic equation. The algebraic equation is expressed in terms of a new (or transformed) variable. (2) The simpler operation of solving the algebraic equation is performed in the transformed domain. (3) The solution in the original domain is obtained by consulting appropriate tables.

6.16. The Laplace Transform: Definition and Usefulness

A typical form of the differential equations encountered in the study of electrical engineering is illustrated by Eq. (6.64)

$$A_2 \frac{d^2 i(t)}{dt^2} + A_1 \frac{di(t)}{dt} + A_0 i(t) - A_{-1} \int i(t)dt = f(t)\,, \qquad (6.64)$$

where *f(t)* is the source function, the coefficients *A* denote constants, and *i(t)* is the solution or desired response. Since Eq. (6.64) involves derivatives as well as an integral, and because of the importance of preserving the identity of the response function in spite of these operations of differentiation and integration, it can reasonably be concluded that the Laplace transform must in some way involve the exponential function.

A time function (current, voltage, e.m.f.) is written *f(t)*. Direct Laplace transform of a function *f(t)* is defined as

$$F(p) = \int_0^{\infty} f(t) e^{-pt} dt \qquad (6.65)$$

where *p* is the intermediate or transformed variable. Note that *p* is part of the exponential function. Equation (6.65) states that to obtain the Laplace transform of a function *f(t)* it must be multiplied by e^{-pt} and then integrated from $t = 0$ to $t = \infty$. In general, the variable *p* is *a complex number* such that $p = a + jb$, where *a* is a real part, and *b* is the imaginary part. To solve the integrodifferential equation by the Laplace transform, it is necessary to apply the Laplace integral to each term of the equation. This converts the original time-domain equation to one expressed entirely in terms of the intermediate variable *p*.

The function *f(t)* is the *inverse transform* of *F*. In Eq. (6.64) the unknown response *i(t)* is identified in the transform domain as *I(p)*, i.e., the correspondence between them is written thus

$$L[f(t)] = F(p). \qquad (6.66)$$

Also, a glance at Eq. (6.65) shows that $L[A_0 i(t)] = A_0 I(p)$. Thus one of the five terms of Eq. (6.64) is transformed. Let's desire to find the Laplace transform of a function *f(t)=A*, where A is a constant. Substituting *A* for *f(t)* in Eq. (6.66) and integrating gives

$$F(p) = \int_0^{\infty} A e^{-pt} dt = A\left(-\frac{1}{p}\right)\int_0^{\infty} d\left(e^{-pt}\right) = -\frac{Ae^{-pt}}{p}\Bigg| = \frac{A}{p}. \qquad (6.67)$$

Thus, the Laplace transform of a constant may be written as

$$L[A] = \frac{A}{p} \quad \text{or} \quad A \doteq \frac{A}{p} \qquad (6.67)$$

where $\doteq$ is the sign of correspondence. Accordingly, the imagine of an exponential function $f(t) = e^{\alpha t}$,

$$F(p)=\int_0^{\infty} e^{\alpha t}e^{-pt}dt=\int_0^{\infty} e^{(\alpha-p)t}dt=\left.\frac{e^{(\alpha-p)t}}{\alpha-p}\right|_0^{\infty}=\frac{1}{\alpha-p}(0-1)=\frac{1}{p-\alpha}$$ Thus

$$e^{\alpha t} \mathrel{.=\!\!^{\cdot}} \frac{1}{p-\alpha}\,. \tag{6.69}$$

If $f(t)=Ae^{\alpha t}$,

$$Ae^{\pm\alpha t} \mathrel{.=\!\!^{\cdot}} \frac{A}{p\mp\alpha}\,. \tag{6.70}$$

In deriving the transform Eq. (6.69), it is taken into account that the real part of the operator p is greater than α, that is a >α. It is only then that the integral converges. This transform (6.69) has a number of important corollaries. Thus, putting $\alpha = j\omega$ gives

$$Ae^{j\omega t} \mathrel{.=\!\!^{\cdot}} \frac{A}{p-j\omega}\,. \tag{6.71}$$

To demonstrate that the Laplace transform is useful for simplifying the solution of integrodifferential equations, let's start by stating that *F(p)* is a transform of *f(t)*. We examine the first derivative transformation, knowing that for t = 0, *f(t) = f(0)*.let us start with the statement that *F(p)* is the transform of *f(t)*.We inquire about the transform of a first derivative $df(t)/dt$, knowing that for t = 0, *f(t)* = *f*(0*)*. Converting the function $df(t)/dt$ by the Laplace transform method gives

$$\int_0^{\infty}\frac{df(t)}{dt}e^{-pt}=\int_0^{\infty} e^{-pt}d\left[f(t)\right].$$

The right side of this equation is readily recognized as the integral of the product of two variables. From the calculus we know that

$$\int u\,dv=u\,v-\int v\,du\,.$$

Putting $e^{-pt}=u$ and $d\left[f(t)\right]=dv$, and integrating by parts gives

$$\int_0^{\infty} e^{-pt}d\left[f(t)\right]=e^{-pt}f(t)\Big\|_0^{\infty}-\int_0^{\infty} f(t)\,d\left[e^{-pt}\right].$$

However, $e^{pt} f(t)\Big|_0^\infty = 0 - f(0) = -f(0)$, and

$$-\int_0^\infty f(t)\, d\left[e^{-pt}\right] = p\int_0^\infty f(t)e^{-pt}dt = pF(p)\,.$$

Thus, the Laplace transform of a derivative term is

$$\frac{df(t)}{dt} \risingdotseq pF(p) - f(0)\,. \tag{6.72}$$

Extension of the Laplace transform to derivatives of higher order is readily accomplished by repeated application of the general procedure implied in Eq. (6.72). Thus, the Laplace transform of the second derivative is

$$\frac{d^2 f(t)}{dt^2} \risingdotseq p^2 F(p) - pf(0) - \left[\frac{df(t)}{dt}\right]_{t=0}. \tag{6.73}$$

Therefore, the Laplace transform of the second derivative of i is

$$\frac{d^2 i(t)}{dt^2} \risingdotseq p^2 I(p) - p\,i(0) - pi'(0)\,.$$

And at last, if $f(t) \risingdotseq F(p)$, then

$$\int_0^t f(t)\;dt \risingdotseq \frac{F(p)}{p}\,. \tag{6.74}$$

6.17. The Laplace Transform of the Voltage Across the Inductance

The Laplace Transform of a current i is $I(p)$. Then the transform of the voltage across the inductance will be $u_L = L\dfrac{di}{dt}$.

To find the imagine of this function we will take the advantage of the Laplace transform (integrating in parts)

$$U_L(p) = \int_0^\infty L\frac{di}{dt}e^{-pt}dt = L\int_0^\infty e^{-pt}di = \begin{vmatrix} w = e^{-pt} & dw = -pe^{-pt}dt \\ dv = di & v = i \end{vmatrix} =$$

$$= iLe^{-pt}\Big|_0^\infty - \int_0^\infty iL(-p)e^{-pt}dt = 0 - Li(0) + pL\int_0^\infty ie^{-pt}dt = -Li(0) + pLI(p).$$

Therefore,

$$u_L = L\frac{di}{dt} \doteq pLI(p) - Li(0) . \qquad (6.75)$$

6.18. The Laplace Transform of the Voltage Across a Capacitor

The voltage across a capacitor, u_C, if often given the form $u_C = \frac{1}{C}\int idt$. But the more exhaustive form is $u_C = u_C(0) + \frac{1}{C}\int_0^t idt$.

In accordance with Eq. (6.74) the transform of $\frac{1}{C}\int idt$ is $\frac{I(p)}{Cp}$, and the transform of the constant $u_C(0)$ is the constant itself divided by *p*.

To find the imagine of this function we will take advantage of Laplace transform (having presented the integral in the form of the sum of integrals and integrating in parts) as

$$U_C(p) = \int_0^\infty \left(u_C(0) + \frac{1}{C}\int_0^t idt \right) e^{-pt}dt = u_C(0)\int_0^\infty e^{-pt}dt + \frac{1}{C}\int_0^\infty\int_0^t idt \cdot e^{-pt}dt =$$

(after transformation) $= \frac{u_C(0)}{p} + \frac{I(p)}{pC}$.

Therefore, the Laplace transform of the capacitive voltage is

$$u_C = \frac{1}{C}\int_0^t idt + u_C(0) \doteq \frac{I(p)}{pC} + \frac{u_C(0)}{p} . \qquad (6.76)$$

The Laplace transform and the respective inverse Laplace transform are referred to as a transform pair.

A formal solution to a linear integrodifferential equation is readily obtained in an intermediate form by Laplace transforming each term. This procedure leads to the represented results.

Once the Laplace transform *F(p)* is found for a specific time function *f*(*t*), it is tabulated for future reference just as is done for logarithms or integrals. Such a tabulation is particularly useful because, as can be shown, the Laplace transform of a time function, if it exists, is unique. The most useful of the transform pairs are listed below in Table 6.1.

Table 6.1

$F(p)$	$f(t)$
$\dfrac{1}{p \pm \alpha}$	$e^{\mp \alpha t}$
$\dfrac{1}{p - j\omega}$	$e^{-j\omega t}$
$\dfrac{1}{p\,(p+\alpha)}$	$\dfrac{1}{\alpha}\,(1-e^{-\alpha t})$
$\dfrac{\alpha}{p\,(p+\alpha)}$	$1-e^{-\alpha t}$
$\dfrac{1}{(p+\alpha)^2}$	$te^{-\alpha t}$
$\dfrac{p}{(p+\alpha)^2}$	$(1-\alpha t)\; e^{-\alpha t}$
$\dfrac{1}{p\,(p+\alpha)^2}$	$\dfrac{t}{a}-\dfrac{1}{a^2}+\dfrac{e^{-\alpha t}}{a^2}$
$\dfrac{1}{(p+\alpha)(p+b)}$	$\dfrac{1}{a-b}(e^{-at}-e^{-bt})$
$\dfrac{p}{(p+\alpha)(p+b)}$	$\dfrac{1}{a-b}(a\,e^{-at}-b\,e^{-bt})$
$\dfrac{1}{p^2+a^2}$	$\dfrac{1}{a}\sin\,(at)$
$\dfrac{p}{p^2+a^2}$	$\cos\,(at)$
$\dfrac{1}{p^2}$	t
$\dfrac{1}{p^3}$	$\dfrac{1}{2}t^2$

6.19. Ohm's Law in Operational Form. Internal Electromotive Forces

Let's consider the electrical branch which consists of e.m.f. source and resistive, inductive and capacitor elements is connected to two nodes *a* and *b*. The current in the branch is directed from node *a* to node *b*.

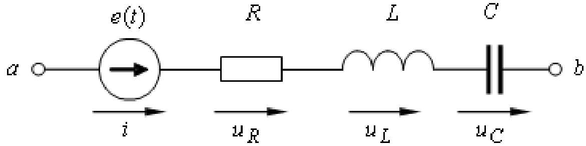


Fig. 6.19. A part of the complex ramified electrical circuit.

The switching in other part of the circuit leads to the transient. Independent initial conditions for a considered section are: the current through an inductor $i(0)=i(0-)$ and voltage across a capacitor $u_C(0)=u_C(0-)$.

The voltage between nodes a and b for the circuit after switching

$$u_{ab}=\varphi_a-\varphi_b=u_R+u_L+u_C-e(t) \qquad (6.77)$$

where $u_R=Ri$; $u_L=L\dfrac{di}{dt}$; $u_C=\dfrac{1}{C}\int\limits_0^t idt+u_C(0)$. After substituting

$$u_{ab}=Ri+L\frac{di}{dt}+\frac{1}{C}\int idt-e(t). \qquad (6.78)$$

Since the Laplace transform is a linear operator, it follows that the transform of a sum is equal to the sum of transforms. On substituting the respective transforms for each of the terms in Eq. (6.78), we get

$$U_{ab}(p)=RI(p)+pLI(p)-Li(0)+\frac{1}{pC}I(p)+\frac{u_C(0)}{p}-E(p). \qquad (6.79)$$

The effect of the above transformation is that instead of a differential equation (6.78) we have the algebraic equation (6.79) relating the current transform *I(p)* to the e.m.f. transform *E(p)* and the voltage transform $U_{ab}(p)$. From Eq. (6.78) it follows that

$$I(p)=\frac{U_{ab}(p)+E(p)+Li(0)-\dfrac{u_C(0)}{p}}{R+pL+\dfrac{1}{pC}} \qquad (6.80)$$

where $Z(p) = R + pL + \dfrac{1}{pC}$ is the *operation impedance* of the branch *ab*. It is similar in structure to the complex impedance of the same branch with p substituted for $j\omega$. The term operation impedance is preferred in this description because resistance, inductance and capacitance are involved.

Eq. (6.80) may be termed Ohm's law in operational form for the circuit branch containing a voltage source under non-zero *i.i.c.s.*

The term $Li(0)$ is the internal e.m.f. due to the energy stored by the magnetic field of an inductive element, produced by the current *i(0)* through the inductance just before a switching. The term $u_C(0)/p$ is the internal e.m.f. due to the energy stored by the electrostatic field of a capacitor, produced by the voltage across the capacitor just before a switching.

In a case, when a branch contains no e.m.f. and we have zero initial conditions at the instant of switching, Eq. (6.80) takes on a simpler form

$$I(p) = \frac{U_{ab}(p)}{Z(p)}\,. \qquad (6.81)$$

Eq. (6.81) is a mathematical expression for Ohm's law for the circuit branch containing no e.m.f., under zero independent initial conditions.

6.20. Kirchhoff's Laws in Operational Form

According to the *first Kirchhoff's law* an algebraic sum of instantaneous values of the currents in any junction of a circuit is equal to zero

$$\sum_{k=1}^{n} i_k = 0\,.$$

Current by Laplace transform can be presented in operation form

$$\sum_{k=1}^{n} I_k(p) = 0\,. \qquad (6.82)$$

According to the *2nd Kirchhoff's law* in a closed loop the sum of the instantaneous voltages is equal to the algebraic sum of electromotive forces:

$$\sum_{k=1}^{n} R_k i_k + \sum_{k=1}^{n} L_k \frac{di_k}{dt} + \sum_{k=1}^{n}\left(\frac{1}{C_k}\int_0^t i_k + u_{Ck}(0)\right) = \sum_{k=1}^{n} e_k\,. \qquad (6.83)$$

Having transferred to operational imagines, we have got the second Kirchhoff's law in the operational form as

$$\sum_{k=1}^{n} R_k I_k(p) + \sum_{k=1}^{n}\left(pL_k I_k(p) - L_k i_{Lk}(0)\right) + \sum_{k=1}^{n}\left(\frac{I_k(p)}{pC} + \frac{u_{Ck}(0)}{p}\right) = \sum_{k=1}^{n} E_k(p)$$

or $\sum_{k=1}^{n} Z_k(p) I_k(p) = \sum_{k=1}^{n}\left(E_k(p) + L_k i_{Lk}(0) - \frac{u_{Ck}(p)}{p}\right)$

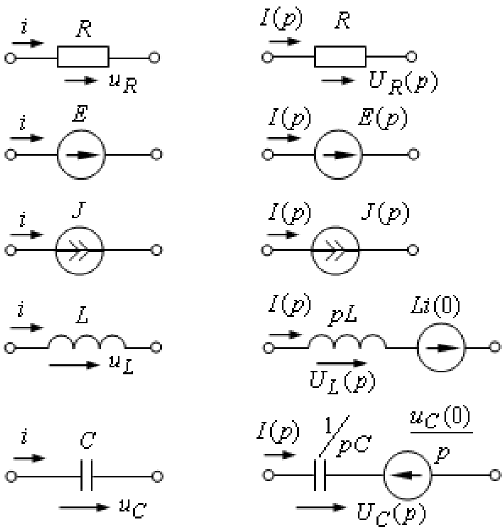
where $i_{Lk}(0)$, $u_{Ck}(0)$ are initial conditions of currents through inductor and voltages across capacitor. At zero i.i.c. the 2nd Kirchhoff's law:

$$\sum_{k=1}^{n} Z_k(p) I_k(p) = \sum_{k=1}^{n} E_k(p) \qquad (6.84)$$

In the case of nonzero initial conditions (the general case), $E_k(p)$ also includes all internal electromotive forces.

6.21. The equivalent operational circuit

The relationships between circuit elements for instantaneous values of currents and voltages and the elements of the operational circuit are presented at the end of this section. The internal e.m.f.s caused by a reserve of energy in reactive ele-ments at the moment of switching, are taken into account here. In case of zero i.i.c. the internal e.m.f.s are equal to zero and there are no internal e.m.f. in the equivalent operational circuit.



6.22. Inverse Laplace Transformation Through Partial Fraction Expansion

Let the image looks like a proper fraction (i.e. polynomial extent *m* in a numerator is less a polynomial *n* in a denominator $m < n$)

$$F(p) = \frac{F_1(p)}{F_2(p)}\,, \tag{6.85}$$

and a numerator and a denominator have no common roots.

To accomplish this in the common case, it is necessary first to find the roots of this equation by any of the standard methods of algebra. For the sake of illustration assume these roots are found to be , ,…and so on to and that each is real. Then Eq. (6-85) may be rewritten as of a partial-fraction expansion (6-86)

To accomplish this in the common case, it is necessary first to find the roots of this equation by any of the standard methods of algebra. For the sake of illustration assume these roots are found to be p_1, p_2, to p_n and that each is real. Eq. (6-85) may be rewritten in terms of a partial fraction

$$\frac{F_1(p)}{F_2(p)} = \frac{A_1}{p-p_1} + \frac{A_2}{p-p_2} + \ldots + \frac{A_n}{p-p_n} = \sum_{k=1}^{n} \frac{A_k}{p-p_k}\,. \tag{6.86}$$

Now since each term on the right side of this expression is identifiable in Table 6-1, a complete description of the time solution merely requires an evaluation of the *n* coefficients. Let us formulate the transformed solution of Eq. (6-86) in completely general terms and we may write

$$F(p) = \frac{F_1(p)}{F_2(p)} = \frac{A_1}{p-p_1} + \frac{A_2}{p-p_2} + \ldots + \frac{A_n}{p-p_n} = \sum_{k=1}^{n} \frac{A_k}{p-p_k}\,. \tag{6.87}$$

For definition of coefficient A_1 we will multiply both parts of Eq. (6.86) by $(p - p_1)$. We will get the following equation

$$\frac{F_1(p)}{F_2(p)}(p-p_1) = A_1 + (p-p_1)\sum_{k=2}^{n} \frac{A_k}{p-p_k}\,. \tag{6.88}$$

At $p \to p_1$, Eq. (6.88) may be rewritten thus

$$A_1 = \frac{F_1(p_1)}{F_2'(p_1)}\,.$$

Analogically one can determine the remaining coefficients as

$$A_k = \frac{F_1(p_k)}{F_2'(p_k)}. \qquad (6.89)$$

Hence, $F(p) = \dfrac{F_1(p)}{F_2(p)} = \sum_{k=1}^{n} \dfrac{A_k}{p - p_k} = \sum_{k=1}^{n} \dfrac{F_1(p_1)}{F_2'(p_1)} \cdot \dfrac{1}{p - p_k}$ (6.90)

As it has been shown earlier that

$$\frac{A_k}{p - p_k} \,\dot{=}\, A_k e^{p_k t},$$

$F_1(p_k)/F_2'(p_k)$ on the right-hand side of Eq. (6.90) are constants, and that the functions of *p* on the right-hand side are the factors $1/(p - p_k)$ corresponding to a time function of form $e^{p_k t}$ (see Eq. 6.69), we may write

$$f(t) = \sum_{k=1}^{n} \frac{F_1(p_k)}{F_2'(p_k)} e^{p_k t}. \qquad (6.91)$$

The inverse transformation with the aid of Eq. (6.91) is based on the expansion of the transform into partial fractions of the form $[F_1(p_k)/F_2'(p_k)][1/(p - p_k)]$, whose corresponding time functions are exponential functions $[F_1(p_k)/F_2'(p_k)]\, e^{p_k t}$.

The expression in the right-hand side of Eq. (6.90) has as many terms as there are roots of the equation *F(p)* = 0. The factors $F_1(p_k)/F_2'(p_k)$ may be likened to the constants of integration of a differential equation in the classical method. If one of the roots of the equation *F(p)* = 0 is *p* = 0, and the corresponding term on the right-hand side of Eq. (6.91) will be

$$f(t) = \frac{F_1(0)}{F_2'(0)} + \sum_{k=2}^{n} \frac{F_1(p_k)}{F_2'(p_k)} e^{p_k t}. \qquad (6.92)$$

The term is the component of the unknown cur-rent or voltage due to a direct-current voltage. If equation has complex conjugate roots, the sum of com-plex conjugate magnitudes is equal to the doubled value of the real part of any of these numbers. And it is enough to calculate the val-ue of this function only for the root and find the real part of this term.

$$\frac{F_1(p_1)e^{p_1 t}}{F_2'(p_1)} + \frac{F_1(p_2)e^{p_2 t}}{F_2'(p_2)} = 2\,\mathrm{Re}\left(\frac{F_1(p_1)e^{p_1 t}}{F_2'(p_1)}\right). \qquad (6.93)$$

The apportionment into vulgar fractions can be made by the method of undetermined coefficients which is related to the solution of the system of equations, that it is not convenient in practice.

Required time function looks like

$$f(t)=e^{p_1 t}\sum_{q=1}^{m}\frac{1}{(q-1)!\cdot(m-q)!}\left(\frac{d^{q-1}}{dp^{q-1}}\left(\frac{F_1(p)}{F_3(p)}\right)\right)_{p=p_1} t^{m-q}+$$

$$+\sum_{k=2}^{n}\frac{F_1(p_k)}{F_2^{'}(p_k)}e^{p_k t}\ . \qquad (6.94)$$

6.23. Common steps in operational analysis

Transient analysis by an operational method is made in the following sequence in connection with the Heaviside expansion theorem:

1. The analysis of a circuit before switching and the definition of independent initial conditions as in a classical method. Note that expansion formula is applicable under any i.i.c.

2. Non-zero i.i.c. are dealt with by introducing "internal" electromotive forces in N(p). Then draw equivalent operational circuit immediately after switching. This formulation of equivalent operational circuit is made by replacement of energy sources and passive elements by their images.

3. Make up the system of equations in an operational form. This system can be generated by any of methods which we use for the calculation of steady-state modes in the linear circuits for the equivalent operational circuit.

4. The solution of this system of equations concerning the images of required currents or voltages by any known method.

5. The definition of originals of required currents or voltages through the use of Eq. (6.91) or table 6.1. If the image of function represents the ratio of two polynomials rather *p*, it is possible to take advantage of expansion theorem for making up of an inverse Laplace transform.

6.24. Step Response of a Direct Current RL Circuit

Consider the transient in *dc* circuit which is presented in Fig. 6.20.

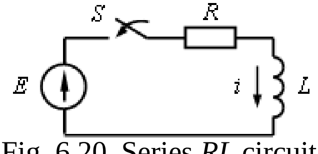


Fig. 6.20. Series *RL* circuit.

Appearing in Fig. 6.20 is the circuit arrangement of an initially deenergized inductor in series with a resistor. It is desired to find the complete solution for the current when the switch is closed.

By Kirchhoff's voltage law one can write

$$E = Ri + L\frac{di}{dt} \tag{6.95}$$

Because the switch in Fig, 6.20 is initially open, it follows that

$$i(0) = 0\,.$$

Since we have a zero i.i.c. here the internal e.m.f. due to the energy stored by the magnetic field is also equal to zero. Then we have the following operational circuit which is presented in Fig. 6.21.

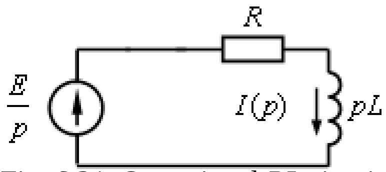


Fig. 6.21. Operational *RL* circuit.

Substituting the respective transforms for each term of Eq. (6.95) gives

$$\frac{E}{p} = I(p)\ (R + pL)\,.$$

Therefore, the transformed solution for the response becomes

$$I(p) = \frac{E/p}{R + pL} = \frac{E/L}{p\,(p + R/L)}\,. \tag{6.96}$$

To obtain the time solution which corresponds to Eq. (6.96) we must perform the inverse Laplace transformation, which requires putting I(p) into a form whose terms are readily identifiable in Table 6.1. Accordingly, we can rewrite this equation as $I(p) = \dfrac{E/L}{p\,(p + R/L)} = \dfrac{E}{L}\cdot\dfrac{1}{p\,(p+\alpha)}$,

where $\alpha = R/L$.

From the table of Laplace transform pairs we find that

$$\frac{1}{p\ (p+\alpha)}\ \;\dot{=}\;\ \frac{1}{\alpha}\,(1 - e^{-\alpha t})$$

By the application of this transform we have the following result

$$i(t) = \frac{E}{L}\left[\frac{1}{R/L}\left(1 - e^{-(R/L)t}\right)\right] = \frac{E}{R} - \frac{E}{R}\,e^{-(R/L)t} \tag{6.97}$$

where the term E/R is a steady-state component of the current, and the term $\frac{E}{R}e^{-(R/L)t}$ is a transient (or free) component of the required current. Thus, by means of Table 6.1 this operation is carried out by proceeding from the *p*-function column to the corresponding time-function column. It is interesting to note in Eq. (6.96) that the presence of *p* in the denominator is associated with the step forcing function, and this in turn is responsible for generating the steady-state solution. The operational impedance ($R + pL$) is associated with the generation of the transient term, as also revealed by Eq. (6.97).

It is useful to formulate the ratio of the output response (in this case, current) to the input forcing function (voltage) of the Laplace-transformed function for zero initial conditions. This ratio is called the *transfer function* of the circuit. In the case of the RL circuit the transfer function can be written as

$$\frac{I(p)}{E(p)} = \frac{1}{Z(p)} = \frac{1}{R + pL}. \qquad (6.98)$$

Right side of Eq. (6.98) indicates, the transfer function indirectly conveys information about the transient response of the circuit. The transfer function involves just the circuit parameters.

6.25. Step Response of a Direct Current RC Circuit

Consider the circuit with a resistor in series with a capacitor.

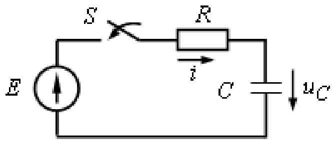


Fig. 6.22. Series RC circuit.

Let us now find the current after the switch S is closed. Taking voltage drops in a clockwise direction as positive the governing differential equation is obtained by KVL to the closed circuit: $E = Ri + \frac{1}{C}\int idt$ (6.99)

Laplace transforming each term yields

$$\frac{E}{p} = I(p)\ (R + \frac{1}{Cp}). \qquad (6.100)$$

As the switch was open, it follows that *uC (0) = 0*. Than e.m.f. due to the energy stored by the electrostatic field of a capacitor also equal to zero. Then we have the following operational circuit described in Fig. 6.23.

Therefore, the transformed solution for the response becomes

$$I(p)=\frac{E/p}{R+\dfrac{1}{Cp}}=\frac{ECp}{p(RCp+1)}=\frac{E}{R}\cdot\frac{1}{\left(p+\dfrac{1}{RC}\right)}=\frac{E}{R}\cdot\frac{1}{(p+\alpha)} \qquad (6.101)$$

where $\alpha=1/RC$.

From the table of Laplace transform pairs we find that

$$\frac{1}{(p+\alpha)} \;\dot{=}\; e^{-\alpha t}.$$

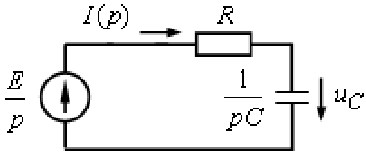


Fig. 6.23. Operational RC circuit

By the application of this transform we have the following result

$$i(t)=\frac{E}{R}\,e^{-\frac{1}{RC}t}. \qquad (6.102)$$

So, by means of inverse Laplace-transformed functions (see Table 6.1) this operation is carried out by proceeding from the *p*-function form to the corresponding time-function form.

6.26. Step Response of Second-Order system (*RLC* Circuit)

Let consider the circuit with series *RLC*. Note, that the capacitor has no initial charge before switching.

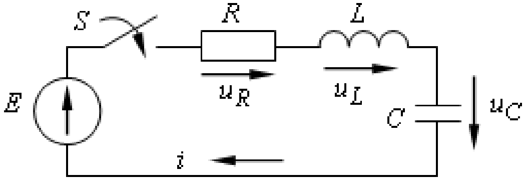


Fig. 6.24

Because the switch in Fig, 6.24 is initially open, voltage $u_C(0)=0$. So, the internal e.m.f. due to the energy stored by the electrostatic field of a capacitor, is also equal to zero and we have the following operational circuit

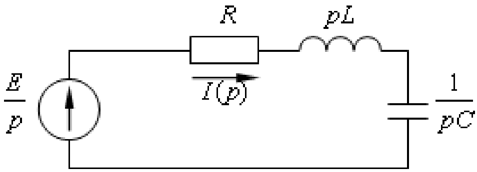


Fig. 6.25. Operational RLC circuit.

The governing differential equation for the circuit is found upon applying Kirchhoff's voltage law. Thus

$$E = Ri + L\frac{di}{dt} + \frac{1}{c}\int idt\ . \qquad (6.103)$$

Equation (6.103) is a second-order differential equation because it involves a derivative and integral term. The Laplace transform gives

$$\frac{E}{p} = I(p)\cdot\left(R + pL + \frac{1}{pC}\right). \qquad (6.104)$$

The function in brackets is the operational impedance of *RLC* circuit. The expression for the current solution in the *p* domain readily becomes

$$I(p) = \frac{CE}{p^2 LC + pRC + 1} = \frac{E/L}{\left(p^2 + p\frac{R}{L} + \frac{1}{LC}\right)}. \qquad (6.105)$$

A glance at Eq. (6.105) shows that the denominator expression does not include an p factor standing alone as was the case for the RL circuit. This means that a constant term in the steady-state solution does not exist.

We must first find the specific roots of the quadratic expression. That is, we must determine those values of p which satisfy the equation

$$R + pL + \frac{1}{pC} = 0\ . \qquad (6.106)$$

This expression is called the *characteristic equation* of the second-order system. The roots of this equation, depend on the circuit parameters. To understand this better, consider that *p1* and *p2* are the roots of the characteristic equation. The expression for the time solution becomes

$$i(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t}\ . \qquad (6.107)$$

Thus, the characteristic manner in which the transient terms decay is solely dependent on the roots p1 and p2. As an example we will consider the transient in the *dc* circuit presented in Figure 6.26.

Example 6.5. In the circuit of Fig.6.26 with parameters E = *60 V*, $R_1 = 10\,\Omega$, $R_2 = 20\,\Omega$, $R_3 = 30\,\Omega$, $L = 0.1\,H$, $C = 0.002\,F$ find the expression for the currents when the switch S is closed.

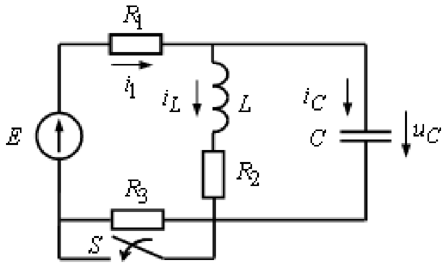


Fig. 6.26

Solution: Let's define independent initial conditions:

$$i_L(0) = i_L(0-) = \frac{E}{R_1 + R_2 + R_3} = \frac{60}{10+20+30} = 1\ A\,.$$

$$u_C(0) = u_C(0-) = R_2 i_L(0-) = 20\,\frac{60}{10+20+30} = 20\ V$$

The equivalent operational circuit after switching is in Fig. 6.27. Let's work out the equations by mesh-current method:

$$\begin{cases} I_{11}(p)(R_1 + R_2 + pL) - I_{22}(R_2 + pl) = \dfrac{E}{p} + Li_L(0) \\ -I_{11}(R_2 + pl) + I_{22}(p)\left(R_2 + pL + \dfrac{1}{pC}\right) = -Li_L(0) - \dfrac{u_C(0)}{p} \end{cases}$$

Solve the system of equations by matrix method. Than required current:

$$I_{11}(p) = I_1(p) = \frac{E + EpC(R_2 + pL) + pLi_L(0) - u_C(0)pC(R_2 + pL)}{p\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} =$$

$$= \frac{E + EpC(R_2 + pL)}{pZ(p)} + \frac{Li_L(0)}{Z(p)} - \frac{Cu_C(0)(R_2 + pL)}{Z(p)}\,.$$

$$I_{22}(p) = I_2(p) = \frac{\left(E(R_2 + pL) - R_1pLi_L(0) - u_C(p)(R_1 + R_2 + pL)\right)C}{\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} =$$

$$= \frac{EC(R_2 + pL)}{Z(p)} - \frac{pLCR_1i_L(0)}{Z(p)} - \frac{Cu_C(p)(R_1 + R_2 + pL)}{Z(p)}\,,$$

where $Z(p) = 0$ is the characteristic equation.

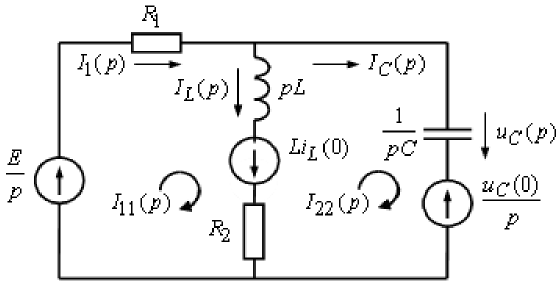


Fig. 6.27. The equivalent operational circuit after switching.

Now we pass from the image of a current to the original with the use of expansion theorem in case of the real different roots.

$$I_1(p) = \frac{E + EpC(R_2 + pL) + pLi_L(0) - u_C(0)pC(R_2 + pL)}{p\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} = \frac{F_1(p)}{pZ(p)};$$

$$I_2(p) = \frac{\left(E(R_2 + pL) - R_1pLi_L(0) - u_C(p)(R_1 + R_2 + pL)\right)C}{\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} = \frac{F_2(p)}{Z(p)}.$$

Let's consider characteristic equation and find its roots:

$$p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2) = 0$$

$$p_{1,2} = \frac{-(CR_1R_2 + L) \pm \sqrt{(CR_1R_2 + L)^2 - 4LCR_1(R_1 + R_2)}}{2LCR_1} =$$

$$= \frac{-(0.002 \cdot 10 \cdot 20 + 0.1) \pm \sqrt{(0.002 \cdot 10 \cdot 20 + 0.1)^2 - 4 \cdot 0.1 \cdot 0.002 \cdot 10(10 + 20)}}{2 \cdot 0.1 \cdot 0.002 \cdot 10} =$$

$$= -150; \quad -100\ s^{-1}.$$

Let's observe the receiving of the time solution for the first current. As the denominator contains a factor p we search original by the formula

$$i_1(t) = \frac{F_1(0)}{Z(0)} + \sum_{k=1}^{2} \frac{F_1(p_k)}{p_k Z'(p_k)} e^{p_k t}.$$

$$F_1(0) = E = 60.$$

$$F_1(-150) = 60 + 60 \cdot (-150) \cdot 0.002 \cdot (20 + (-150) \cdot 0.1) + (-150) \cdot 0.1 \cdot 1 -$$
$$- 20 \cdot (-150) \cdot 0.002 \cdot (20 + (-150) \cdot 0.1) = -15.$$

$$F_1(-100) = 60 + 60 \cdot (-100) \cdot 0.002 \cdot (20 + (-100) \cdot 0.1) + (-100) \cdot 0.1 \cdot 1 -$$
$$- 20 \cdot (-100) \cdot 0.002 \cdot (20 + (-100) \cdot 0.1) = -30;$$

$$Z(0) = R_1 + R_2 = 10 + 20 = 30\,;$$

$$Z'(p) = 2pLCR_1 + (CR_1R_2 + L)\,;$$

$$Z'(-150) = 2\cdot(-150)\cdot 0.1\cdot 0.002\cdot 10 + (0.002\cdot 10\cdot 20 + 0.1) = -0.1\,;$$

$$Z'(-100) = 2\cdot(-100)\cdot 0.1\cdot 0.002\cdot 10 + (0.002\cdot 10\cdot 20 + 0.1) = 0.1\,;$$

$$i_1(t) = \frac{F_1(0)}{Z(0)} + \frac{F_1(p_1)}{p_1 Z'(p_1)}e^{p_1 t} + \frac{F_1(p_2)}{p_2 Z'(p_2)}e^{p_2 t} =$$

$$= \frac{60}{20} + \frac{(-15)}{(-150)(-0.1)}\cdot e^{-150t} + \frac{(-30)}{(-100)(0.1)}\cdot e^{-100t} =$$

$$= 2 - 1\cdot e^{-150t} + 3\cdot e^{-100t}\ \text{A}.$$

Let's find time solution for the 2nd current. As the denominator does not contain a factor p we search the current original by the formula

$$i_2(t) = \sum_{k=1}^{2}\frac{F_2(p_k)}{Z'(p_k)}e^{p_k t}\,;$$

$$F_2(-150) = (60\cdot(20 + (-150)\cdot 0.1) - 10\cdot(-150)\cdot 0.1\cdot 1)\cdot 0.002 -$$
$$- 20\cdot(10 + 20 + (-150)\cdot 0.1)\cdot 0.002 = 0.3\,;$$

$$F_2(-100) = (60\cdot(20 + (-100)\cdot 0.1) - 10\cdot(-100)\cdot 0.1\cdot 1)\cdot 0.002 -$$
$$- 20\cdot(10 + 20 + (-100)\cdot 0.1)\cdot 0.002 = 0.6\,;$$

$$i_2(t) = \frac{F_2(p_1)}{Z'(p_1)}e^{p_1 t} + \frac{F_2(p_2)}{Z'(p_2)}e^{p_2 t} = \frac{0.3}{(-0.1)}\cdot e^{-150t} + \frac{0.6}{(0.1)}\cdot e^{-100t} =$$

$$= -3\cdot e^{-150t} + 6\cdot e^{-100t}\ A.$$

Consider the case with two *conjugate roots*. The roots of the characteristic equation and a form of a free component depend on the parameters of a circuit. Suppose, that one of parameters has changed (for example, inductance has increased four times and became equal $L = 0.4\ H$). As i.i.c.s don't depend on the inductance, they remained invariable.

In this case the roots of the characteristic equation look like

$$p_{1,2} = \frac{-(0.002\cdot 10\cdot 20 + 0.4) \pm \sqrt{(0.002\cdot 10\cdot 20 + 0.4)^2 - 4\cdot 0.4\cdot 0.002\cdot 10(10 + 20)}}{2\cdot 0.4\cdot 0.002\cdot 10} =$$

$$= -50 + j35.4;\quad -5 - j35.4\ \text{s}^{-1}.$$

Find the time function for the current in the first branch:

$$i_1(t)=\frac{F_1(0)}{Z(0)}+2\,\mathrm{Re}\left(\frac{F_1(p_1)}{p_1 Z'(p_1)}e^{p_1 t}\right);$$

$$F_1(0)=E=60\,;$$

$$F_1(-50+j35.4)=60+60\cdot(-50+j35.4)\cdot 0.002\cdot(20+(-50+j35.4)\cdot 0.4)+$$
$$+(-50+j35.4)\cdot 0.4\cdot 1-20\cdot(-50+j35.4)\cdot 0.002\times$$
$$\times(20+(-50+j35.4)\cdot 0.4)=-0.10+j42.48\,;$$

$$Z(0)=R_1+R_2=10+20=30\,;$$

$$Z'(p)=2pLCR_1+(CR_1R_2+L)\,;$$

$$Z'(-50+j35.4)=2\cdot(-50+j35.4)\cdot 0.4\cdot 0.002\cdot 10+(0.002\cdot 10\cdot 20+0.4)=$$
$$=-j0.566\,;$$

$$i_1(t)=\frac{60}{30}+2\,\mathrm{Re}\left(\frac{-0.10+j42.48}{(-50+j35.4)(-j0.566)}e^{-(50+j35.4)t}\right)=$$
$$=2+2.448e^{-50t}\sin(35.4t+55^{o})\,A\ .$$

Find the time function for the current in the second branch:

$$i_2(t)=2\,\mathrm{Re}\left(\frac{F_2(p_k)}{Z'(p_k)}e^{p_k t}\right);$$

$$F_2(-50+j35.4)=[60\cdot(20+(-50+j35.4)\cdot 0.4)-10\cdot(-50+j35.4)\cdot 0.4\cdot 1)-$$
$$-20\cdot(10+20+(-50+j35.4)\cdot 0.4)]\cdot 0.002=-j0.850\,;$$

$$i_2(t)=2\,\mathrm{Re}\left(\frac{-j0.850}{-j0.566}e^{(-50+j35.4)t}\right)=2\,\mathrm{Re}\left(1.5e^{j0^{o}}e^{(-50+j35.4)t}\right)=$$
$$=3e^{-50t}\cos(35.4t)=3e^{-50t}\sin(35.4t+90^{o})\ A.$$