Some function approximations on the real axis in spaces of summable functions

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The presented work is dedicated to determination of the approximate properties of the truncated Fourier series and of the interpolating Lagrange polynomials in the spaces $L_{p\rho}$, $\rho(x) = \left(1+x^2\right)^{-1}$, $p > 1$, L_p, $p \geq 2$ on the real axis R by systems of functions

$$\omega_k(x) = \left(\frac{x-i}{x+i}\right)^k, \qquad k = 0, \pm 1, \ldots, \tag{1}$$

$$\psi_k(x) = \frac{2i}{x+i}\left(\frac{x-i}{x+i}\right)^k, \qquad k = 0, \pm 1, \ldots. \tag{2}$$

Let P_n (Q_n) is the operator that assigns to each function $f(x) \in L_{p\rho}$, $p > 1$ ($f_1(x) \in L_p$, $p \geq 2$) the segment of the Fourier series by the function system (1) ((2), respectively).

Theorem 1. If $f(x) \in L_{p\rho}$, $p > 1$, or $f(x) \in C$, then

$$\|f(x) - (P_n f)(x)\|_{L_{p\rho}} \to 0, \qquad n \to \infty;$$

if $f(x) \in L_{p\rho}^{(r)}$, $r \geq 1$, $p > 1$, then

$$\|f(x) - (P_n f)(x)\|_{L_{p\rho}} \leq d_1(n+1)^{-r};$$

if $f(x) \in C^{(r)}$, $r \geq 1$, then

$$\|f(x) - (P_n f)(x)\|_{L_{p\rho}} \leq d_2(n+1)^{-r}, \qquad p > 1;$$

if $f(x) \in H_\alpha^{(r)}$, $r \geq 0$, $0 < \alpha \leq 1$, then

$$\|f(x) - (P_n f)(x)\|_{L_{p\rho}} \leq d_3(n+1)^{-r-\alpha}, \qquad p > 1.$$

Here and further below d_i are certain constants independent on n.

Theorem 2. If $f_1(x) \in L_p$, $p \geq 2$, or $f_1(x) \in C_0$, then

$$\|f_1(x) - (Q_n f_1)(x)\|_{L_p} \to 0, \qquad n \to \infty, \qquad p \geq 2;$$

if $f_1(x) \in L_p^{(r)}$, $r \geq 1$, $p \geq 2$, then

$$\left\| f_1(x) - \left(Q_n f_1\right)(x) \right\|_{L_p} \leq d_4(n+1)^{-r};$$

if $f_1(x) \in C_{01}^{(r)}$, $r \geq 1$, then

$$\left\| f_1(x) - \left(Q_n f_1\right)(x) \right\|_{L_p} \leq d_5(n+1)^{-r}, \qquad p \geq 2;$$

if $f_1(x) \in \overset{\circ}{H}{}_{\alpha 1}^{(r)}$, $r \geq 0$, $0 < \alpha \leq 1$, then

$$\left\| f_1(x) - \left(Q_n f_1\right)(x) \right\|_{L_p} \leq d_6(n+1)^{-r-\alpha}, \qquad p \geq 2.$$

Theorems 1 and 2 establish sufficient conditions for the convergence of segments of Fourier series by the function systems (1) and (2) respectively in the spaces $L_{p\rho}$, $\rho(x) = \left(1+x^2\right)^{-1}$, $p > 1$, L_p, $p \geq 2$, on the real axis R, depending on the structural properties of approximated functions.

Let $f(x) \in C$. Then its Lagrange interpolation polynomial on the basic functions of the system (1) is defined as [1]:

$$\left(L_n f\right)(x) = \sum_{k=-n}^{n} \alpha_k \omega_k(x),$$

$$\alpha_k = \frac{1}{2n+1} \sum_{j=-n}^{n} f\left(x_j\right) e^{-\frac{2\pi i}{2n+1} jk},$$

where

$$x_j = -\operatorname{ctg} \frac{\pi}{2n+1} j, \qquad j = -n, \ldots, n. \tag{3}$$

Let $f_1(x) \in C_{01}$. Then its Lagrange interpolation polynomial on the basic functions of the system (1) is defined as [2]:

$$\left(l_n f_1\right)(x) = \sum_{k=-n}^{n} \alpha_k^{(1)} \omega_k(x),$$

$$\alpha_k^{(1)} = \frac{1}{2n+1} \sum_{j=-n}^{n} f_1\left(x_j\right) e^{-\frac{2\pi i}{2n+1} jk},$$

where x_j – interpolation points (3). In this case, it can be expressed as [2]:

$$\left(l_n f_1\right)(x) = -\sum_{k=-n}^{n} \gamma_k \psi_k(x),$$

where $\gamma_k = \alpha_1^{(1)} + \alpha_2^{(1)} + \ldots + \alpha_{k+1}^{(1)}$.

Theorem 3. If $f(x) \in C$, then

$$\left\| f(x) - \left(L_n f\right)(x) \right\|_{L_{p\rho}} \to 0, \qquad n \to \infty, \qquad p > 1;$$

if $f(x) \in C^{(r)}$, $r \geq 1$, then

$$\left\| f(x) - \left(L_n f\right)(x) \right\|_{L_{p\rho}} \leq d_7 (n+1)^{-r}, \qquad p > 1;$$

if $f(x) \in H_\alpha^{(r)}$, $r \geq 0$, $0 < \alpha \leq 1$ then

$$\left\| f(x) - \left(L_n f\right)(x) \right\|_{L_{p\rho}} \leq d_8 (n+1)^{-r-\alpha}, \qquad p > 1.$$

Theorem 4. If $f_1(x) \in C_{01}$, then

$$\left\| f_1(x) - \left(l_n f_1\right)(x) \right\|_{L_p} \to 0, \qquad n \to \infty, \qquad p \geq 2;$$

if $f_1(x) \in C_{01}^{(r)}$, $r \geq 1$ then

$$\left\| f_1(x) - \left(l_n f_1\right)(x) \right\|_{L_p} \leq d_9 (n+1)^{-r}, \qquad p \geq 2;$$

if $f_1(x) \in \overset{\circ}{H}{}_{\alpha 1}^{(r)}$, $r \geq 0$, $0 < \alpha \leq 1$, then

$$\left\| f_1(x) - \left(l_n f_1\right)(x) \right\|_{L_p} \leq d_{10} (n+1)^{-r-\alpha}, \qquad p \geq 2.$$

Theorems 3 and 4 establish sufficient conditions for convergence in the spaces $L_{p\rho}$, $\rho(x) = \left(1 + x^2\right)^{-1}$, $p > 1$ and L_p, $p \geq 2$, Lagrange interpolation polynomials on the real axis R by interpolation points (3) to the function to be approximated, respectively, of the spaces C and C_{01}.

REFERENCES

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