MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
NATIONAL UNIVERSITY
ZAPORIZHZHIA POLYTECHNIC

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BRIEF PRACTICAL COURSE OF HIGHER MATHEMATICS FOR ENGINEERING STUDENTS

Part 1
Linear Algebra. Analytic Geometry. Differential Calculus

Dedicated to the125th Anniversary of
National University Zaporizhzhia Polytechnic

Textbook

Zaporizhzhia • NU Zaporizhzhia Polytechnic • 2025

UDC 517(076)
A 62

Recommended for publication by the Academic Council
National University Zaporizhzhia Polytechnic
(minutes No 3 of 21.10, 2025)

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A62 Antonenko, N. M.
Brief Practical Course of Higher Mathematics for Engineering Students. Part 1. Linear Algebra. Analytic Geometry. Differential Calculus: textbook / N. M. Antonenko, A. V. Fasoliak. – Zaporizhzhia : NU Zaporizhzhia Polytechnic, 2025. – 135 c.
ISBN 978-617-529-471-0

The textbook provides concise theoretical material and solutions to typical problems on Linear Algebra, Vector Algebra, Analytical Geometry, Limits and Continuity of Functions of a Single Variable, Differential Calculus of Functions of a Single Variable, and Differential Calculus of Functions of Several Independent Variables. Each chapter includes practice exercises with answers to assist readers in applying the covered principles. It has been developed for students of technical specialties.

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ISBN 978-617-529-471-0

CONTENTS

INTRODUCTION

The textbook has been developed for students of technical specialties, who study Higher Mathematics in English. At National University Zaporizhzhia Polytechnic, Higher Mathematics in English is traditionally taught for students majoring in G3 Electrical Engineering. The primary objective of the current textbook is to facilitate comprehension and mastery of fundamental ideas and concepts in higher mathematics. Given the importance of mathematical knowledge and skills in technical sciences, a structured approach that combines essential theoretical knowledge and methods for solving practical problems has been proposed.

The first part of the textbook consists of the following chapters: Linear Algebra, Vector Algebra, Analytical Geometry, Limits and Continuity of Functions of a Single Variable, Differential Calculus of Functions of a Single Variable and Differential Calculus of Functions of Several Variables. Each chapter provides essential theoretical material, including definitions, concepts, equations and formulas, necessary for students to understand the approaches and methods of solving the problems in the current textbook. They also contain the solutions of the typical problems to illustrate application of the considered theoretical material. Students are provided with training exercises at the end of each chapter. These exercises are developed to assist students in comprehending the material in greater depth and to practice the knowledge and skills they have acquired. Each exercise contains the answer, enabling students to check their solutions and detect pieces of material which needs to be studied additionally. Some materials from the lectures of V.P. Chumachenko [4, 5] have been used in the preparation of this textbook.

The current textbook can be used for practical classes or as supplementary material for Higher Mathematics lectures conducted in English. In addition, it can be utilized by students to study and master the main ideas, concepts, and methods of linear and vector algebra, analytical geometry, and differential calculus.

1 LINEAR ALGEBRA

Matrices

A ***matrix*** is a rectangular array of numbers (or functions) enclosed in parentheses.

If a matrix has m rows and n columns then it is called an $m\times n$ ***matrix***. An $m\times n$ matrix is written as follows:

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ or } A=\left(a_{ij}\right),$$

where a_{ij} is called the ***entries*** or ***elements*** of the matrix, $i=\overline{1,m}$, $j=\overline{1,n}$.

A matrix which has as many rows as columns is called a ***square matrix***.

The diagonal of a square matrix A containing the entries a_{11}, a_{22}, …, a_{nn} is called the ***principal (main) diagonal***.

A square matrix with ones on the main diagonal and zeros elsewhere is called an ***identity*** (***unit***) ***matrix***. It is denoted by I .

A square matrix in which all entries above (below) the main diagonal are all zero is called a ***lower*** (***upper) triangular matrix***.

Matrix operations

1. Scalar multiplication. The ***product of any number*** k ***by any matrix*** $A=\left(a_{ij}\right)$ is a matrix $kA=\left(ka_{ij}\right)$ obtained by multiplying each entry of A by k .

2. Addition (Subtraction). ***Addition (subtraction)*** is defined only for matrices of the same dimensions. If $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$, then their ***sum*** (***difference***) is obtained by adding (subtracting) the corresponding entries: $A\pm B=\left(a_{ij}\pm b_{ij}\right)$.

3. Matrix multiplication. The ***product*** $C=AB$, where $A=\left(a_{ik}\right)$ and $B=\left(b_{kj}\right)$, is defined if and only if the number of columns of the first factor A coincides with the number of rows of the second factor B. By definition, the entry of the product matrix located in the i-th row and the j-th column is equal to the sum of products of the corresponding entries of the i-th row of A and the j-th column of B:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}.$$

4. Transposition. The ***transpose*** A^T of an $m \times n$ matrix A is an $n \times m$ matrix obtained by replacing the rows of A by its columns.

Problem 1.1. Matrices $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 0 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 \\ 3 & 1 \\ 0 & 5 \end{pmatrix}$, $F = \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix}$, $G = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $H = \begin{pmatrix} 4 & 7 \end{pmatrix}$ are given. Find:

a) $(C + 2D) \cdot F$; **c)** $D \cdot F$ and $F \cdot D$;

b) $A^T \cdot B - 5I$; **d)** $(A \cdot G)^T - H \cdot F$.

Solution.

a) *Step 1.* $2D = 2 \cdot \begin{pmatrix} -2 & 0 \\ 3 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 6 & 2 \\ 0 & 10 \end{pmatrix}$.

Step 2. $C + 2D = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 0 \\ 6 & 2 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 6 & 1 \\ 3 & 14 \end{pmatrix}$.

Step 3. $(C + 2D) \cdot F = \begin{pmatrix} -2 & 1 \\ 6 & 1 \\ 3 & 14 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -2 \cdot 2 + 1 \cdot 4 & -2 \cdot 0 + 1 \cdot (-3) \\ 6 \cdot 2 + 1 \cdot 4 & 6 \cdot 0 + 1 \cdot (-3) \\ 3 \cdot 2 + 14 \cdot 4 & 3 \cdot 0 + 14 \cdot (-3) \end{pmatrix} =$

$= \begin{pmatrix} 0 & -3 \\ 16 & -3 \\ 62 & -42 \end{pmatrix}$.

b) *Step 1.* $A^T = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & -2 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 4 & -2 \end{pmatrix}$.

Step 2. $A^T \cdot B = \begin{pmatrix} 2 & 1 \\ -3 & 0 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 5 \\ 3 & 0 & -2 \end{pmatrix} =$

$$= \begin{pmatrix} 2\cdot1+1\cdot3 & 2\cdot4+1\cdot0 & 2\cdot5+1\cdot(-2) \\ -3\cdot1+0\cdot3 & -3\cdot4+0\cdot0 & -3\cdot5+0\cdot(-2) \\ 4\cdot1+(-2)\cdot3 & 4\cdot4+(-2)\cdot0 & 4\cdot5+(-2)\cdot(-2) \end{pmatrix} = \begin{pmatrix} 5 & 8 & 8 \\ -3 & -12 & -15 \\ -2 & 16 & 24 \end{pmatrix}.$$

Step 3. $5I = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Step 4. $A^T \cdot B - 5I = \begin{pmatrix} 5 & 8 & 8 \\ -3 & -12 & -15 \\ -2 & 16 & 24 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 8 & 8 \\ -3 & -17 & -15 \\ -2 & 16 & 19 \end{pmatrix}$.

c) $D \cdot F = \begin{pmatrix} -2 & 0 \\ 3 & 1 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} -2\cdot2+0\cdot4 & -2\cdot0+0\cdot(-3) \\ 3\cdot2+1\cdot4 & 3\cdot0+1\cdot(-3) \\ 0\cdot2+5\cdot4 & 0\cdot0+5\cdot(-3) \end{pmatrix} =$

$$= \begin{pmatrix} -4 & 0 \\ 10 & -3 \\ 20 & -15 \end{pmatrix}.$$

The product $F \cdot D$ is not defined because the number of columns of the first factor F doesn't coincide with the number of rows of the second factor D.

d) *Step 1.* $A \cdot G = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot3+(-3)\cdot(-2)+4\cdot1 \\ 1\cdot3+0\cdot(-2)+(-2)\cdot1 \end{pmatrix} = \begin{pmatrix} 16 \\ 1 \end{pmatrix}$.

Step 2. $(A \cdot G)^T = \begin{pmatrix} 16 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 16 & 1 \end{pmatrix}$.

Step 3. $H \cdot F = \begin{pmatrix} 4 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 4\cdot2+7\cdot4 & 4\cdot0+7\cdot(-3) \end{pmatrix} = \begin{pmatrix} 36 & -21 \end{pmatrix}$.

Step 4. $(A\cdot G)^T - H\cdot F = (16 \quad 1) - (36 \quad -21) = (-20 \quad 22)$.

Answer: **a)** $\begin{pmatrix} 0 & -3 \\ 16 & -3 \\ 62 & -42 \end{pmatrix}$; **b)** $\begin{pmatrix} 0 & 8 & 8 \\ -3 & -17 & -15 \\ -2 & 16 & 19 \end{pmatrix}$;

c) $D\cdot F = \begin{pmatrix} -4 & 0 \\ 10 & -3 \\ 20 & -15 \end{pmatrix}$, $F\cdot D$ is not defined; **d)** $(-20 \quad 22)$.

Determinants

A determinant of the n-th order is a numerical expression associated with an $n\times n$ matrix.

1. A ***second-order determinant*** is specified by the equality

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

2. A ***third-order determinant*** is specified by the equality (the ***triangle rule)***

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} -$$

$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}.$$

3. The ***determinant of the*** n ***-th order*** is written as

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

A ***minor*** M_{ij} of an entry a_{ij} in a determinant of order n is a determinant of order $n-1$, obtained by deleting the i-th row and the j-th column in the original determinant.

A ***cofactor*** A_{ij} of an entry a_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

A ***determinant of any order*** is equal to the sum of the products of the entries of any row (column) and their corresponding cofactors.

The expansion of the determinant by the entries of the first row and second column has the following form:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} ,$$

$$\det A = a_{21}A_{21} + a_{22}A_{22} + \ldots + a_{2n}A_{2n} .$$

Properties of determinants

1. The value of a determinant remains unchanged if its rows (columns) are replaced by the respective columns (rows).

2. The sign of a determinant changes if any two rows (columns) are interchanged.

3. A common factor of the entries of any row (column) can be taken outside the determinant.

4. If any two rows (columns) of a determinant are identical, then the determinant is equal to zero.

5. If all the entries in any row or column of a determinant are zero, then the determinant is equal to zero.

6. If corresponding entries in two rows (columns) of a determinant are proportional, the value of the determinant is zero.

7. If all entries in a row or column of a determinant are expressed as the sum of two terms, then the determinant can be expressed as the sum of two separate determinants. For example:

$$\begin{vmatrix} a+f & b+e \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} f & e \\ c & d \end{vmatrix} .$$

8. The determinant does not change its value if each entry of one row (column) is adding to or subtracting from the corresponding entries of another row (column) multiplied by the same number.

9. The value of the determinant of a triangular matrix (upper triangular or lower triangular matrix) is equal to the product of the entries on the principal diagonal.

10. The sum of the products of the entries of any row (column) and the cofactors of the corresponding entries of a different row (column) is equal to zero.

Problem 1.2. Calculate the determinants:

a) $\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix}$; **b)** $\begin{vmatrix} 2 & 1 & -3 \\ -5 & 4 & 8 \\ 7 & 9 & 6 \end{vmatrix}$.

Solution.

a) $\begin{vmatrix} 2 & -4 \\ 6 & 8 \end{vmatrix} = 2 \cdot 8 - (-4) \cdot 6 = 40$.

b) Using the triangle rule, we obtain:

$$\begin{vmatrix} 2 & 1 & -3 \\ -5 & 4 & 8 \\ 7 & 9 & 6 \end{vmatrix} = 2 \cdot 4 \cdot 6 + 1 \cdot 8 \cdot 7 + (-5) \cdot 9 \cdot (-3) - (-3) \cdot 4 \cdot 7 -$$

$$-1 \cdot (-5) \cdot 6 - 8 \cdot 9 \cdot 2 = 209.$$

Answer: **a)** 40; **b)** 209.

Problem 1.3. Solve the equations:

a) $\begin{vmatrix} 2x+3 & 2 \\ x^2 & x-4 \end{vmatrix} = 0$; **b)** $\begin{vmatrix} x & 2 & 2 \\ 3 & x-2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$.

Solution.

a) $\begin{vmatrix} 2x+3 & 2 \\ x^2 & x-4 \end{vmatrix} = 0$, $(2x+3)(x-4) - 2x^2 = 0$, $-5x - 12 = 0$, $x = -2.4$.

b) Using the triangle rule, we get:

$$\begin{vmatrix} x & 2 & 2 \\ 3 & x-2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0,$$

$$x \cdot (x-2) \cdot 1 + 2 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 2 - 2 \cdot (x-2) \cdot 2 - 2 \cdot 3 \cdot 1 - 1 \cdot 1 \cdot x = 0,$$

$$x^2 - 7x + 12 = 0,\ x_1 = 3,\ x_2 = 4.$$

Answer: **a)** -2.4; **b)** 3; 4.

Problem 1.4. Calculate the determinant $\begin{vmatrix} 3 & 0 & 2 & -1 \\ 1 & -2 & 3 & 2 \\ 2 & 0 & -1 & 0 \\ -1 & 3 & 4 & 2 \end{vmatrix}$:

a) using expansion by the first row;
b) using expansion by the second column;
c) reducing it to the triangular form;
d) reducing it to the second order determinant.

Solution.

a) Let's expand the determinant by the first row:

$$\Delta = \begin{vmatrix} 3 & 0 & 2 & -1 \\ 1 & -2 & 3 & 2 \\ 2 & 0 & -1 & 0 \\ -1 & 3 & 4 & 2 \end{vmatrix} = 3 \cdot A_{11} + 0 \cdot A_{12} + 2 \cdot A_{13} + (-1) \cdot A_{14}.$$

Since the coefficient of A_{12} is zero, we calculate only the cofactors A_{11}, A_{13} and A_{14}:

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} -2 & 3 & 2 \\ 0 & -1 & 0 \\ 3 & 4 & 2 \end{vmatrix} = (-2) \cdot (-1) \cdot 2 + 3 \cdot 0 \cdot 3 + 0 \cdot 4 \cdot 2 -$$

$$-2 \cdot (-1) \cdot 3 - 3 \cdot 0 \cdot 2 - (-2) \cdot 4 \cdot 0 = 10,$$

$$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 1 & -2 & 2 \\ 2 & 0 & 0 \\ -1 & 3 & 2 \end{vmatrix} = 20,$$

$$A_{14} = (-1)^{1+4} M_{14} = -\begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ -1 & 3 & 4 \end{vmatrix} = -35,$$

$$\Delta = 3 \cdot 10 + 0 + 2 \cdot 20 + (-1) \cdot (-35) = 105.$$

b) Let's expand the determinant by the second column:

$$\Delta = \begin{vmatrix} 3 & 0 & 2 & -1 \\ 1 & -2 & 3 & 2 \\ 2 & 0 & -1 & 0 \\ -1 & 3 & 4 & 2 \end{vmatrix} = 0 \cdot A_{12} + (-2) \cdot A_{22} + 0 \cdot A_{32} + 3 \cdot A_{42}.$$

Since the coefficients of A_{12} and A_{32} are zero, we calculate only the cofactors A_{22}, A_{42}:

$$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix}3 & 2 & -1\\ 2 & -1 & 0\\ -1 & 4 & 2\end{vmatrix}=-21,$$

$$A_{42}=(-1)^{4+2}M_{42}=\begin{vmatrix}3 & 2 & -1\\ 1 & 3 & 2\\ 2 & -1 & 0\end{vmatrix}=21,$$

$$\Delta=0+(-2)\cdot(-21)+0+3\cdot 21=105.$$

c) By applying properties 2, 3 and 8, we reduce the given determinant to the triangular form. Then, by using property 9, we compute it:

$$\Delta=\begin{vmatrix}3 & 0 & 2 & -1\\ 1 & -2 & 3 & 2\\ 2 & 0 & -1 & 0\\ -1 & 3 & 4 & 2\end{vmatrix}\begin{matrix}r_1\leftrightarrow r_2\\ \\ \\ \\ \end{matrix}\overset{pr.2}{=}-\begin{vmatrix}1 & -2 & 3 & 2\\ 3 & 0 & 2 & -1\\ 2 & 0 & -1 & 0\\ -1 & 3 & 4 & 2\end{vmatrix}\begin{matrix} \\ r_2\to r_2-3r_1\\ r_3\to r_3-2r_1\\ r_4\to r_4+r_1\end{matrix}\overset{pr.8}{=}$$

$$\overset{pr.8}{=}-\begin{vmatrix}1 & -2 & 3 & 2\\ 0 & 6 & -7 & -7\\ 0 & 4 & -7 & -4\\ 0 & 1 & 7 & 4\end{vmatrix}\begin{matrix} \\ r_2\leftrightarrow r_4\\ \\ \\ \end{matrix}\overset{pr.2}{=}\begin{vmatrix}1 & -2 & 3 & 2\\ 0 & 1 & 7 & 4\\ 0 & 4 & -7 & -4\\ 0 & 6 & -7 & -7\end{vmatrix}\begin{matrix} \\ \\ r_3\to r_3-4r_2\\ r_4\to r_4-6r_2\end{matrix}\overset{pr.8}{=}$$

$$\overset{pr.8}{=}\begin{vmatrix}1 & -2 & 3 & 2\\ 0 & 1 & 7 & 4\\ 0 & 0 & -35 & -20\\ 0 & 0 & -49 & -31\end{vmatrix}\overset{pr.3}{=}-5\cdot\begin{vmatrix}1 & -2 & 3 & 2\\ 0 & 1 & 7 & 4\\ 0 & 0 & 7 & 4\\ 0 & 0 & -49 & -31\end{vmatrix}\begin{matrix} \\ \\ \\ r_4\to r_4+7r_3\end{matrix}\overset{pr.8}{=}$$

$$\overset{pr.8}{=}-5\cdot\begin{vmatrix}1 & -2 & 3 & 2\\ 0 & 1 & 7 & 4\\ 0 & 0 & 7 & 4\\ 0 & 0 & 0 & -3\end{vmatrix}\overset{pr.9}{=}-5\cdot 1\cdot 1\cdot 7\cdot(-3)=105.$$

d) Using property 8, we make the element $a_{42}=3$ zero, and then expand the resulting determinant by the second column:

$$\Delta=\begin{vmatrix}3&0&2&-1\\1&-2&3&2\\2&0&-1&0\\-1&3&4&2\end{vmatrix}\begin{matrix}\\ r_2\to r_2+r_4\\ \\ \\ \end{matrix}\overset{pr.8}{=}\begin{vmatrix}3&0&2&-1\\0&1&7&4\\2&0&-1&0\\-1&3&4&2\end{vmatrix}\begin{matrix}\\ \\ \\ r_4\to r_4-3r_2\end{matrix}\overset{pr.8}{=}$$

$$\overset{pr.8}{=}\begin{vmatrix}3&0&2&-1\\0&1&7&4\\2&0&-1&0\\-1&0&-17&-10\end{vmatrix}=1\cdot A_{22}=\begin{vmatrix}3&2&-1\\2&-1&0\\-1&-17&-10\end{vmatrix}.$$

By applying property 8 to the obtained third-order determinant, we make the element $a_{33}=-10$ zero and then expand the resulting determinant by the third column:

$$\begin{vmatrix}3&2&-1\\2&-1&0\\-1&-17&-10\end{vmatrix}\begin{matrix}\\ \\ r_3\to r_3-10r_1\end{matrix}\overset{pr.8}{=}\begin{vmatrix}3&2&-1\\2&-1&0\\-31&-37&0\end{vmatrix}=-1\cdot A_{13}=$$

$$=-\begin{vmatrix}2&-1\\-31&-37\end{vmatrix}=-(2\cdot(-37)-(-1)\cdot(-31))=105.$$

Answer: **a)** 105; **b)** 105; **c)** 105; **d)** 105.

Inverse matrix

Let A be a square matrix. A matrix A^{-1} is called an ***inverse matrix of*** A if $A\cdot A^{-1}=A^{-1}\cdot A=I$, where I is an identity matrix.

If $\det A\neq 0$, the inverse matrix of A exists and can be found using the following formula:

$$A^{-1}=\frac{1}{\det A}\cdot A^{*}, \qquad (1.1)$$

where $A^{*}=\left(A_{ij}\right)^{T}=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1n}\\A_{21}&A_{22}&\cdots&A_{2n}\\ \cdot&\cdot&\cdot&\cdot\\A_{n1}&A_{n2}&\cdots&A_{nn}\end{pmatrix}^{T}$ is the ***adjoint matrix of*** A, A_{ij} are cofactors of a_{ij} in $\det A$.

Solution of matrix equations

Let A and B be the given matrices, and let X be the unknown matrix.

The solution of the equation $A\cdot X = B$ is given by the formula

$$X = A^{-1}\cdot B\,. \tag{1.2}$$

The solution of the equation $X\cdot A = B$ is given by the formula

$$X = B\cdot A^{-1}\,. \tag{1.3}$$

Problem 1.5. Solve the matrix equations:

a) $\begin{pmatrix} 1 & -4 \\ 5 & -2 \end{pmatrix}\cdot X = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$; **b)** $X\cdot\begin{pmatrix} 3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -5 & 2 \end{pmatrix}$.

Solution.

a) The given matrix equation has the form $A\cdot X = B$, where $A = \begin{pmatrix} 1 & -4 \\ 5 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$.

Let's find the inverse matrix of A:

1) we calculate $\det A$: $\det A = \begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} = 1\cdot(-2)-(-4)\cdot 5 = 18$;

2) we find the cofactors of all its entries:

$$A_{11} = (-1)^{1+1}\cdot(-2) = -2\,,\quad A_{12} = (-1)^{1+2}\cdot 5 = -5\,,$$

$$A_{21} = (-1)^{2+1}\cdot(-4) = 4\,,\quad A_{22} = (-1)^{2+2}\cdot 1 = 1\,;$$

3) we find the adjoint matrix of A:

$$A^{*} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{T} = \begin{pmatrix} -2 & -5 \\ 4 & 1 \end{pmatrix}^{T} = \begin{pmatrix} -2 & 4 \\ -5 & 1 \end{pmatrix};$$

4) by (1.1), the inverse of matrix A is $A^{-1} = \dfrac{1}{18}\begin{pmatrix} -2 & 4 \\ -5 & 1 \end{pmatrix}$.

Using formula (1.2), we get:

$$X = \frac{1}{18}\begin{pmatrix} -2 & 4 \\ -5 & 1 \end{pmatrix}\cdot\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -2\cdot 6+4\cdot 2 & -2\cdot(-1)+4\cdot 3 \\ -5\cdot 6+1\cdot 2 & -5\cdot(-1)+1\cdot 3 \end{pmatrix} =$$

$$= \frac{1}{18}\begin{pmatrix} -4 & 14 \\ -28 & 8 \end{pmatrix} = \begin{pmatrix} -\frac{2}{9} & \frac{7}{9} \\ -\frac{14}{9} & \frac{4}{9} \end{pmatrix}.$$

b) The given matrix equation has the form $X \cdot A = B$, where $A = \begin{pmatrix} 3 & 1 \\ -2 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ -5 & 2 \end{pmatrix}$.

Let's find the inverse matrix of A:

1) we calculate $\det A$:

$$\det A = \begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix} = 3 \cdot (-4) - 1 \cdot (-2) = -10 ;$$

2) we find the cofactors of all its entries:

$$A_{11} = (-1)^{1+1} \cdot (-4) = -4 , \quad A_{12} = (-1)^{1+2} \cdot (-2) = 2 ,$$

$$A_{21} = (-1)^{2+1} \cdot 1 = -1 , \quad A_{22} = (-1)^{2+2} \cdot 3 = 3 ;$$

3) we find the adjoint matrix of A:

$$A^* = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} -4 & 2 \\ -1 & 3 \end{pmatrix}^T = \begin{pmatrix} -4 & -1 \\ 2 & 3 \end{pmatrix} ;$$

4) by (1.1), the inverse of matrix A is $A^{-1} = -\frac{1}{10} \begin{pmatrix} -4 & -1 \\ 2 & 3 \end{pmatrix}$.

Using formula (1.3), we get:

$$X = \begin{pmatrix} 3 & 0 \\ -5 & 2 \end{pmatrix} \cdot \left(-\frac{1}{10} \right) \cdot \begin{pmatrix} -4 & -1 \\ 2 & 3 \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} 3 \cdot (-4) + 0 \cdot 2 & 3 \cdot (-1) + 0 \cdot 3 \\ -5 \cdot (-4) + 2 \cdot 2 & -5 \cdot (-1) + 2 \cdot 3 \end{pmatrix} =$$

$$= -\frac{1}{10} \begin{pmatrix} -12 & -3 \\ 24 & 11 \end{pmatrix} = \begin{pmatrix} 1.2 & 0.3 \\ -2.4 & -1.1 \end{pmatrix} .$$

Answer: **a)** $X = \begin{pmatrix} -\frac{2}{9} & \frac{7}{9} \\ -\frac{14}{9} & \frac{4}{9} \end{pmatrix}$; **b)** $X = \begin{pmatrix} 1.2 & 0.3 \\ -2.4 & -1.1 \end{pmatrix}$.

Solution of system of three linear equations in three unknowns

A system of three linear equations in three unknowns x_1, x_2, x_3 has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases} \qquad (1.4)$$

where a_{ij} are given numbers, called the ***coefficients of the system***, b_i are also given numbers, called the ***free terms*** of the system, $i=\overline{1,3}$, $j=\overline{1,3}$.

System of equations (1.4) can be written in the ***matrix form***:

$$AX = B, \tag{1.5}$$

where $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is called the ***coefficient matrix***, $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the ***column vector of the unknowns***, $B=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the ***column vector of the free terms***.

If the system of equations has at least one solution then it is called ***consistent***. If the system doesn't have solution, it is called ***inconsistent***.

Matrix solution of the system. If $\det A \neq 0$, then the solution of (1.5) has the following form:

$$X = A^{-1}B, \tag{1.6}$$

where A^{-1} is the inverse of matrix A.

Cramer's rule. If $\det A \neq 0$, then the system (1.4) is consistent and has a unique solution, which is represented by the formulas:

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ x_3 = \frac{\Delta_3}{\Delta}, \tag{1.7}$$

where $\Delta = \det A$ is the determinant of the coefficient matrix, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$ are the determinants of the matrices obtained by replacing the j-th column of A with the column vector of the free terms B.

Gaussian elimination. Gauss' method consists in consecutive elimination of unknowns. The procedure for solving the system is as follows:

1) write the ***augmented matrix*** $\widetilde{A} = (A|B)$ of the given system;

2) reduce $\widetilde{A}$ to the ***echelon form*** using elementary row operations;
3) write down the linear system corresponding to the augmented matrix obtained in step 2;
4) solve the system obtained in step 3, starting from the last equation.

Remark. Elementary row operations of matrices include: interchanging any two rows; multiplying a row by a non-zero number; adding or subtracting a row multiplied by a non-zero number to another row.

Problem 1.6. The system of linear equations $\begin{cases} 2x_1 - 3x_2 + x_3 = -1, \\ x_1 - 2x_2 - 3x_3 = 6, \\ 4x_1 + 2x_2 + x_3 = 8. \end{cases}$ is given. Solve the system using:

a) the matrix method; **b)** Cramer's rule; **c)** Gaussian elimination.

Solution.

Write the system in the matrix form:

$$AX = B,$$

where $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & -3 \\ 4 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

a) Let's find the inverse matrix A^{-1}:

1) we calculate the determinant of the coefficient matrix:

$$\det A = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -2 & -3 \\ 4 & 2 & 1 \end{vmatrix} = 57;$$

2) we find the cofactors of all its entries:

$$A_{11} = (-1)^{1+1} \begin{vmatrix} -2 & -3 \\ 2 & 1 \end{vmatrix} = -2 \cdot 1 - (-3) \cdot 2 = 4,$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} = -(1 \cdot 1 - (-3) \cdot 4) = -13,$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix} = 1 \cdot 2 - (-2) \cdot 4 = 10,$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}-3 & 1\\ 2 & 1\end{vmatrix}=5,\qquad A_{31}=(-1)^{3+1}\begin{vmatrix}-3 & 1\\ -2 & -3\end{vmatrix}=11,$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix}2 & 1\\ 4 & 1\end{vmatrix}=-2,\qquad A_{32}=(-1)^{3+2}\begin{vmatrix}2 & 1\\ 1 & -3\end{vmatrix}=7,$$

$$A_{23}=(-1)^{2+3}\begin{vmatrix}2 & -3\\ 4 & 2\end{vmatrix}=-16,\qquad A_{33}=(-1)^{3+3}\begin{vmatrix}2 & -3\\ 1 & -2\end{vmatrix}=-1;$$

3) we find the adjoint matrix of A:

$$A^*=\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T=\begin{pmatrix}4 & -13 & 10\\ 5 & -2 & -16\\ 11 & 7 & -1\end{pmatrix}^T=\begin{pmatrix}4 & 5 & 11\\ -13 & -2 & 7\\ 10 & -16 & -1\end{pmatrix};$$

4) by (1.1), the inverse matrix of A is $A^{-1}=\dfrac{1}{57}\begin{pmatrix}4 & 5 & 11\\ -13 & -2 & 7\\ 10 & -16 & -1\end{pmatrix}$.

By (1.6), we obtain:

$$X=\frac{1}{57}\begin{pmatrix}4 & 5 & 11\\ -13 & -2 & 7\\ 10 & -16 & -1\end{pmatrix}\cdot\begin{pmatrix}-1\\ 6\\ 8\end{pmatrix}=\frac{1}{57}\begin{pmatrix}4\cdot(-1)+5\cdot 6+11\cdot 8\\ -13\cdot(-1)-2\cdot 6+7\cdot 8\\ 10\cdot(-1)-16\cdot 6-1\cdot 8\end{pmatrix}=$$

$$=\frac{1}{57}\begin{pmatrix}-4+30+88\\ 13-12+56\\ -10-96-8\end{pmatrix}=\frac{1}{57}\begin{pmatrix}114\\ 57\\ -114\end{pmatrix}=\begin{pmatrix}2\\ 1\\ -2\end{pmatrix}.$$

b) From a) $\Delta=\det A=57$. Let's find Δ_1, Δ_2, Δ_3:

$$\Delta_1=\begin{vmatrix}-1 & -3 & 1\\ 6 & -2 & -3\\ 8 & 2 & 1\end{vmatrix}=114,\ \Delta_2=\begin{vmatrix}2 & -1 & 1\\ 1 & 6 & -3\\ 4 & 8 & 1\end{vmatrix}=57,$$

$$\Delta_3=\begin{vmatrix}2 & -3 & -1\\ 1 & -2 & 6\\ 4 & 2 & 8\end{vmatrix}=-114.$$

Using formulas (1.7), we obtain:

$$x_1=\frac{\Delta_1}{\Delta}=\frac{114}{57}=2,\ x_2=\frac{\Delta_2}{\Delta}=\frac{57}{57}=1,\ x_3=\frac{\Delta_3}{\Delta}=\frac{-114}{57}=-2.$$

c) We write the augmented matrix of the system:

$$\widetilde{A}=\left(\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & -2 & -3 & 6 \\ 4 & 2 & 1 & 8 \end{array}\right).$$

By using elementary row operations, we reduce it to echelon form:

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & -2 & -3 & 6 \\ 4 & 2 & 1 & 8 \end{array}\right) r_1 \leftrightarrow r_2 \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 6 \\ 2 & -3 & 1 & -1 \\ 4 & 2 & 1 & 8 \end{array}\right) \begin{array}{l} \\ r_2 \to r_2 - 2r_1 \sim \\ r_3 \to r_3 - 4r_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 6 \\ 0 & 1 & 7 & -13 \\ 0 & 10 & 13 & -16 \end{array}\right) r_3 \to r_3 - 10r_2 \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 6 \\ 0 & 1 & 7 & -13 \\ 0 & 0 & -57 & 114 \end{array}\right) r_3 \to \frac{r_3}{-57} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 6 \\ 0 & 1 & 7 & -13 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

The last matrix corresponds to the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 - 3x_3 = 6, \\ \quad\quad x_2 + 7x_3 = -13, \\ \quad\quad\quad\quad x_3 = -2. \end{cases}$$

Starting from the last equation, we solve the last system:

$$\begin{cases} x_3 = -2, \\ x_2 = -13 - 7x_3 = -13 - 7\cdot(-2) = 1, \\ x_1 = 6 + 2x_2 + 3x_3 = 6 + 2\cdot 1 + 3\cdot(-2) = 2. \end{cases} \Rightarrow \begin{cases} x_1 = 2, \\ x_2 = 1, \\ x_3 = -2. \end{cases}$$

Answer: **a)** $x_1 = 2$, $x_2 = 1$, $x_3 = -2$; **b)** $x_1 = 2$, $x_2 = 1$, $x_3 = -2$; **c)** $x_1 = 2$, $x_2 = 1$, $x_3 = -2$.

Investigation of a system of m linear equations in n unknowns

A system of m ***linear equations in*** n ***unknowns*** has the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where x_j are the unknowns, a_{ij} are the given numbers, called the ***coefficients of the system***, b_i are also the given numbers, known as the ***free terms*** of the system, $i = \overline{1, m}$, $j = \overline{1, n}$.

Kronecker-Capelli's theorem. A system of m linear equations in n unknowns is consistent if and only if $rank(A) = rank(\widetilde{A}) = r$.

If $r = n$, the system has exactly one solution, which can be found using any of the methods described above.

If $r < n$, then the system has an infinite number of solutions, which depend on $n - r$ parameters.

Gauss' method is the universal method for solving systems of m linear equations in n unknowns.

Remark. The rank of a matrix is equal to the number of non-zero rows after reducing it to echelon form using elementary row operations.

Problem 1.7. Solve the systems, provided they are consistent:

a) $\begin{cases} 3x_1 + 9x_2 - 4x_3 = 5, \\ 4x_1 + 11x_2 - 7x_3 = 20, \\ x_1 + 2x_2 - 3x_3 = 4. \end{cases}$ **b)** $\begin{cases} x_1 - 4x_2 + 5x_3 = 7, \\ 2x_1 - 7x_2 + 7x_3 = 12, \\ x_1 - 3x_2 + 2x_3 = 5. \end{cases}$

c) $\begin{cases} 3x_1 + 2x_2 + 5x_4 = 8, \\ x_1 + 2x_2 - 4x_3 + 7x_4 = 4, \\ 2x_1 + x_2 + x_3 + 2x_4 = 5, \\ x_1 + x_2 - x_3 + 3x_4 = 3. \end{cases}$

Solution.

a) Write the augmented matrix of the system and reduce it to echelon form using elementary row operations:

$$\widetilde{A}=\left(\begin{array}{ccc|c}3&9&-4&5\\4&11&-7&20\\1&2&-3&4\end{array}\right)\begin{matrix}r_1\leftrightarrow r3\\ \\ \end{matrix}\sim\left(\begin{array}{ccc|c}1&2&-3&4\\4&11&-7&20\\3&9&-4&5\end{array}\right)\begin{matrix}\\ r_2\to r_2-4r_1\\ r_3\to r_3-3r_1\end{matrix}\sim$$

$$\sim\left(\begin{array}{ccc|c}1&2&-3&4\\0&3&5&4\\0&3&5&-7\end{array}\right)\begin{matrix}\\ \\ r_3\to r_3-r_2\end{matrix}\sim\left(\begin{array}{ccc|c}1&2&-3&4\\0&3&5&4\\0&0&0&-11\end{array}\right).$$

$$rank(A)=2,\ rank(\widetilde{A})=3.$$

So, $rank(A)\neq rank(\widetilde{A})$. According to Kronecker-Capelli's theorem, the given system is inconsistent.

b) Let's write the augmented matrix of the system and reduce it to echelon form using elementary row operations:

$$\widetilde{A}=\left(\begin{array}{ccc|c}1&-4&5&7\\2&-7&7&12\\1&-3&2&5\end{array}\right)\begin{matrix}\\ r_2\to r_2-2r_1\\ r_3\to r_3-r_1\end{matrix}\sim\left(\begin{array}{ccc|c}1&-4&5&7\\0&1&-3&-2\\0&1&-3&-2\end{array}\right)\begin{matrix}\\ \\ r_3\to r_3-r_2\end{matrix}\sim$$

$$\sim\left(\begin{array}{ccc|c}1&-4&5&7\\0&1&-3&-2\\0&0&0&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-4&5&7\\0&1&-3&-2\end{array}\right).$$

$rank(A)=rank(\widetilde{A})=2$. According to Kronecker-Capelli's theorem, the considered system is consistent. The solution will depend on $n-r=3-2=1$ parameter.

The last matrix corresponds to the following system of linear equations:

$$\begin{cases}x_1-4x_2+5x_3=7,\\ \qquad x_2-3x_3=-2.\end{cases}\Rightarrow\begin{cases}x_1=7+4x_2-5x_3,\\ x_2=-2+3x_3.\end{cases}$$

$$\begin{cases}x_1=7+4(-2+3x_3)-5x_3=-1+7x_3,\\ x_2=-2+3x_3.\end{cases}\Rightarrow\begin{cases}x_1=-1+7x_3,\\ x_2=-2+3x_3.\end{cases}$$

Let's take $x_3=t$, then $\begin{cases}x_1=-1+7t,\\ x_2=-2+3t,\\ x_3=t,\end{cases}$ where $t\in R$.

c) Let's write the augmented matrix of the system and reduce it to echelon form using elementary row operations:

$$\widetilde{A}=\left(\begin{array}{cccc|c}3&2&0&5&8\\1&2&-4&7&4\\2&1&1&2&5\\1&1&-1&3&3\end{array}\right)\begin{matrix}r_1\leftrightarrow r_2\\ \\ \\ \\ \end{matrix}\sim\left(\begin{array}{cccc|c}1&2&-4&7&4\\3&2&0&5&8\\2&1&1&2&5\\1&1&-1&3&3\end{array}\right)\begin{matrix} \\ r_2\to r_2-3r_1\\ r_3\to r_3-2r_1\\ r_4\to r_4-r_1\end{matrix}\sim$$

$$\sim\left(\begin{array}{cccc|c}1&2&-4&7&4\\0&-4&12&-16&-4\\0&-3&9&-12&-3\\0&-1&3&-4&-1\end{array}\right)\begin{matrix}r_2\to-\frac{1}{4}\cdot r_2\\ r_3\to-\frac{1}{3}\cdot r_3\\ r_4\to-1\cdot r_4\end{matrix}\sim\left(\begin{array}{cccc|c}1&2&-4&7&4\\0&1&-3&4&1\\0&1&-3&4&1\\0&1&-3&4&1\end{array}\right)\begin{matrix} \\ \\ r_3\to r_3-r_2\\ r_4\to r_4-r_2\end{matrix}\sim$$

$$\sim\left(\begin{array}{cccc|c}1&2&-4&7&4\\0&1&-3&4&1\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)\sim\left(\begin{array}{cccc|c}1&2&-4&7&4\\0&1&-3&4&1\end{array}\right).$$

$rank(A)=rank(\widetilde{A})=2$. According to Kronecker-Capelli's theorem, the given system is consistent. The solution will depend on $n-r=4-2=2$ parameters.

The last matrix corresponds to the following system of linear equations:

$$\begin{cases}x_1+2x_2-4x_3+7x_4=4,\\ x_2-3x_3+4x_4=1.\end{cases}\Rightarrow\begin{cases}x_1=4-2x_2+4x_3-7x_4,\\ x_2=1+3x_3-4x_4.\end{cases}$$

$$\begin{cases}x_1=4-2\cdot(1+3x_3-4x_4)+4x_3-7x_4=2-2x_3+x_4,\\ x_2=1+3x_3-4x_4.\end{cases}\quad\begin{cases}x_1=2-2x_3+x_4,\\ x_2=1+3x_3-4x_4.\end{cases}$$

Let's take $x_3=t$, $x_4=s$, then $\begin{cases}x_1=2-2t+s,\\ x_2=1+3t-4s,\\ x_3=t,x_4=s,\end{cases}$ where $t\in R$, $s\in R$.

Answer: **a)** the system is inconsistent; **b)** $\begin{cases}x_1=-1+7t,\\ x_2=-2+3t,\\ x_3=t,\end{cases}$ where $t\in R$; **c)** $\begin{cases}x_1=2-2t+s,\\ x_2=1+3t-4s,\\ x_3=t,x_4=s,\end{cases}$ where $t\in R$, $s\in R$.

Homogeneous systems of linear equations

A homogeneous system of m **_linear equations in_** n **_unknowns_** has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0, \\ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0. \end{cases}$$

The homogeneous system is always consistent because it always has the trivial solution. If the rank of the coefficient matrix r is equal to the number of unknowns n, the system has only the trivial solution. If $r < n$, the system has infinitely many solutions, which can be found by Gauss' method.

Problem 1.8. Solve the homogeneous systems:

a) $\begin{cases} 7x_1 + 6x_2 - 7x_3 = 0, \\ 3x_1 + 2x_2 - 5x_3 = 0, \\ x_1 - 8x_3 = 0. \end{cases}$ **b)** $\begin{cases} 5x_1 - 4x_2 - x_3 = 0, \\ x_1 - 2x_2 - 9x_3 = 0, \\ 3x_1 - 3x_2 - 5x_3 = 0. \end{cases}$

c) $\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0, \\ 2x_1 + 7x_2 - 4x_3 + 3x_4 = 0, \\ 3x_1 + 9x_2 - 5x_3 + 4x_4 = 0. \end{cases}$

Solution.

a) Let's find the rank of the coefficient matrix:

$$A = \begin{pmatrix} 7 & 6 & -7 \\ 3 & 2 & -5 \\ 1 & 0 & -8 \end{pmatrix} \begin{matrix} r_1 \leftrightarrow r_3 \\ \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & -8 \\ 3 & 2 & -5 \\ 7 & 6 & -7 \end{pmatrix} \begin{matrix} \\ r_2 \to r_2 - 3r_1 \\ r_3 \to r_3 - 7r_1 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & -8 \\ 0 & 2 & 19 \\ 0 & 6 & 49 \end{pmatrix} \begin{matrix} \\ \\ r_3 \to r_3 - 3 \cdot r_2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & -8 \\ 0 & 2 & 19 \\ 0 & 0 & 8 \end{pmatrix}.$$

$rank(A) = r = 3$, $n = 3$, $r = n$. Hence, the given system has only a trivial solution $x_1 = x_2 = x_3 = 0$.

b) Let's find the rank of the coefficient matrix:

$$A=\begin{pmatrix}5 & -4 & -1\\ 1 & -2 & -9\\ 3 & -3 & -5\end{pmatrix}\begin{matrix}r_1\leftrightarrow r_2\\ \\ \\ \end{matrix}\sim\begin{pmatrix}1 & -2 & -9\\ 5 & -4 & -1\\ 3 & -3 & -5\end{pmatrix}\begin{matrix}\\ r_2\to r_2-5r_1\\ r_3\to r_3-3r_1\end{matrix}\sim$$

$$\sim\begin{pmatrix}1 & -2 & -9\\ 0 & 6 & 44\\ 0 & 3 & 22\end{pmatrix}\begin{matrix}\\ r_2\to 0.5\cdot r_2\\ r_3\to r_3-0.5\cdot r_2\end{matrix}\sim\begin{pmatrix}1 & -2 & -9\\ 0 & 3 & 22\\ 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & -2 & -9\\ 0 & 3 & 22\end{pmatrix}.$$

$rank(A)=r=2$, $n=3$, $r<n$. Hence, the system has infinitely many solutions and it is equivalent to the following system:

$$\begin{cases}x_1-2x_2-9x_3=0,\\ 3x_2+22x_3=0.\end{cases}\Rightarrow\begin{cases}x_1=-\dfrac{17}{3}x_3,\\ x_2=-\dfrac{22}{3}x_3.\end{cases}$$

Setting $x_3=3t$, we obtain the solution in the form: $\begin{cases}x_1=-17t,\\ x_2=-22t,\quad t\in R.\\ x_3=3t,\end{cases}$

c) We have the homogeneous system of three equations in four unknowns. Let's reduce the coefficient matrix to echelon form:

$$\begin{pmatrix}1 & 2 & -1 & 1\\ 2 & 7 & -4 & 3\\ 3 & 9 & -5 & 4\end{pmatrix}\begin{matrix}\\ r_2\to r_2-2r_1\\ r_3\to r_3-3r_1\end{matrix}\sim\begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 3 & -2 & 1\\ 0 & 3 & -2 & 1\end{pmatrix}\begin{matrix}\\ \\ r_3\to r_3-r_2\end{matrix}\sim$$

$$\sim\begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 3 & -2 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 2 & -1 & 1\\ 0 & 3 & -2 & 1\end{pmatrix}.$$

The given system is equivalent to the following system: $\begin{cases}x_1+2x_2-x_3+x_4=0,\\ \quad 3x_2-2x_3+x_4=0.\end{cases}$ Its solution is $\begin{cases}x_1=-\dfrac{1}{3}x_3-\dfrac{1}{3}x_4,\\ x_2=\dfrac{2}{3}x_3-\dfrac{1}{3}x_4.\end{cases}$

Setting $x_3=3t$, $x_4=3s$, we obtain the solution in the form:

$$\begin{cases} x_1 = -t - s, \\ x_2 = 2t - s, \\ x_3 = 3t, \\ x_4 = 3s, \end{cases} \quad t \in R\,,\ s \in R\,.$$

Answer: **a)** $x_1 = x_2 = x_3 = 0$; **b)** $\begin{cases} x_1 = -17t, \\ x_2 = -22t, \quad t \in R\,; \\ x_3 = 3t. \end{cases}$

c) $\begin{cases} x_1 = -t - s, \\ x_2 = 2t - s, \\ x_3 = 3t, \\ x_4 = 3s, \end{cases} \quad t \in R\,,\ s \in R\,.$

Exercises

1.1. The matrices $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ 4 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix}$, $F = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ are given. Find:

a) $C \cdot \left(A + 2B^T\right)$; **c)** AB, BA, AC, CA, BC, CB, FG, GF;

b) $A \cdot B - 5D^T$; **d)** $B \cdot A - (G \cdot F)^T$.

Answer: **a)** $\begin{pmatrix} 15 & 7 & 23 \\ -4 & 7 & 13 \end{pmatrix}$; **b)** $\begin{pmatrix} -7 & -17 \\ 29 & 15 \end{pmatrix}$; **c)** $AB = \begin{pmatrix} 3 & 3 \\ 29 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 7 & -5 \\ 6 & -2 & 10 \\ 7 & 7 & 5 \end{pmatrix}$, $CA = \begin{pmatrix} 5 & 3 & 5 \\ 8 & -5 & 15 \end{pmatrix}$, $BC = \begin{pmatrix} 7 & 0 \\ -2 & 6 \\ 7 & 7 \end{pmatrix}$, $FG = 5$, $GF = \begin{pmatrix} 12 & -8 & 4 \\ 6 & -4 & 2 \\ -9 & 6 & -3 \end{pmatrix}$, AC and CB aren't defined; **d)** $\begin{pmatrix} -12 & 1 & 4 \\ 14 & 2 & 4 \\ 3 & 5 & 8 \end{pmatrix}$.

1.2. Calculate the determinants:

a) $\begin{vmatrix} 3 & 6 \\ -7 & -5 \end{vmatrix}$;   **b)** $\begin{vmatrix} a-b & a-2b \\ a & a-b \end{vmatrix}$;   **c)** $\begin{vmatrix} 4 & 2 & -5 \\ -3 & 1 & 3 \\ -4 & -2 & 6 \end{vmatrix}$.

Answer: **a)** 27; **b)** b^2; **c)** 10.

1.3. Solve the equations:

a) $\begin{vmatrix} x-2 & x \\ x-8 & x+2 \end{vmatrix} = 0$;   **b)** $\begin{vmatrix} 5 & 3 & x \\ x+1 & 2 & 3 \\ 3 & 1 & -2 \end{vmatrix} = 0$.

Answer: **a)** 0.5; **b)** -2; 1.

1.4. Calculate the determinant $\begin{vmatrix} 2 & 3 & 0 & -4 \\ -1 & 0 & -2 & 3 \\ 5 & 2 & 1 & 0 \\ 1 & 0 & -1 & 6 \end{vmatrix}$:

a) using expansion by the first row;
b) using expansion by the second column;
c) reducing it to the triangular form;
d) reducing it to the second order determinant.

Answer: -48.

1.5. Solve the matrix equations:

a) $\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 1 \\ 5 & -2 \end{pmatrix}$;   **b)** $X \cdot \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & -2 \end{pmatrix}$.

Answer: **a)** $X = \begin{pmatrix} \frac{7}{11} & \frac{6}{11} \\ \frac{19}{11} & -\frac{1}{11} \end{pmatrix}$; **b)** $X = \begin{pmatrix} \frac{15}{11} & -\frac{1}{11} \\ \frac{14}{11} & -\frac{9}{11} \end{pmatrix}$.

1.6. The system of linear equations $\begin{cases} -2x+3y-5z=3, \\ x-2y+3z=-2, \\ 4x-y+2z=-4. \end{cases}$ is given.

Solve the system using:

a) matrix method; **b)** Cramer's rule; **c)** Gaussian elimination.

Answer: **a)** $x_1=-1$, $x_2=2$, $x_3=1$; **b)** $x_1=-1$, $x_2=2$, $x_3=1$; **c)** $x_1=-1$, $x_2=2$, $x_3=1$.

1.7. Solve the systems provided they are consistent:

a) $\begin{cases} 5x_1 + x_2 - 6x_3 = 5, \\ 4x_1 + 3x_2 - 7x_3 = -3, \\ x_1 - 2x_2 + x_3 = 2. \end{cases}$ **b)** $\begin{cases} 2x_1 - x_2 + 4x_3 = -2, \\ 7x_1 - 5x_2 + 3x_3 = 1, \\ 5x_1 - 4x_2 - x_3 = 3. \end{cases}$

c) $\begin{cases} 2x_1 + x_2 + 5x_3 + 2x_4 = 5, \\ x_1 + x_2 + 4x_3 = 3, \\ 3x_1 + x_2 + 6x_3 + 4x_4 = 7, \\ x_1 + 2x_2 + 7x_3 - 2x_4 = 4. \end{cases}$

Answer: **a)** the system is inconsistent; **b)** $\begin{cases} x_1 = -\frac{11}{3} - \frac{17}{3}t, \\ x_2 = -\frac{16}{3} - \frac{22}{3}t, \\ x_3 = t, \end{cases}$

where $t \in R$; **c)** $\begin{cases} x_1 = 2 - t - 2s, \\ x_2 = 1 - 3t + 2s, \\ x_3 = t, x_4 = s, \end{cases}$ where $t \in R$, $s \in R$.

1.8. Solve the homogeneous systems:

a) $\begin{cases} 4x_1 - 3x_2 + 4x_3 = 0, \\ 5x_1 - x_2 + 3x_3 = 0, \\ 3x_1 - 5x_2 + 4x_3 = 0. \end{cases}$ **b)** $\begin{cases} 2x_1 - x_2 + 5x_3 = 0, \\ x_1 + 3x_2 - 2x_3 = 0, \\ 4x_1 + 5x_2 + x_3 = 0. \end{cases}$

c) $\begin{cases} 3x_1 - 5x_2 + 5x_3 + 14x_4 = 0, \\ 2x_1 - 2x_2 + 3x_3 + 10x_4 = 0, \\ 4x_1 - 8x_2 + 7x_3 + 18x_4 = 0. \end{cases}$

Answer: **a)** $x_1 = x_2 = x_3 = 0$; **b)** $\begin{cases} x_1 = -13t, \\ x_2 = 9t, \\ x_3 = 7t, \end{cases}$ $t \in R$;

c) $\begin{cases} x_1 = -5t - 11s, \\ x_2 = t - s, \\ x_3 = 4t, \\ x_4 = 2s, \end{cases}$ $t \in R$, $s \in R$.

2 ELEMENTS OF VECTOR ALGEBRA

Vectors

A ***vector*** is a directed line segment (arrow). Vectors are denoted by $\overline{a}$, $\overline{AB}$ (A is the initial point and B is the terminal point of the vector) or $\vec{a}$, $\overrightarrow{AB}$.

The length of the vector represents the ***magnitude*** or the ***length*** of the vector. The magnitude of a vector $\overline{a}$ is denoted by $|\overline{a}|$.

A vector of length 1 is called a ***unit vector***.

A vector whose magnitude equals 0 is called a ***zero vector*** and is denoted by $\overline{0}$.

Two vectors are said to be ***parallel*** (***collinear***) if they lie on the same line or on parallel lines. This is denoted by $\overline{a} \parallel \overline{b}$.

Two vectors $\overline{a}$ and $\overline{b}$ are called ***equal*** if they are collinear and have the same magnitude and direction.

Vectors are said to be ***coplanar*** if they are parallel to a plane or they lie in the same plane.

Given a vector $\overline{a} = \overline{AB}$ and an axis l. Let A_1 and B_1 be the orthogonal projections of A and B on l. The ***projection*** of the vector $\overline{a}$ on the axis l, denoted by $proj_l\,\overline{a}$, is defined as the length of the vector $\overline{A_1B_1}$, taken with a plus sign if $\overline{A_1B_1}$ has the same direction as the axis l and with a minus sign if $\overline{A_1B_1}$ has the opposite direction to l (Fig. 2.1). It can be calculated by the formula:

$$proj_l\,\overline{a} = |\overline{a}|\cos\varphi,$$

where φ is the angle between the vector and the axis.

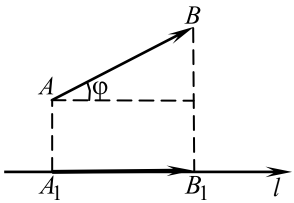


Figure 2.1

Let's consider a Cartesian coordinate system $Oxyz$. Take an arbitrary vector $\bar{a}$ and translate it so that its initial point coincides with the origin (Fig. 2.2). Let points $A_1(a_1,0,0)$, $A_2(0,a_2,0)$, $A_3(0,0,a_3)$ be the orthogonal projections of the terminal point of $\bar{a}$ on the x-, y-, and z-axes, respectively, and let A' be its orthogonal projection on the xy-plane. Denote by $\bar{i}$, $\bar{j}$, and $\bar{k}$ the unit vectors in the positive directions of the x-, y-, and z-axes, respectively ($|\bar{i}|=|\bar{j}|=|\bar{k}|=1$, $\bar{i}\perp\bar{j}$, $\bar{i}\perp\bar{k}$, $\bar{j}\perp\bar{k}$). From Fig. 2.2, it follows that

$$\bar{a}=\overline{OA_1}+\overline{OA_2}+\overline{OA_3}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}\ .$$

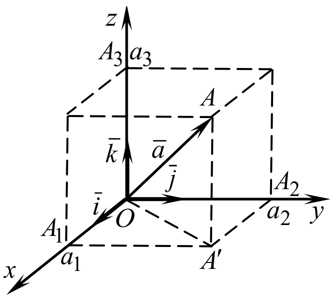


Figure 2.2

The numbers a_1, a_2, a_3 in the last equality are called the ***coordinates*** of the vector. Instead of $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, we can write simply $\bar{a}=(a_1,a_2,a_3)$.

If $A(x_1,y_1,z_1)$ is the initial point and $B(x_2,y_2,z_2)$ is the terminal point of the vector $\bar{a}=\overline{AB}$, then its ***coordinates*** are

$$\overline{AB}=(x_2-x_1,y_2-y_1,z_2-z_1). \qquad (2.1)$$

The ***magnitude*** or the ***length*** of the vector $\bar{a}=(a_1,a_2,a_3)$ is specified by the formula

$$|\bar{a}|=\sqrt{a_1^2+a_2^2+a_3^2}\ . \qquad (2.2)$$

The coordinates of the unit vector $\bar{a}^0$ in the direction of $\bar{a}$ are given by the formula

$$\bar{a}^0 = \frac{\bar{a}}{|\bar{a}|} = \left(\frac{a_1}{|\bar{a}|}, \frac{a_2}{|\bar{a}|}, \frac{a_3}{|\bar{a}|}\right). \qquad (2.3)$$

Let a vector $\bar{a} = (a_1, a_2, a_3)$ make angles α, β and γ with the positive directions of the x-, y- and z-axes, respectively. The cosines of these angles are called the ***direction cosines***. They can be determined by the following formulas:

$$\cos\alpha = \frac{a_1}{|\bar{a}|}, \ \cos\beta = \frac{a_2}{|\bar{a}|}, \ \cos\gamma = \frac{a_3}{|\bar{a}|}. \qquad (2.4)$$

Vector operations

Let $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$ be the given vectors, and let m be a scalar. Then:

1) $m\bar{a} = (m a_1, m a_2, m a_3)$;

2) $\bar{a} \pm \bar{b} = (a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3)$.

Two vectors are ***parallel*** or ***collinear*** if they lie on the same line or parallel lines.

Test for parallelism: $\bar{a} \parallel \bar{b} \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

Problem 2.1. A vector $\bar{a} = (1, 3, -2)$ and three points $A(-2, 1, 3)$, $B(2, 3, -1)$ and $C(-3, 1, 4)$ are given. Find:

a) $|\bar{a}|$, $|\overline{AB}|$, $|\overline{CA}|$; **b)** the unit vector $\bar{a}^0$ and the direction cosines of $\bar{a}$.

Solution.

a) Using formula (2.1), we find the coordinates of vectors $\overline{AB}$, $\overline{CA}$:

$$\overline{AB} = (2-(-2), 3-1, -1-3) = (4, 2, -4),$$
$$\overline{CA} = (-2-(-3), 1-1, 3-4) = (1, 0, -1).$$

By (2.2), we find the lengths of the vectors $\bar{a}$, $\overline{AB}$ and $\overline{CA}$:

$$|\bar{a}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14},$$

$$|\overline{AB}| = \sqrt{4^2 + 2^2 + (-4)^2} = 6, \qquad |\overline{CA}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}.$$

b) Using formulas (2.3) and (2.4), we find the unit vector $\bar{a}^0$ and the direction cosines of $\bar{a}$:

$$\bar{a}^0 = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right),$$

$$\cos\alpha = \frac{1}{\sqrt{14}}, \ \cos\beta = \frac{3}{\sqrt{14}}, \ \cos\gamma = -\frac{2}{\sqrt{14}}.$$

Answer: **a)** $\sqrt{14}$, 6, $\sqrt{2}$; **b)** $\bar{a}^0 = \left(\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}} \right)$,

$\cos\alpha = \frac{1}{\sqrt{14}}$, $\cos\beta = \frac{3}{\sqrt{14}}$, $\cos\gamma = -\frac{2}{\sqrt{14}}$.

Problem 2.2. The vectors $\bar{a} = (-2, 1, 3)$ and $\bar{b} = (1, 2, -5)$ are given. Find:

a) $|\bar{a} + \bar{b}|$; **b)** $|\bar{a} - \bar{b}|$; **c)** $|2\bar{a} - 3\bar{b}|$.

Solution.

a) $\bar{a} + \bar{b} = (-2+1, 1+2, 3+(-5)) = (-1, 3, -2)$,

$|\bar{a} + \bar{b}| = \sqrt{(-1)^2 + 3^2 + (-2)^2} = \sqrt{1+9+4} = \sqrt{14}$.

b) $\bar{a} - \bar{b} = (-2-1, 1-2, 3-(-5)) = (-3, -1, 8)$,

$|\bar{a} - \bar{b}| = \sqrt{(-3)^2 + (-1)^2 + 8^2} = \sqrt{9+1+64} = \sqrt{74}$.

c) $2\bar{a} - 3\bar{b} = 2\cdot(-2,1,3) - 3\cdot(1,2,-5) = (-4, 2, 6) - (3, 6, -15) =$
$= (-4-3, 2-6, 6-(-15)) = (-7, -4, 21)$,

$|2\bar{a} - 3\bar{b}| = \sqrt{(-7)^2 + (-4)^2 + 21^2} = \sqrt{49+16+441} = \sqrt{506}$.

Answer: **a)** $\sqrt{14}$; **b)** $\sqrt{74}$; **c)** $\sqrt{506}$.

Scalar (dot) product of two vectors

The ***scalar (dot) product*** of two vectors $\bar{a}$ and $\bar{b}$, denoted by $\bar{a} \cdot \bar{b}$, and defined as:

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\alpha, \qquad (2.5)$$

where α is the angle between vectors $\bar{a}$ and $\bar{b}$.

If $\overline{a}=(a_1,a_2,a_3)$ and $\overline{b}=(b_1,b_2,b_3)$, then

$$\overline{a}\cdot\overline{b}=a_1\cdot b_1+a_2\cdot b_2+a_3\cdot b_3. \qquad (2.6)$$

The properties of the scalar product:

1) $\overline{a}\cdot\overline{b}=\overline{b}\cdot\overline{a}$;

2) $\overline{a}\cdot\overline{a}=|\overline{a}|^2$, $|\overline{a}|=\sqrt{\overline{a}\cdot\overline{a}}$;

3) $(m\overline{a})\cdot\overline{b}=\overline{a}\cdot(m\overline{b})=m(\overline{a}\cdot\overline{b})$;

4) $\overline{a}\cdot(\overline{b}+\overline{c})=\overline{a}\cdot\overline{b}+\overline{a}\cdot\overline{c}$;

5) $\overline{a}\cdot\overline{b}=\overline{a}\cdot proj_{\overline{a}}\overline{b}=\overline{b}\cdot proj_{\overline{b}}\overline{a}$.

The cosine of the angle between two vectors can be found by the following formula:

$$\cos(\overline{a},\hat{}\,\overline{b})=\frac{\overline{a}\cdot\overline{b}}{|\overline{a}|\cdot|\overline{b}|}. \qquad (2.7)$$

Test for perpendicularity: $\overline{a}\perp\overline{b} \Leftrightarrow \overline{a}\cdot\overline{b}=0$.

Physical applications of the scalar product. The ***work*** A done by the force $\overline{F}$ when an object is displaced by a vector $\overline{S}$ is defined as:

$$A=\overline{F}\cdot\overline{S}. \qquad (2.8)$$

Problem 2.3. Find the scalar product of vectors $\overline{a}$ and $\overline{b}$ if $|\overline{a}|=8$, $|\overline{b}|=3\sqrt{2}$ and the angle between $\overline{a}$ and $\overline{b}$ is equal to $45°$.

Solution.

By (2.5), $\overline{a}\cdot\overline{b}=8\cdot 3\sqrt{2}\cdot\cos 45°=8\cdot 3\sqrt{2}\cdot\frac{\sqrt{2}}{2}=24$.

Answer: 24.

Problem 2.4. Find the scalar product of vectors $\overline{a}=(3,-1,5)$ and $\overline{b}=(-2,4,8)$.

Solution.

By (2.6), $\overline{a}\cdot\overline{b}=3\cdot(-2)+(-1)\cdot 4+5\cdot 8=30$.

Answer: 30.

Problem 2.5. $\overline{m}=2\overline{a}-\overline{b}$, $\overline{n}=\overline{a}+4\overline{b}$, $|\overline{a}|=2$, $|\overline{b}|=\sqrt{3}$, $(\overline{a},\hat{}\,\overline{b})=\pi/6$ are given. Find:

a) $\overline{m}\cdot\overline{n}$; **b)** $\cos(\overline{m},\hat{}\,\overline{n})$; **c)** $proj_{\overline{n}}\,\overline{m}$.

Solution.

a) Using the properties of the scalar product and formula (2.5), we obtain:

$$\overline{m}\cdot\overline{n}=\left(2\overline{a}-\overline{b}\right)\cdot\left(\overline{a}+4\overline{b}\right)=2\overline{a}\cdot\overline{a}+2\overline{a}\cdot4\overline{b}-\overline{b}\cdot\overline{a}-\overline{b}\cdot4\overline{b}=$$

$$=2|\overline{a}|^2+8\overline{a}\cdot\overline{b}-\overline{a}\cdot\overline{b}-4|\overline{b}|^2=2|\overline{a}|^2+7|\overline{a}||\overline{b}|\cos\left(\overline{a},\hat{}\,\overline{b}\right)-4|\overline{b}|^2=$$

$$=2\cdot4+7\cdot2\cdot\sqrt{3}\cdot\frac{\sqrt{3}}{2}-4\cdot3=17\,.$$

b) By property 2 of the scalar product and formula (2.5), we get:

$$|\overline{m}|=\sqrt{\left(2\overline{a}-\overline{b}\right)^2}=\sqrt{4|\overline{a}|^2-4\overline{a}\cdot\overline{b}+|\overline{b}|^2}=$$

$$=\sqrt{4|\overline{a}|^2-4|\overline{a}|\cdot|\overline{b}|\cos\left(\overline{a},\hat{}\,\overline{b}\right)+|\overline{b}|^2}=$$

$$=\sqrt{4\cdot2^2-4\cdot2\cdot\sqrt{3}\cdot\frac{\sqrt{3}}{2}+\left(\sqrt{3}\right)^2}=\sqrt{7}\,,$$

$$|\overline{n}|=\sqrt{\left(\overline{a}+4\overline{b}\right)^2}=\sqrt{|\overline{a}|^2+8\overline{a}\cdot\overline{b}+16|\overline{b}|^2}=$$

$$=\sqrt{|\overline{a}|^2+8|\overline{a}|\cdot|\overline{b}|\cos\left(\overline{a},\hat{}\,\overline{b}\right)+16|\overline{b}|^2}=$$

$$=\sqrt{2^2+8\cdot2\cdot\sqrt{3}\cdot\frac{\sqrt{3}}{2}+16\cdot\left(\sqrt{3}\right)^2}=2\sqrt{19}\,.$$

Using (2.7), we find $\cos\left(\overline{m},\hat{}\,\overline{n}\right)$:

$$\cos\left(\overline{m},\hat{}\,\overline{n}\right)=\frac{\overline{m}\cdot\overline{n}}{|\overline{m}|\cdot|\overline{n}|}=\frac{17}{\sqrt{7}\cdot2\sqrt{19}}=\frac{17\sqrt{133}}{266}\,.$$

c) By property 5, we get:

$$proj_{\overline{n}}\,\overline{m}=\frac{\overline{m}\cdot\overline{n}}{|\overline{n}|}=\frac{17}{2\sqrt{19}}=\frac{17\sqrt{19}}{38}\,.$$

Answer: **a)** 17; **b)** $\frac{17\sqrt{133}}{266}$; **c)** $\frac{17\sqrt{19}}{38}$.

Problem 2.6. Find the value of the parameter p that satisfy the required condition:

a) $\overline{a}=(-6,14,4)$, $\overline{b}=(9,-21,p)$, and $\overline{a}\parallel\overline{b}$;

b) $\overline{a}=(2,p,-3)$, $\overline{b}=(-7,20,-5)$, and $\overline{a}\perp\overline{b}$.

Solution.

a) Using the test for parallelism, we obtain:

$$\frac{-6}{9}=\frac{14}{-21}=\frac{4}{p}\Rightarrow\frac{-6}{9}=\frac{4}{p}\Rightarrow -6p=36\Rightarrow p=-6.$$

b) Using the test for perpendicularity, we get:

$$2\cdot(-7)+p\cdot 20+(-3)\cdot(-5)=0\Rightarrow 20p=-1\Rightarrow p=-\frac{1}{20}=-0.05.$$

Answer: **a)** $p=-6$; **b)** $p=-0.05$.

Problem 2.7. The points $A(2,1,-3)$, $B(3,-1,5)$, and the force $\overline{F}=(2,-4,5)$ are given. Find the work done by the force $\overline{F}$ that moves an object from the point A to the point B along the straight line.

Solution.

Let's find the coordinates of the vector $\overline{AB}$:

$$\overline{AB}=(3-2,\,-1-1,\,5-(-3))=(1,-2,8).$$

According to formula (2.8), $A=\overline{F}\cdot\overline{AB}=2\cdot 1+(-4)\cdot(-2)+5\cdot 8=50$.

Answer: 50.

Vector product or cross product of two vectors

The ***vector product*** of two vectors $\overline{a}$ and $\overline{b}$ is a vector $\overline{c}$ (Fig. 2.3) such that:

1) $\overline{c}\perp\overline{a}$ and $\overline{c}\perp\overline{b}$;
2) $|\overline{c}|=|\overline{a}||\overline{b}|\sin\alpha$, where α is the angle between $\overline{a}$ and $\overline{b}$;
3) $\overline{a}$, $\overline{b}$ and $\overline{c}$, in this order, form a right-handed triple of vectors.

It is denoted as $\overline{a}\times\overline{b}$ or $[\overline{a},\overline{b}]$.

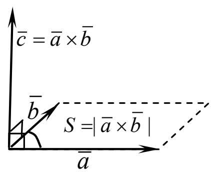
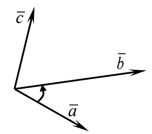
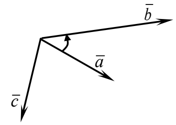


Fig. 2.3

Remark. Any ordered triple of vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ with a common origin in space forms a ***right-handed triple*** if these vectors are not in the same plane and the shortest turn from $\bar{a}$ to $\bar{b}$ as seen from the tip of $\bar{c}$ is counterclockwise. Otherwise, the triple is ***a left-handed triple***.



Right-handed triple



Left-handed triple

If $\bar{a} = (a_1, a_2, a_3)$ and $\bar{b} = (b_1, b_2, b_3)$ then

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \bar{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \bar{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \bar{k} . \qquad (2.9)$$

The properties of the vector product:

1) $\bar{b} \times \bar{a} = -\bar{a} \times \bar{b}$;

2) $(m\bar{a}) \times \bar{b} = \bar{a} \times (m\bar{b}) = m\,(\bar{a} \times \bar{b})$;

3) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$;

4) In the case of nonzero vectors, $\bar{a} \times \bar{b} = \bar{0} \Leftrightarrow \bar{a} \parallel \bar{b}$;

5) $\bar{a} \times \bar{a} = \bar{0}$.

Geometrical interpretation. The area of the parallelogram formed by the vectors $\bar{a}$ and $\bar{b}$ is equal to the magnitude of the vector $\bar{a} \times \bar{b}$: $S = |\bar{a} \times \bar{b}|$.

The area of the triangle formed by the vectors $\bar{a}$ and $\bar{b}$ is given by the formula $S_\Delta = \frac{1}{2}|\bar{a} \times \bar{b}|$.

Physical applications of the vector product. The ***moment*** $\overline{M}$ of the force $\overline{F}$ applied at the point A, with respect to the point B, is defined as

$$\overline{M} = \overline{BA} \times \overline{F} . \qquad (2.10)$$

Problem 2.8. The vectors $\bar{a}=(2,-1,3)$ and $\bar{b}=(-3,1,-5)$ are given. Find $\bar{a}\times\bar{b}$, $|\bar{a}\times\bar{b}|$ and the area S of the parallelogram formed by the vectors $\bar{a}$ and $\bar{b}$.

Solution.

Using formula (2.9), we get:

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & -1 & 3\\ -3 & 1 & -5\end{vmatrix}=\begin{vmatrix}-1 & 3\\ 1 & -5\end{vmatrix}\bar{i}-\begin{vmatrix}2 & 3\\ -3 & -5\end{vmatrix}\bar{j}+\begin{vmatrix}2 & -1\\ -3 & 1\end{vmatrix}\bar{k}=$$

$$=2\bar{i}+1\bar{j}-1\bar{k}=(2,1,-1).$$

$$|\bar{a}\times\bar{b}|=\sqrt{2^2+1^2+(-1)^2}=\sqrt{6},$$

$$S=|\bar{a}\times\bar{b}|=\sqrt{6}\ \text{(units}^2\text{)}.$$

Answer: $\bar{a}\times\bar{b}=(2,1,-1)$, $|\bar{a}\times\bar{b}|=\sqrt{6}$, $S=\sqrt{6}$ units^2.

Problem 2.9. $\bar{m}=2\bar{a}-\bar{b}$, $\bar{n}=\bar{a}+4\bar{b}$, $|\bar{a}|=2$, $|\bar{b}|=\sqrt{3}$, $(\bar{a},\hat{\ }\bar{b})=\pi/6$ are given. Find the area S of the parallelogram formed by the vectors $\bar{m}$ and $\bar{n}$.

Solution.

By the properties of the vector product, we obtain:

$$\bar{m}\times\bar{n}=(2\bar{a}-\bar{b})\times(\bar{a}+4\bar{b})=2\bar{a}\times\bar{a}+2\bar{a}\times4\bar{b}-\bar{b}\times\bar{a}-\bar{b}\times4\bar{b}=$$

$$=\bar{0}+8\bar{a}\times\bar{b}+\bar{a}\times\bar{b}-\bar{0}=9\bar{a}\times\bar{b}.$$

$$S=|\bar{m}\times\bar{n}|=9|\bar{a}\times\bar{b}|=9\cdot|\bar{a}||\bar{b}|\sin(\bar{a},\hat{\ }\bar{b})=9\cdot2\cdot\sqrt{3}\cdot\frac{1}{2}=9\sqrt{3}\ \text{(units}^2\text{)}.$$

Answer: **a)** 17; **b)** $9\sqrt{3}$ units^2.

Problem 2.10. Two points $A(2,1,-3)$, $B(3,-1,5)$, and the force $\bar{F}=(2,-4,5)$ are given. Find the magnitude of the moment $\bar{M}$ of the force $\bar{F}$, applied at the point A, with respect to the point B.

Solution.

Let's find the coordinates of the vector $\overline{BA}$:

$$\overline{BA}=(2-3,1-(-1),-3-5)=(-1,2,-8).$$

By formula (2.10), we have:

$$\overline{M}=\overline{BA}\times\overline{F}=\begin{vmatrix}\bar i & \bar j & \bar k\\ -1 & 2 & -8\\ 2 & -4 & 5\end{vmatrix}=\begin{vmatrix}2 & -8\\ -4 & 5\end{vmatrix}\bar i-\begin{vmatrix}-1 & -8\\ 2 & 5\end{vmatrix}\bar j+\begin{vmatrix}-1 & 2\\ 2 & -4\end{vmatrix}\bar k=$$

$$=-22\,\bar i-11\,\bar j+0\,\bar k\,.$$

$$\left|\overline{M}\right|=\sqrt{(-22)^2+(-11)^2+0^2}=\sqrt{605}=11\sqrt{5}\,.$$

Answer: $11\sqrt{5}$.

Mixed triple product

The ***mixed triple product*** or the ***scalar triple product*** of three vectors $\bar a$, $\bar b$ and $\bar c$, written as $\bar a\,\bar b\,\bar c$, is defined as $\bar a\,\bar b\,\bar c=(\bar a\times\bar b)\cdot\bar c$.

If $\bar a=(a_1,a_2,a_3)$, $\bar b=(b_1,b_2,b_3)$, $\bar c=(c_1,c_2,c_3)$ then

$$\bar a\,\bar b\,\bar c=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}. \qquad (2.11)$$

Properties of the mixed product:

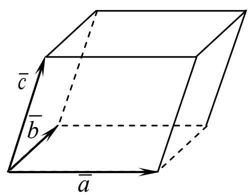
1) $\bar a\,\bar b\,\bar c=\bar b\,\bar c\,\bar a=\bar c\,\bar a\,\bar b$;

2) $\bar a\,\bar b\,\bar c=-\bar b\,\bar a\,\bar c=-\bar c\,\bar b\,\bar a=-\bar a\,\bar c\,\bar b$;

3) three non-zero vectors $\bar a$, $\bar b$, $\bar c$ are ***coplanar*** if and only if $\bar a\,\bar b\,\bar c=0$;

4) if $\bar a\,\bar b\,\bar c>0$, then the vectors $\bar a$, $\bar b$, $\bar c$ form a right-handed triple of vectors, otherwise they form a left-handed triple of vectors.

Geometrical interpretation. The absolute value of the mixed triple product of three vectors $\bar a$, $\bar b$, $\bar c$ is equal to the volume of the parallelepiped formed by these vectors.



The volume of a tetrahedron formed by the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ is given by the formula $V = \frac{1}{6}\left|\bar{a}\,\bar{b}\,\bar{c}\right|$.

Problem 2.11. The vectors $\bar{a} = (2, 1, -1)$, $\bar{b} = (3, 0, -2)$, and $\bar{c} = (1, -1, 2)$ are given.

a) Find $\bar{a}\bar{b}\bar{c}$;

b) determine whether the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar;

c) if the given vectors aren't coplanar, determine the type of the triple they form.

Solution.

a) Using formula (2.11), we obtain:

$$\bar{a}\bar{b}\bar{c} = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 1 & -1 & 2 \end{vmatrix} = -9.$$

b) Since $\bar{a}\bar{b}\bar{c} = -9 \neq 0$, the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ are not coplanar.

c) Since $\bar{a}\bar{b}\bar{c} = -9 < 0$, the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ form a left-handed triple.

Answer: **a)** -9; **b)** the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ are not coplanar; **c)** the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ form a left-handed triple.

Problem 2.12. Four points $A(1, 1, 0)$, $B(-1, 0, 3)$, $C(0, 4, -2)$, and $D(4, 3, -5)$ are given. Determine whether they lie in the same plane.

Solution. Four points A, B, C, and D lie in the same plane if the vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are coplanar. By property 3, three non-zero vectors are coplanar if and only if their mixed triple product is equal to zero. Let's find the coordinates of the vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and compute their mixed triple product $\overline{AB}\,\overline{AC}\,\overline{AD}$:

$$\overline{AB} = (-1-1, 0-1, 3-0) = (-2, -1, 3),$$
$$\overline{AC} = (0-1, 4-1, -2-0) = (-1, 3, -2),$$
$$\overline{AD} = (4-1, 3-1, -5-0) = (3, 2, -5),$$

$$\overline{AB}\,\overline{AC}\,\overline{AD} = \begin{vmatrix} -2 & -1 & 3 \\ -1 & 3 & -2 \\ 3 & 2 & -5 \end{vmatrix} = 0 .$$

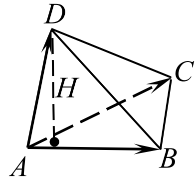
Since $\overline{AB}\,\overline{AC}\,\overline{AD} = 0$, the vectors $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ are coplanar and the points A, B, C, and D lie in the same plane.

Answer: the given points A, B, C, and D lie in the same plane.

Problem 2.13. The tetrahedron is given by its vertices $A(2,5,-3)$, $B(6,4,-13)$, $C(5,3,-13)$, and $D(1,2,2)$. Find the volume of the tetrahedron and the altitude from the point D to the base ABC.

Solution.

Let's find the coordinates of the vectors on which the tetrahedron is constructed:



$$\overline{AB} = (6-2, 4-5, -13-(-3)) = (4,-1,-10),$$

$$\overline{AC} = (5-2, 3-5, -13-(-3)) = (3,-2,-10),$$

$$\overline{AD} = (1-2, 2-5, 2-(-3)) = (-1,-3,5).$$

The volume of the tetrahedron $ABCD$ constructed on the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ is 1/6 of the parallelepiped constructed on these vectors:

$$V_{ABCD} = \frac{1}{6}\left|\overline{AB}\,\overline{AC}\,\overline{AD}\right| . \qquad (2.12)$$

By formula (2.11), we obtain:

$$\overline{AB}\,\overline{AC}\,\overline{AD} = \begin{vmatrix} 4 & -1 & -10 \\ 3 & -2 & -10 \\ -1 & -3 & 5 \end{vmatrix} = -45 .$$

Using formula (2.12), we find the volume of the tetrahedron $ABCD$:

$$V_{ABCD} = \frac{1}{6}\cdot|-45| = \frac{45}{6} = 7.5 \text{ (units}^3\text{)}.$$

On the other hand, the volume of this tetrahedron can also be calculated by the formula:

$$V_{ABCD} = \frac{1}{3}\cdot S_{base}\cdot H = \frac{1}{3}\cdot S_{ABC}\cdot H .$$

From the last formula, $H = \dfrac{3V_{ABCD}}{S_{ABC}}$.

Let's find the area of the triangle ABC:

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 4 & -1 & -10\\ 3 & -2 & -10\end{vmatrix}=\begin{vmatrix}-1 & -10\\ -2 & -10\end{vmatrix}\bar{i}-\begin{vmatrix}4 & -10\\ 3 & -10\end{vmatrix}\bar{j}+\begin{vmatrix}4 & -1\\ 3 & -2\end{vmatrix}\bar{k}=$$

$$=-10\bar{i}+10\bar{j}-5\bar{k},$$

$$\left|\overline{AB}\times\overline{AC}\right|=\sqrt{(-10)^2+10^2+(-5)^2}=15,$$

$$S_{ABC}=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|=\frac{1}{2}\cdot 15=7.5 \text{ (units}^2\text{)}.$$

Finally, $H=\dfrac{3\cdot 7.5}{7.5}=3$ (units).

Answer: 7.5 units^3, 3 units.

Exercises

2.1. A vector $\bar{a}=(2,-1,-2)$ and three points $A(-3,2,4)$, $B(2,-1,5)$ and $C(5,-2,3)$ are given. Find:

a) $|\bar{a}|$, $\left|\overline{AC}\right|$, $\left|\overline{BA}\right|$; **b)** the unit vector $\bar{a}^0$ and the direction cosines of $\bar{a}$.

Answer: **a)** 3, 9, $\sqrt{35}$; **b)** $\bar{a}^0=\left(\frac{2}{3},-\frac{1}{3},-\frac{2}{3}\right)$, $\cos\alpha=\frac{2}{3}$, $\cos\beta=-\frac{1}{3}$, $\cos\gamma=-\frac{2}{3}$.

2.2. The vectors $\bar{a}=(2,-1,-2)$ and $\bar{b}=(-6,3,-2)$ are given. Find:

a) $\left|\bar{a}+\bar{b}\right|$; **b)** $\left|\bar{a}-\bar{b}\right|$; **c)** $\left|2\bar{b}-3\bar{a}\right|$.

Answer: **a)** 6; **b)** $4\sqrt{5}$; **c)** $\sqrt{409}$.

2.3. Find the scalar product of the vectors $\bar{a}$ and $\bar{b}$ if $|\bar{a}|=6$, $|\bar{b}|=4\sqrt{3}$ and the angle between $\bar{a}$ and $\bar{b}$ is equal to $30°$.

Answer: 36.

2.4. Find the scalar product of vectors $\bar{a}=(-2,3,-4)$ and $\bar{b}=(5,7,-2)$.

Answer: 19.

2.5. $\overline{m} = \overline{a} - 2\overline{b}$, $\overline{n} = 3\overline{a} - \overline{b}$, $|\overline{a}| = 2$, $|\overline{b}| = 3\sqrt{2}$, $(\overline{a},\hat{} \overline{b}) = 3\pi/4$ are given. Find:

a) $\overline{m} \cdot \overline{n}$; **b)** $\cos(\overline{m},\hat{} \overline{n})$; **c)** $proj_{\overline{n}}\, \overline{m}$.

Answer: **a)** 90; **b)** $\frac{3\sqrt{10}}{10}$; **c)** $3\sqrt{10}$.

2.6. Find the value of the parameter p that satisfy the required condition:

a) $\overline{a} = (12, p, -8)$, $\overline{b} = (-15, 30, 10)$, and $\overline{a} \parallel \overline{b}$;

b) $\overline{a} = (-5, 3, 2)$, $\overline{b} = (p, -6, 7)$, and $\overline{a} \perp \overline{b}$.

Answer: **a)** -24; **b)** -0.8.

2.7. The points $A(3,-2,4)$, $B(4,-3,7)$, and the force $\overline{F} = (3,-2,4)$ are given. Find the work done by the force $\overline{F}$ that moves an object from the point A to the point B along the straight line.

Answer: 17.

2.8. The vectors $\overline{a} = (3,-2,-10)$ and $\overline{b} = (-4,1,10)$ are given. Find $\overline{a} \times \overline{b}$, $|\overline{a} \times \overline{b}|$ and the area S of the parallelogram formed by the vectors $\overline{a}$ and $\overline{b}$.

Answer: $\overline{a} \times \overline{b} = (-10, 10, -5)$, $|\overline{a} \times \overline{b}| = 15$, $S = 15$ units^2.

2.9. $\overline{m} = \overline{a} - 2\overline{b}$, $\overline{n} = 3\overline{a} - \overline{b}$, $|\overline{a}| = 2$, $|\overline{b}| = 3\sqrt{2}$, $(\overline{a},\hat{} \overline{b}) = 3\pi/4$ are given. Find the area S of the parallelogram formed by the vectors $\overline{m}$ and $\overline{n}$.

Answer: 30 units^2.

2.10. The points $A(3,-2,4)$, $B(4,-3,7)$, and the force $\overline{F} = (3,-2,4)$ are given. Find the magnitude of the moment $\overline{M}$ of the force $\overline{F}$, applied at the point A, with respect to the point B.

Answer: $\sqrt{30}$.

2.11. The vectors $\overline{a} = (3,-2,1)$, $\overline{b} = (0,4,-2)$ and $\overline{c} = (-1,2,1)$ are given.

a) Find $\overline{a}\overline{b}\overline{c}$;

b) determine whether the vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ are coplanar;

c) if the given vectors aren't coplanar, determine the type of the triple they form.

Answer: **a)** 24; **b)** the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ aren't coplanar; **c)** the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ form a right-handed triple.

2.12. Four points $A(2,-1,-2)$, $B(1,0,3)$, $C(3-2,-7)$, $D(-1,3,2)$ are given. Determine whether they lie in the same plane.

Answer: the points A, B, C, and D lie in the same plane.

2.13. The tetrahedron is given by its vertices $A(4,0,1)$, $B(-1,-7,-3)$, $C(-5,-10,-1)$ and $D(2,-11,4)$. Find the volume of the tetrahedron and the altitude from the point D to the base ABC.

Answer: 45.5 units^3, 7 units.

3 ANALYTICAL GEOMETRY

Distance between two points

Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is calculated by the formula:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$

Division of a Segment in a Given Ratio

A line segment, whose endpoints are $A(x_1, y_1)$ and $B(x_2, y_2)$, is divided by a point $C(x, y)$ in the ratio λ if $|AC|/|CB| = \lambda$. The coordinates of the point $C(x, y, z)$ can be found by the following formulas:

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \ y = \frac{y_1 + \lambda y_2}{1 + \lambda} \ .$$

If $\lambda = 1$ then $C(x, y)$ is the ***midpoint*** of the segment AB, and

$$x = \frac{x_1 + x_2}{2}, \ y = \frac{y_1 + y_2}{2} \ .$$

Straight line on the plane

1. ***General equation:*** $Ax + By + C = 0$,

where A, B, $C \in R$, $\overline{n} = (A, B)$ is a normal vector of the line.

The ***normal vector*** of the straight line is the vector which is perpendicular to the line.

2. ***Slope-intercept form of equation:*** $y = kx + b$,

where b is the y-intercept and k is the ***slope*** of the straight line.

3. ***Point-slope equation:*** $y - y_0 = k(x - x_0)$,

where $M_0(x_0, y_0)$ is the point on the line.

4. ***Intercept form of equation:*** $\frac{x}{a} + \frac{y}{b} = 1$,

where a and b is the x-intercept and y-intercept, respectively.

5. ***Two-point form of equation or an equation of a line passing through two given points*** $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ has the following form:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}.$$

The ***distance*** from the point $M_0(x_0, y_0)$ to the line $Ax+By+C=0$ is calculated by the following formula:

$$d = \frac{|A\cdot x_0 + B\cdot y_0 + C|}{\sqrt{A^2+B^2}}. \qquad (3.1)$$

Angle between two lines

1) If two lines in the xy-plane are given by their equations in the slope-intercept form $y=k_1x+b_1$ and $y=k_2x+b_2$, and φ is the angle between the lines, then

$$\tan\varphi = \left|\frac{k_2-k_1}{1+k_1\cdot k_2}\right|. \qquad (3.2)$$

Test for parallelism: $k_1=k_2$.

Test for perpendicularity: $k_1k_2=-1$.

2) If two lines in the xy-plane are given by their equations in the general form $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$, and φ is the angle between the lines, then

$$\cos\varphi = \frac{|\bar{n}_1\cdot\bar{n}_2|}{|\bar{n}_1||\bar{n}_2|} = \frac{|A_1A_2+B_1B_2|}{\sqrt{A_1^2+B_1^2}\sqrt{A_2^2+B_2^2}}.$$

Test for parallelism: $\frac{A_1}{A_2}=\frac{B_1}{B_2}$.

Test for perpendicularity: $A_1A_2+B_1B_2=0$.

Problem 3.1. Let ABC be a triangle with the vertices at the points $A(1,2)$, $B(2,-3)$ and $C(-3,1)$ in the xy-plane. Sketch the drawing. Find:

a) the length of the side BC and its equation;

b) the equation of the median AM from the vertex A;

c) the length of the altitude AH from the vertex A and its equation;

d) the equation of the line l passing through the point A parallel to the line BC;

e) the tangent of the angle B.

Solution.

Fig. 3.1 shows the given triangle.

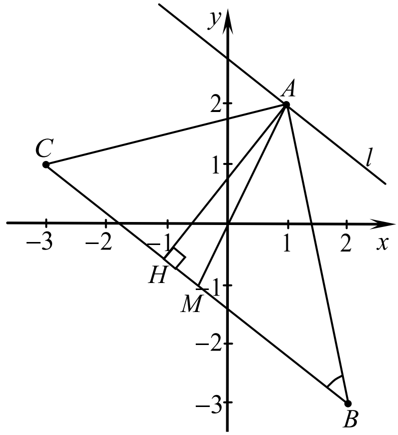


Figure 3.1

a) Let's find the length of BC:

$$BC=\sqrt{(x_C-x_B)^2+(y_C-y_B)^2}=\sqrt{(-3-2)^2+(1-(-3))^2}=\sqrt{41}.$$

Using the equation of a line passing through two given points, we find the equation of BC:

$$BC:\ \frac{x-x_B}{x_C-x_B}=\frac{y-y_B}{y_C-y_B},\ \frac{x-2}{-3-2}=\frac{y-(-3)}{1-(-3)},\ \frac{x-2}{-5}=\frac{y+3}{4},$$

$4x+5y+7=0$ is a general equation of BC,

$y=-\frac{4}{5}x-\frac{7}{5}$ is a slope-intercept form equation of BC.

b) Let $M(x_M, y_M)$ be the midpoint of the segment BC:

$$x_M=\frac{x_B+x_C}{2}=\frac{2+(-3)}{2}=-\frac{1}{2},\ y_M=\frac{y_B+y_C}{2}=\frac{-3+1}{2}=-1,$$

$$M\left(-\frac{1}{2},-1\right).$$

Using the equation of a line passing through two given points, we find the equation of AM:

$$AM:\ \frac{x-x_A}{x_M-x_A}=\frac{y-y_A}{y_M-y_A},\ \frac{x-1}{-\frac{1}{2}-1}=\frac{y-2}{-1-2},\ \frac{x-1}{-\frac{3}{2}}=\frac{y-2}{-3},\ y=2x.$$

c) The length of the altitude from the vertex A is equals to the distance from the point A to the line BC. By formula (3.1), we find the altitude AH:

$$d=AH=\frac{|Ax_A+By_A+C|}{\sqrt{A^2+B^2}}=\frac{|4\cdot 1+5\cdot 2+7|}{\sqrt{4^2+5^2}}=\frac{21}{\sqrt{41}}=\frac{21\sqrt{41}}{41}.$$

The slope of BC from a) equals $k_{BC}=-\frac{4}{5}$.

$$AH\perp BC\Rightarrow k_{AH}\cdot k_{BC}=-1,\ k_{AH}=-\frac{1}{k_{BC}}=\frac{5}{4}.$$

Using the point-slope equation, we find the equation of the altitude AH:

$$y=y_A+k_{AH}(x-x_A),\ y=2+\frac{5}{4}(x-1)=\frac{5}{4}x+\frac{3}{4}.$$

d) $l \parallel BC \Rightarrow k_l=k_{BC}=-\frac{4}{5}$. Using the point-slope equation, we find the equation of the line l:

$$y=y_A+k_l(x-x_A)\Rightarrow y=2-\frac{4}{5}(x-1)\Rightarrow y=-\frac{4}{5}x+\frac{14}{5}.$$

e) By formula (3.2), $\tan B=\frac{k_{BC}-k_{AB}}{1+k_{AB}\cdot k_{BC}}$. From a) $k_{BC}=-\frac{4}{5}$. Using the equation of a line passing through two given points, we find the equation of AB and its slope:

$$AB:\ \frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A},\ \frac{x-1}{2-1}=\frac{y-2}{-3-2},\ \frac{x-1}{1}=\frac{y-2}{-5},$$

$$y=-5x+7,\ k_{AB}=-5.$$

$$\tan B=\frac{-\frac{4}{5}-(-5)}{1+\left(-\frac{4}{5}\right)\cdot(-5)}=\frac{21}{25}.$$

Answer: fig. 3.1; **a)** $BC=\sqrt{41}$, $BC: y=-\frac{4}{5}x-\frac{7}{5}$; **b)** $AM: y=2x$; **c)** $AH=\frac{21\sqrt{41}}{41}$, $AH: y=\frac{5}{4}x+\frac{3}{4}$; **d)** $l: y=-\frac{4}{5}x+\frac{14}{5}$; **e)** $\tan B=\frac{21}{25}$.

Plane

1. ***General equation:***

$$Ax+By+Cz+D=0,$$

where $A,B,C\in R$, $\bar{n}=(A,B,C)$ is the normal vector of the plane.

The ***normal vector*** of the plane is the vector that is perpendicular to the given plane.

2. ***The equation of the plane passing through a given point*** $M_0(x_0,y_0,z_0)$:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0. \qquad (3.3)$$

3. ***Three-point form of equation.*** The equation of a plane passing through three given points $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$, $M_3(x_3,y_3,z_3)$ that do not lie on the same line:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0. \qquad (3.4)$$

4. ***The intercept form of the equation:***

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1, \qquad (3.5)$$

where a, b, c are the x-, y-, z-intercepts, respectively.

The ***angle*** ψ ***between two planes*** $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is equal to the angle between their normal vectors $\bar{n}_1=(A_1,B_1,C_1)$ and $\bar{n}_2=(A_2,B_2,C_2)$. The cosine of the angle ψ is calculated by the formula:

$$\cos\psi=\left|\cos(\bar{n}_1,\hat{\ }\bar{n}_2)\right|=\frac{|\bar{n}_1\cdot\bar{n}_2|}{|\bar{n}_1|\cdot|\bar{n}_2|}, \qquad (3.6)$$

where $0\le\psi\le 90°$.

Test for parallelism: $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}\neq\frac{D_1}{D_2}$.

Test for perpendicularity: $\bar{n}_1\cdot\bar{n}_2=0$ or $A_1A_2+B_1B_2+C_1C_2=0$.

Straight line in space

1. ***Parametric equations:***

$$\begin{cases} x=x_0+l\,t, \\ y=y_0+m\,t, \\ z=z_0+n\,t, \end{cases} \tag{3.7}$$

where $t\in R$ is a parameter, $\bar{s}=(l,m,n)$ is a direction vector of the line, the point $M_0(x_0,y_0,z_0)$ belongs to the line.

A ***direction vector*** of the line is the vector that is parallel to the given line or lies on it.

2. ***Canonical equations*:**

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}, \tag{3.8}$$

where $\bar{s}=(l,m,n)$ is a direction vector of the line, the point $M_0(x_0,y_0,z_0)$ belongs to the line.

3. ***Two-point form of equations*** or ***equations of a line passing through two given points*** $M_1(x_1,y_1,z_1)$ and $M_2(x_2,y_2,z_2)$ has the following form:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \tag{3.9}$$

4. ***The general equations of the line***. In general case, the straight line is the result of intersection of two planes:

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \\ A_2x+B_2y+C_2z+D_2=0, \end{cases}$$

where the normal vectors of the planes, $\bar{n}_1=(A_1,B_1,C_1)$ and $\bar{n}_2=(A_2,B_2,C_2)$, are not collinear. The direction vector $\bar{s}$ of the line is given by the vector $\bar{s}=\bar{n}_1\times\bar{n}_2$.

The ***distance*** between the point $M_0(x_0,y_0,z_0)$ and the plane $Ax+By+Cz+D=0$ is given by the following formula:

$$d = \frac{|A \cdot x_0 + B \cdot y_0 + C \cdot z_0 + D|}{\sqrt{A^2 + B^2 + C^2}}. \quad (3.10)$$

The ***cosine of the angle*** α ***between two straight lines*** is equal to the one between their direction vectors. It is calculated by the formula

$$\cos\alpha = \left|\cos(\bar{s}_1, \hat{\ } \bar{s}_2)\right| = \frac{|\bar{s}_1 \cdot \bar{s}_2|}{|\bar{s}_1| \cdot |\bar{s}_2|}, \quad (3.11)$$

where $\bar{s}_1$, $\bar{s}_2$ are the direction vectors of the given lines, $0 \le \alpha \le 90°$.

The ***sine of the angle*** φ ***between a line and a plane*** can be determined using the formula

$$\sin\varphi = \left|\cos(\bar{n}, \hat{\ } \bar{s})\right| = \frac{|\bar{n} \cdot \bar{s}|}{|\bar{n}| \cdot |\bar{s}|}, \quad (3.12)$$

where $\bar{s}$ is the direction vector of the line, $\bar{n}$ is the normal vector of the plane, $0 \le \varphi \le 90°$.

Problem 3.2. The points $A(1,2,-1)$, $B(-2,4,3)$, $C(3,4,-2)$, and $D(-2,5,-1)$ are given. Find:

a) the equation of the plane (ABC) in the intercept form and sketch the drawing;

b) the equation of the line l_1 passing through the point C parallel to the line AB;

c) the equation of the plane Q_1 passing through the point C perpendicular to the line AB;

d) the equation of the line l_2 passing through the point D perpendicular to the plane (ABC);

e) the equation of the plane Q_2 passing through the point D parallel to the plane (ABC);

f) the distance from the point D to the plane (ABC);

g) the point P of intersection of the line l_2 and the plane (ABC);

h) the angle between the lines AB and CD;

i) the angle between the line CD and the plane (ABC).

Solution.

a) We substitute the coordinates of the points A, B, C into equation (3.4):

$$\begin{vmatrix} x-1 & y-2 & z-(-1) \\ -2-1 & 4-2 & 3-(-1) \\ 3-1 & 4-2 & -2-(-1) \end{vmatrix} = 0, \qquad \begin{vmatrix} x-1 & y-2 & z+1 \\ -3 & 2 & 4 \\ 2 & 2 & -1 \end{vmatrix} = 0.$$

We expand the determinant by the first row:

$$(x-1)\cdot\begin{vmatrix} 2 & 4 \\ 2 & -1 \end{vmatrix} - (y-2)\cdot\begin{vmatrix} -3 & 4 \\ 2 & -1 \end{vmatrix} + (z+1)\cdot\begin{vmatrix} -3 & 2 \\ 2 & 2 \end{vmatrix} = 0,$$

$$(x-1)\cdot(-10)-(y-2)\cdot(-5)+(z+1)\cdot(-10)=0,$$

$$-10x+5y-10z-10=0,\ 2x-y+2z+2=0.$$

So, $2x-y+2z+2=0$ is the general equation of (ABC). The normal vector of the plane (ABC) is $\bar{n}=(2,-1,2)$.

Let's write the equation of (ABC) in the intercept form (3.5):

$$2x-y+2z=-2 \,\Big|\cdot\frac{1}{-2}, \qquad \frac{x}{-1}+\frac{y}{2}+\frac{z}{-1}=1.$$

Fig. 3.2 shows the graph of the plane (ABC).

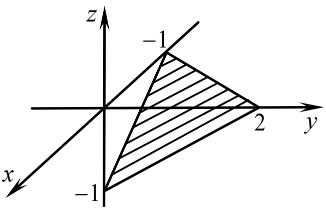


Figure 3.2

b) $l_1 \parallel AB \Rightarrow \bar{s}_{l_1}=\bar{s}_{AB}$. Let's find the direction vector of AB:

$$\bar{s}_{AB}=\overline{AB}=(-2-1,4-2,3-(-1))=(-3,2,4).$$

Using equations (3.8), we find the canonical equations of l_1:

$$\frac{x-3}{-3}=\frac{y-4}{2}=\frac{z-(-2)}{4},\ \frac{x-3}{-3}=\frac{y-4}{2}=\frac{z+2}{4}.$$

c) Let $\bar{n}_1$ be the normal vector of the plane Q_1 passing through the point C perpendicular to the line AB.

$$AB\perp Q_1 \Rightarrow \bar{n}_1=\bar{s}_{AB}=(-3,2,4).$$

Using equation (3.3), we get the equation of the plane Q_1:

$$Q_1:\ -3\cdot(x-3)+2\cdot(y-4)+4\cdot(z-(-2))=0,$$

$$-3x+2y+4z+9=0,\ 3x-2y-4z-9=0.$$

d) Let $\bar{s}_{l_2}$ be the direction vector of the line l_2 passing through the point D perpendicular to the plane (ABC). $l_2 \perp (ABC) \Rightarrow \bar{s}_{l_2}=\bar{n}=(2,-1,2)$. Let's find the equations of l_2 in the parametric form (3.7):

$$\begin{cases} x=-2+2t, \\ y=5-t, \\ z=-1+2t. \end{cases}$$

e) Let $\bar{n}_2$ be a normal vector of the plane Q_2 passing through the point D parallel to the plane (ABC). $Q_2 \parallel (ABC) \Rightarrow \bar{n}_2=\bar{n}=(2,-1,2)$. Using equation (3.3), we find the equation of the plane Q_2:

$$2\cdot(x-(-2))-1\cdot(y-5)+2\cdot(z-(-1))=0,\ 2x-y+2z+11=0.$$

f) By formula (3.10), the distance d from the point D to the plane (ABC) is

$$d=\frac{|2\cdot(-2)-1\cdot 5+2\cdot(-1)+2|}{\sqrt{2^2+(-1)^2+2^2}}=3.$$

g) The point P of intersection of the line l_2 and the plane (ABC) can be found by solving the following system:

$$\begin{cases} (ABC): 2x-y+2z+2=0, \\ l_2: \begin{cases} x=-2+2t, \\ y=5-t, \\ z=-1+2t. \end{cases} \end{cases}$$

We substitute x, y, z in the first equation of the system with their expressions in terms of t from the last three equations and solve for t:

$$2(-2+2t)-(5-t)+2(-1+2t)+2=0,\ t=1.$$

Substituting $t=1$ into the last three equations of the system, we determine the coordinates of the point P:

$$\begin{cases} x = -2 + 2 \cdot 1, \\ y = 5 - 1, \\ z = -1 + 2 \cdot 1. \end{cases} \Rightarrow \begin{cases} x = 0, \\ y = 4, \\ z = 1. \end{cases} \Rightarrow P(0,4,1).$$

h) From b), we know the direction vector of the line AB: $\bar{s}_{AB} = \overline{AB} = (-3,2,4)$. Let's find the direction vector of the line CD: $\bar{s}_{CD} = \overline{CD} = (-2-3, 5-4, -1-(-2)) = (-5;1;1)$. Using formula (3.11), we obtain:

$$\cos\alpha = \frac{\left|\overline{AB} \cdot \overline{CD}\right|}{\left|\overline{AB}\right| \cdot \left|\overline{CD}\right|} = \frac{\left|-3 \cdot (-5) + 2 \cdot 1 + 4 \cdot 1\right|}{\sqrt{(-3)^2 + 2^2 + 4^2} \cdot \sqrt{(-5)^2 + 1^2 + 1^2}} =$$

$$= \frac{21}{\sqrt{29} \cdot \sqrt{27}} = \frac{7\sqrt{87}}{87}.$$

$$\alpha = \arccos\frac{7\sqrt{87}}{87} \approx 41^\circ.$$

i) By (3.12),

$$\sin\varphi = \frac{\left|\bar{n} \cdot \bar{s}_{CD}\right|}{\left|\bar{n}\right| \cdot \left|\bar{s}_{CD}\right|} = \frac{\left|2 \cdot (-5) + (-1) \cdot 1 + 2 \cdot 1\right|}{\sqrt{2^2 + (-1)^2 + 2^2} \cdot \sqrt{(-5)^2 + 1^2 + 1^2}} = \frac{\sqrt{3}}{3},$$

$$\varphi = \arcsin\frac{\sqrt{3}}{3} \approx 35^\circ.$$

Answer: **a)** $(ABC): \frac{x}{-1} + \frac{y}{2} + \frac{z}{-1} = 1$, fig. 3.2; **b)** $l_1: \frac{x-3}{-3} = \frac{y-4}{2} = \frac{z+2}{4}$; **c)** $3x - 2y - 4z - 9 = 0$; **d)** $l_2: \begin{cases} x = -2 + 2t, \\ y = 5 - t, \\ z = -1 + 2t. \end{cases}$; **e)** $2x - y + 2z + 11 = 0$; **f)** 3; **g)** $P(0,4,1)$; **h)** $\arccos\frac{7\sqrt{87}}{87} \approx 41^\circ$; **i)** $\arcsin\frac{\sqrt{3}}{3} \approx 35^\circ$.

Problem 3.3. Reduce the general equations of the line $\begin{cases} 2x - y - 3z - 6 = 0, \\ 2y + z - 4 = 0. \end{cases}$ to the canonical form.

Solution.

The given line is the result of intersection of two planes with normal vectors $\bar{n}_1 = (2,-1,-3)$ and $\bar{n}_2 = (0,2,1)$. Let's find the direction vector of the line:

$$\bar{s} = \bar{n}_1 \times \bar{n}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -1 & -3 \\ 0 & 2 & 1 \end{vmatrix} = 5\bar{i} - 2\bar{j} + 4\bar{k} .$$

To find the point that belongs to the line we take $z = 0$ in the system and solve it:

$$\begin{cases} 2x - y - 3z - 6 = 0, \\ 2y + z - 4 = 0, \\ z = 0. \end{cases} \Rightarrow \begin{cases} x = 4, \\ y = 2, \\ z = 0. \end{cases} \Rightarrow M(4,2,0).$$

The canonical form of the equations has the following form:

$$\frac{x-4}{5} = \frac{y-2}{-2} = \frac{z}{4} .$$

Answer: $\frac{x-4}{5} = \frac{y-2}{-2} = \frac{z}{4}$.

Problem 3.4. Find the angle between two planes $2x - 3y + z - 5 = 0$ and $5x - 2z + 4 = 0$.

Solution.

The normal vectors of the planes are $\bar{n}_1 = (2,-3,1)$ and $\bar{n}_2 = (5,0,-2)$. By formula (3.6), we find the angle ψ between the given planes:

$$\cos\psi = \frac{|2\cdot 5 - 3\cdot 0 + 1\cdot(-2)|}{\sqrt{2^2 + (-3)^2 + 1^2}\cdot\sqrt{5^2 + 0^2 + (-2)^2}} = \frac{4\sqrt{406}}{203} ,$$

$$\psi = \arccos\frac{4\sqrt{406}}{203} \approx 67^\circ .$$

Answer: $\arccos\frac{4\sqrt{406}}{203} \approx 67^\circ$.

Quadratic curves

A ***quadratic curve*** is a set of points in the xy-plane whose coordinates (x, y) satisfy the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0, \qquad (3.13)$$

where a_{11}, a_{12}, a_{22}, a_1, a_2, a_0 are real numbers.

Equation (3.13) can represent a circle, an ellipse, a hyperbola, or a parabola. In degenerate cases, it may also represent a single point, an empty set, a straight line, or a pair of straight lines.

By applying a rotation and translation of the coordinate system, equation (3.13) can be transformed into canonical form [4].

Circle

A ***circle*** is the set of all points in a plane that are at the same distance from a given point, called the ***center***. This distance is known as the ***radius***. The equation of a circle with center at the point $C(x_0, y_0)$ and radius R has the following form:

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

Ellipse

An ***ellipse*** is the set of all points for which the sum of the distances from two fixed points, called the ***foci***, is a constant. This constant is usually denoted by $2a$ and it must be larger than the distance $2c$ between the foci. The ellipse with foci at the points $F_1(-c,0)$ and $F_2(c,0)$ is shown in Fig. 3.3. The points $A_1(-a,0)$, $A_2(a,0)$, $B_1(0,-b)$, $B_2(0,b)$ are called the ***vertices*** of the ellipse.

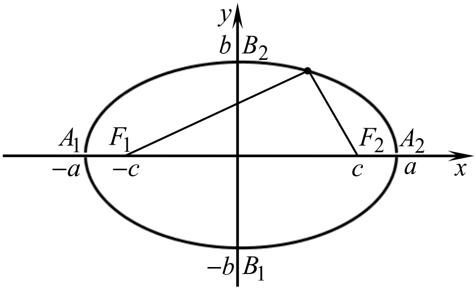


Figure 3.3

The canonical equation of the ellipse has the following form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $b^2=a^2-c^2$, a and b are termed the ***major semi-axis*** and the ***minor semi-axis*** of the ellipse, respectively.

The number $e=c/a$ is called the ***eccentricity of the ellipse***. For an ellipse, $0\le e<1$. If $a=b$ the ellipse becomes a circle, and its $e=0$.

Straight lines $D_1: x=-a/e$ and $D_2: x=a/e$ are called the ***directrices*** of the ellipse.

Hyperbola

A ***hyperbola*** is the set of all points for which the difference of the distances from two fixed points, called ***foci***, is a constant. This constant is usually denoted by $2a$ and it must be smaller than the distance $2c$ between the foci.

The hyperbola whose foci are $F_1(-c,0)$ and $F_2(c,0)$ is shown in Fig. 3.4.

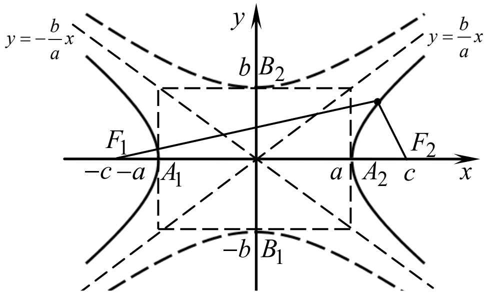


Figure 3.4

The canonical equation of the hyperbola has the following form:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where $b^2=c^2-a^2$, a and b are known as the ***real*** and the ***conjugate semi-axes***, respectively.

The points $A_1(-a,0)$, $A_2(a,0)$ are called the ***vertices*** of the hyperbola.

The hyperbola has two ***asymptotes***: $y = \pm \frac{b}{a} x$.

The ***eccentricity*** of the hyperbola is defined in the same way as for the ellipse. For hyperbola, $e > 1$.

The straight lines $D_1 : x = -a/e$ and $D_2 : x = a/e$ are called the ***directrices*** of the hyperbola.

The hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called the ***conjugate hyperbola***. Its vertices are $B_1(0,-b)$, $B_2(0,b)$. The ***conjugate hyperbola*** is represented by a dashed line in Fig. 3.4.

Parabola

A ***parabola*** is the set of all points that are equidistant from a fixed point, called the ***focus***, and a fixed line, called the ***directrix.*** Fig. 3.5 shows the parabola which has the equation of the directrix $x = -p/2$, $(p > 0)$ and with focus at the point $F(p/2,0)$.

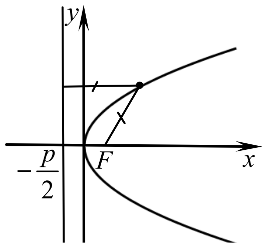


Figure 3.5

The canonical equation of the ***parabola*** has the following form:

$$y^2 = 2px .$$

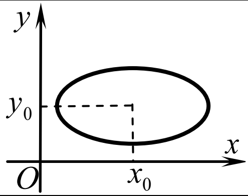
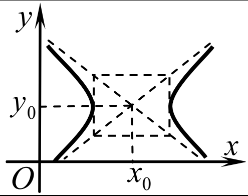
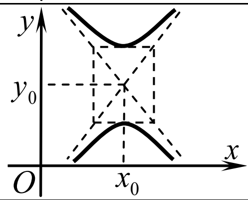
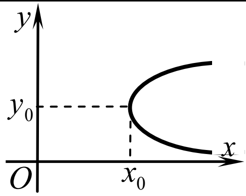
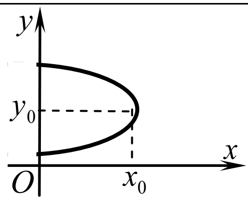
where the number p is called the ***parameter***.

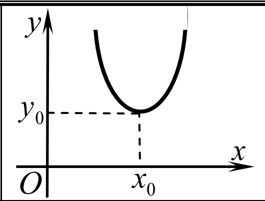
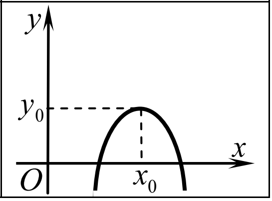
The axis of symmetry is the x-axis (Fig. 3.5). The ***vertex*** of a parabola is the point where it intersects its axis of symmetry.

The following equation represents a parabola whose axis of symmetry is the y-axis:

$$x^2 = 2py .$$

Equations and graphs of quadratic curves with the center at the point (x_0, y_0)

	The type of the curve	Equation	Graph
1	Ellipse	$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$	
2	Hyperbola	$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$	
		$\frac{(y-y_0)^2}{b^2}-\frac{(x-x_0)^2}{a^2}=1$	
3	Parabola	$(y-y_0)^2=2p(x-x_0)$, the equation of the axis of symmetry is $y=y_0$	
		$(y-y_0)^2=-2p(x-x_0)$, the equation of the axis of symmetry is $y=y_0$	

		$(x-x_0)^2 = 2p(y-y_0)$, the equation of the axis of symmetry is $x = x_0$	
		$(x-x_0)^2 = -2p(y-y_0)$, the equation of the axis of symmetry is $x = x_0$	

Problem 3.5. Reduce the equations of the given curves to their canonical form. If the curve is an ellipse or a hyperbola, find its center and its semi-axes. If the curve is a parabola, find its vertex, the intersection points with the coordinate axes, the equation of its axis of symmetry. Sketch the graphs of the curves.

a) $9x^2 + 25y^2 + 72x + 100y + 19 = 0$;

b) $9x^2 + 4y^2 + 36x - 8y + 4 = 0$;

c) $x^2 - 4y^2 - 6x - 8y + 1 = 0$;

d) $25x^2 - 4y^2 - 250x + 8y + 721 = 0$;

e) $y^2 - 4x + 2y + 6 = 0$;

f) $x^2 - 4x - 2y + 10 = 0$.

Solution.

a) $9x^2 + 25y^2 + 72x + 100y + 19 = 0$.

Let's complete the perfect squares:

$$9\left(x^2 + 8x + 16 - 16\right) + 25\left(y^2 + 4y + 4 - 4\right) + 19 = 0,$$

$$9\left(x^2 + 8x + 16\right) - 144 + 25\left(y^2 + 4y + 4\right) - 100 + 19 = 0,$$

$$9(x+4)^2 + 25(y+2)^2 = 225.$$

We divide both sides of the last equation by 225:

$$\frac{(x+4)^2}{25} + \frac{(y+2)^2}{9} = 1.$$

Thus, the given equation describes the ellipse with the center at the point $O_1(-4,-2)$. The major semi-axis equals $a=5$, and the minor semi-axis is $b=3$. Fig. 3.6 shows the graph of the ellipse.

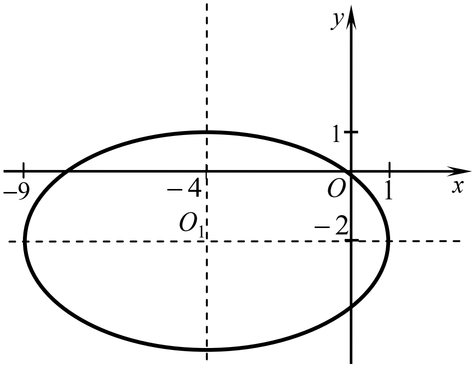


Figure 3.6

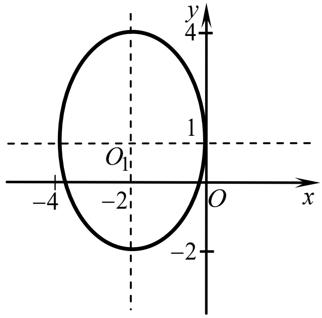


Figure 3.7

b) $9x^2+4y^2+36x-8y+4=0$.

Let's complete the perfect squares:

$$9\left(x^2+4x+4-4\right)+4\left(y^2-2y+1-1\right)+4=0,$$

$$9\left(x^2+4x+4\right)-36+4\left(y^2-2y+1\right)-4+4=0,\ 9(x+2)^2+4(y-1)^2=36.$$

We divide both sides of the last equation by 36:

$$\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}=1.$$

The given equation describes the ellipse with the center at the point $O_1(-2,1)$. The major semi-axis equals $a=2$, and the minor semi-axis is $b=3$. Fig. 3.7 shows the graph of the ellipse.

c) $x^2-4y^2-6x-8y+1=0$.

Let's complete the perfect squares:

$$\left(x^2-6x+9-9\right)-4\left(y^2+2y+1-1\right)+1=0,$$

$$\left(x^2-6x+9\right)-9-4\left(y^2+2y+1\right)+4+1=0,\ (x-3)^2-4(y+1)^2=4.$$

We divide both sides of the last equation by 4:

$$\frac{(x-3)^2}{4}-\frac{(y+1)^2}{1}=1.$$

Thus, the given equation describes the hyperbola with the center at the point $O_1(3,-1)$. The semi-axes are $a=2$, $b=1$. Fig. 3.8 shows the graph of the hyperbola.

d) $25x^2-4y^2-250x+8y+721=0$.

Let's complete the perfect squares:

$$25\left(x^2-10x+25-25\right)-4\left(y^2-2y+1-1\right)+721=0,$$

$$25\left(x^2-10x+25\right)-625-4\left(y^2-2y+1\right)+4+721=0,$$

$$25(x-5)^2-4(y-1)^2=-100.$$

We divide both sides of the last equation by –100:

$$\frac{(y-1)^2}{25}-\frac{(x-5)^2}{4}=1.$$

Thus, the given equation describes the hyperbola with the center at the point $O_1(5,1)$. The semi-axes are $a=2$, $b=5$. Fig. 3.9 shows the graph of the hyperbola.

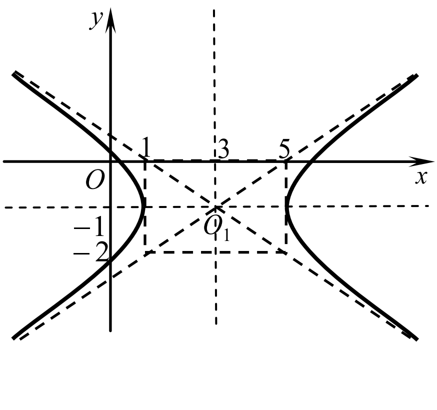


Figure 3.8

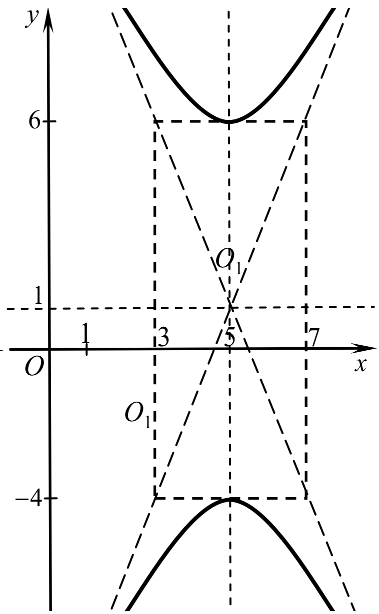


Figure 3.9

e) $y^2 - 4x + 2y + 6 = 0$.

Let's complete the perfect square:

$$\left(y^2 + 2y + 1\right) - 1 = 4x - 6, \ (y+1)^2 = 4x - 5, \ (y+1)^2 = 4\left(x - \frac{5}{4}\right).$$

This equation describes the parabola with the vertex at the point $O_1\left(\frac{5}{4}, -1\right)$.

Let's find the intersection points with the coordinate axes:

if $x = 0$, then $y^2 + 2y + 6 = 0$, $D = -20$, this equation has no solution, the curve doesn't intersect the y-axis;

if $y = 0$, then $-4x + 6 = 0$, $x = \frac{3}{2}$, the curve intersects the x-axis at the point $\left(\frac{3}{2}, 0\right)$.

The axis of symmetry is the line $y + 1 = 0$, $y = -1$, which is parallel to the x-axis. Fig. 3.10 shows the graph of the parabola.

f) $x^2 - 4x - 2y + 10 = 0$.

Let's complete the perfect square:

$$\left(x^2 - 4x + 4\right) - 4 = 2y - 10, \ (x-2)^2 = 2y - 6, \ (x-2)^2 = 2(y-3).$$

This equation describes the parabola with the vertex at the point $O_1(2,3)$.

Let's find the intersection points with the coordinate axes:

if $x = 0$, then $-2y + 10 = 0$, $y = 5$, the curve intersects the y-axis at the point $(0,5)$;

if $y = 0$, then $x^2 - 4x + 10 = 0$, $D = -24$, this equation has no solution, the curve doesn't intersect the x-axis.

The axis of symmetry is the line $x - 2 = 0$, $x = 2$, which is parallel to the x-axis. Fig. 3.11 shows the graph of the parabola.

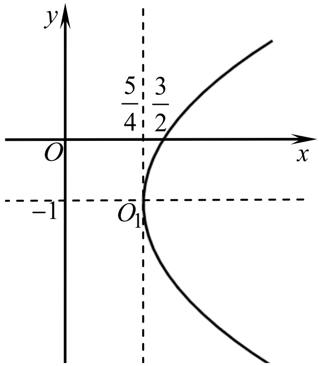


Figure 3.10

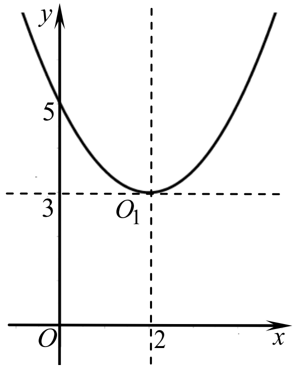


Figure 3.11

Answer: **a)** $\frac{(x+4)^2}{25}+\frac{(y+2)^2}{9}=1$ is the ellipse; the center is $O_1(-4,-2)$; the semi-axes are $a=5$, $b=3$; fig. 3.6;

b) $\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}=1$ is the ellipse; the center is $O_1(-2,1)$; the semi-axes are $a=2$, $b=3$; fig. 3.7;

c) $\frac{(x-3)^2}{4}-\frac{(y+1)^2}{1}=1$ is the hyperbola; the center is $O_1(3,-1)$, the semi-axes are $a=2$, $b=1$; fig. 3.8;

d) $\frac{(y-1)^2}{25}-\frac{(x-5)^2}{4}=1$ is the hyperbola; the center is $O_1(1,5)$, the semi-axes are $a=2$, $b=5$; fig. 3.9;

e) $(y+1)^2=4\left(x-\frac{5}{4}\right)$ is the parabola; the vertex is $O_1\left(\frac{5}{4},-1\right)$; it doesn't intersect the y-axis, the curve intersects the x-axis at the point $\left(\frac{3}{2},0\right)$; the axis of the symmetry is $y=-1$; fig. 3.10;

f) $(x-2)^2=2(y-3)$ is the parabola; the vertex is $O_1(2,3)$; it doesn't intersect the x-axis, the curve intersects the y-axis at the point $(0,5)$; the axis of the symmetry is $x=2$; fig. 3.11.

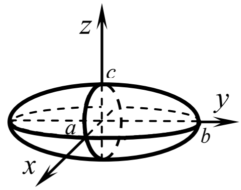
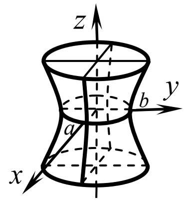
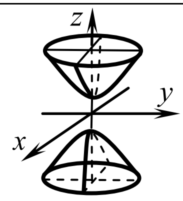
Second-order surfaces

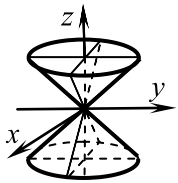
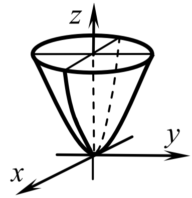
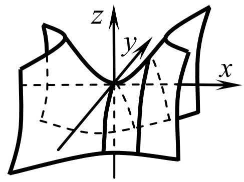
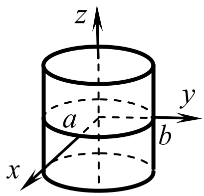
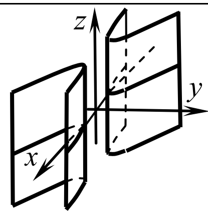
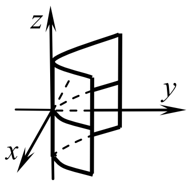
The ***equation of a second-order surface*** has the following form:

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz +$$
$$+2a_1x + 2a_2y + 2a_3z + a_0 = 0\,. \qquad (3.14)$$

In general, the equation (3.14) may define a degenerate surface: a point, a plane, a pair of planes, a straight line, or an empty set. If the surface is non-degenerate, then, by applying a rotation and translation of the coordinate system, equation (3.14) can be reduced to canonical form [4]. The table below presents the canonical equations of second-order surfaces and their graphs.

Canonical equations of the second-order surfaces and its graphs

	The type of the surface	Equation	Graph
1	Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	
2	Hyperboloid of one sheet	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	
3	Hyperboloid of two sheets	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	

4	Second-order cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	
5	Elliptic paraboloid	$\frac{x^2}{p}+\frac{y^2}{q}=2z$ $(p>0,\ q>0)$	
6	Hyperbolic paraboloid	$\frac{x^2}{p}-\frac{y^2}{q}=2z$ $(p>0,\ q>0)$	
7	Elliptic cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	
8	Hyperbolic cylinder	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	
9	Parabolic cylinder	$x^2=2py$ $(p>0)$	

Problem 3.6. Reduce the equation of the surface to the canonical form. Determine the type of the surface and sketch the drawing.

a) $36x^2+16y^2+9z^2+72x-64y+54z+37=0$;

b) $x^2+4x-4y+16=0$;

c) $9x^2+36y^2-4z^2-18x+216y+16z+317=0$.

Solution.

a) $36x^2+16y^2+9z^2+72x-64y+54z+37=0$.

Let's complete the perfect squares:

$$36\left(x^2+2x+1-1\right)+16\left(y^2-4y+4-4\right)+9\left(z^2+6z+9-9\right)+37=0,$$

$$36\left(x^2+2x+1\right)-36+16\left(y^2-4y+4\right)-64+9\left(z^2+6z+9\right)-81+37=0,$$

$$36(x+1)^2+16(y-2)^2+9(z+3)^2=144.$$

Dividing both sides of the last equation by 144, we obtain the equation:

$$\frac{(x+1)^2}{4}+\frac{(y-2)^2}{9}+\frac{(z+3)^2}{16}=1.$$

Taking the point $O_1(-1,2,-3)$ as a new origin, we perform a parallel translation of the coordinate axes. The corresponding transformation of the coordinates $X=x+1$, $Y=y-2$ and $Z=z+3$ results in the equation $\frac{X^2}{4}+\frac{Y^2}{9}+\frac{Z^2}{16}=1$. Hence, the surface is an ellipsoid. Fig. 3.12 shows the graph of the ellipsoid.

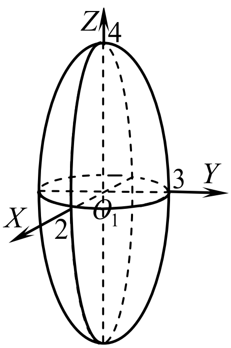


Figure 3.12

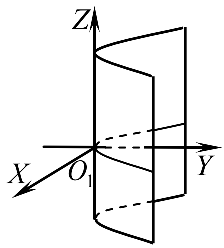


Figure 3.13

b) $x^2+4x-4y+16=0$.

Let's complete the perfect square:

$$\left(x^2+4x+4\right)-4=4y-16,\ (x+2)^2=4y-12,\ (x+2)^2=4(y-3).$$

Taking the point $O_1(-2,3,0)$ as a new origin, we perform a parallel translation of the coordinate axes. The corresponding transformation of the coordinates $X=x+2$, $Y=y-3$ and $Z=z$ results in the equation $X^2=4Y$. Hence, the surface is a parabolic cylinder. Fig. 3.13 shows the graph of the cylinder.

c) $9x^2+36y^2-4z^2-18x+216y+16z+317=0$.

Let's complete the perfect squares:

$$9\left(x^2-2x+1-1\right)+36\left(y^2+6y+9-9\right)-4\left(z^2-4z+4-4\right)+317=0,$$

$$9\left(x^2-2x+1\right)-9+36\left(y^2+6y+9\right)-324-4\left(z^2-4z+4\right)+16+317=0,$$

$$9(x-1)^2+36(y+3)^2=4(z-2)^2.$$

Dividing both sides of the last equation by 36, we obtain the equation:

$$\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1}=\frac{(z-2)^2}{9}.$$

Taking the point $O_1(1,-3,2)$ as a new origin, we perform a parallel translation of the coordinate axes. The corresponding transformation of the coordinates $X=x-1$, $Y=y+3$ and $Z=z-2$ results in the equation $\frac{X^2}{4}+\frac{Y^2}{1}=\frac{Z^2}{9}$. Hence, the surface is a cone. Fig. 3.14 shows the graph of the cone.

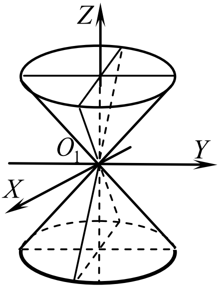


Figure 3.14

Answer: **a)** the ellipsoid $\frac{X^2}{4}+\frac{Y^2}{9}+\frac{Z^2}{16}=1$, fig. 3.12; **b)** the parabolic cylinder $X^2=4Y$, fig. 3.13; **c)** the cone $\frac{X^2}{4}+\frac{Y^2}{1}=\frac{Z^2}{9}$, fig. 3.14.

Exercises

3.1. Let ABC be a triangle with the vertices at the points $A(-2,-1)$, $B(2,6)$ and $C(8,-2)$ in the xy-plane. Sketch the drawing. Find:

a) the length of the side BC and its equation;

b) the equation of the median AM from the vertex A;

c) the length of the altitude AH from the vertex A and its equation;

d) the equation of the line l passing through the point A parallel to BC;

e) the tangent of the angle B.

Answer: fig. 3.15; **a)** $BC=10$, $BC: y=-\frac{4}{3}x+\frac{26}{3}$; **b)** $AM: y=\frac{3}{7}x-\frac{1}{7}$; **c)** $AH=7.4$, $AH: y=\frac{3}{4}x+\frac{1}{2}$; **d)** $l: y=-\frac{4}{3}x-\frac{11}{3}$; **e)** $\tan B=\frac{37}{16}$.

3.2. The points $A(1,3,-1)$, $B(2,-5,2)$, $C(3,-1,-1)$ and $D(3,4,1)$ are given. Find:

a) the equation of the plane (ABC) in the intercept form and sketch the drawing;

b) the equation of the line l_1 passing through the point C parallel to the line AB;

c) the equation of the plane passing through the point C perpendicular to the line AB;

d) the equation of the line l_2 passing through the point D perpendicular to the plane (ABC);

e) the equation of the plane passing through the point D parallel to the plane (ABC);

f) the distance from the point D to the plane (ABC);

g) the point P of intersection of the line l_2 and the plane (ABC);

h) the angle between the lines AB and CD;

i) the angle between the line CD and the plane (ABC).

Answer: **a)** $(ABC): \frac{x}{3/2} + \frac{y}{3} + \frac{z}{3/2} = 1$, fig. 3.16; **b)** $l_1: \frac{x-3}{1} = \frac{y+1}{-8} = \frac{z+1}{3}$; **c)** $x - 8y + 3z - 8 = 0$; **d)** $l_2: \begin{cases} x = 3 + 2t, \\ y = 4 + t, \\ z = 1 + 2t. \end{cases}$;
e) $2x + y + 2z - 12 = 0$; **f)** 3; **g)** $P(1,3,-1)$; **h)** $\arccos\frac{17\sqrt{2146}}{1073} \approx 43°$;
i) $\arcsin\frac{3\sqrt{29}}{29} \approx 34°$.

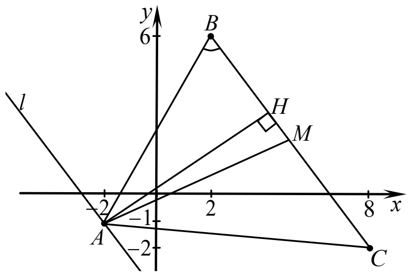


Figure 3.15

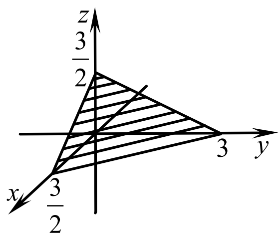


Figure 3.16

3.3. Reduce the general equation of the line $\begin{cases} 3x + 2y - z - 2 = 0, \\ x - 4y + 3z - 3 = 0. \end{cases}$ to the canonical form.

Answer: $\frac{x-1}{1} = \frac{y+0.5}{-5} = \frac{z}{-7}$.

3.4. Find the angle between the planes $x - 2y + 5z - 4 = 0$ and $2x + 3y - z + 3 = 0$.

Answer: $\arccos\frac{3\sqrt{105}}{70} \approx 64°$.

3.5. Reduce the equations of the curves to their canonical form. If the curve is an ellipse or a hyperbola, find its center and its semi-axes. If the curve is a parabola, find its vertex, the intersection points with the coordinate axes, the equation of its axis of symmetry. Sketch the graphs of the curves.

a) $x^2+4y^2+8x-16y+16=0$;

b) $4x^2-25y^2+16x+50y-109=0$;

c) $-9x^2+4y^2-36x-8y-68=0$;

d) $4x^2+12x+10y-11=0$;

e) $y^2+2x-2y+6=0$.

***Answer*:** **a)** $\frac{(x+4)^2}{16}+\frac{(y-2)^2}{4}=1$ is the ellipse; the center is $O_1(-4,2)$; the semi-axes are $a=4$, $b=2$; fig. 3.17;

b) $\frac{(x+2)^2}{25}-\frac{(y-1)^2}{4}=1$ is the hyperbola; the center is $O_1(-2,1)$, the semi-axes are $a=5$, $b=2$; fig. 3.18;

c) $\frac{(x+2)^2}{4}-\frac{(y-1)^2}{9}=-1$ is the hyperbola; the center is $O_1(-2,1)$, the semi-axes are $a=2$, $b=3$; fig. 3.19;

d) $\left(x+\frac{3}{2}\right)^2=-\frac{5}{2}(y-2)$ is the parabola; the vertex is $O_1\left(-\frac{3}{2},2\right)$;

the curve intersects the x-axis at the points $\left(\frac{-3-2\sqrt{5}}{2};0\right)$ and $\left(\frac{-3+2\sqrt{5}}{2};0\right)$, the curve intersects the y-axis at the point $\left(0;\frac{11}{10}\right)$; the axis of the symmetry is $x=-\frac{3}{2}$; fig. 3.20;

e) $(y-1)^2=-2\left(x+\dfrac{5}{2}\right)$ is the parabola; the vertex is $O_1\left(-\dfrac{5}{2};1\right)$; the curve intersects the x-axis at the point $(-3;0)$; it doesn't intersect the y-axis; the axis of the symmetry is $y=1$; fig. 3.21.

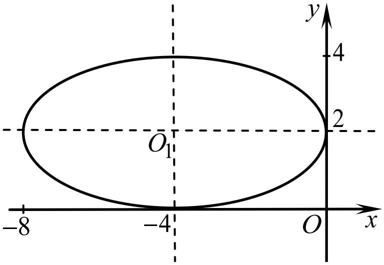


Figure 3.17

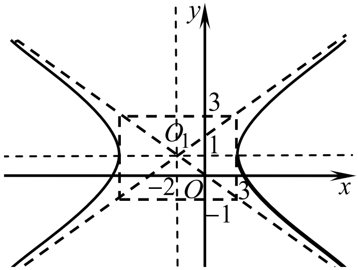


Figure 3.18

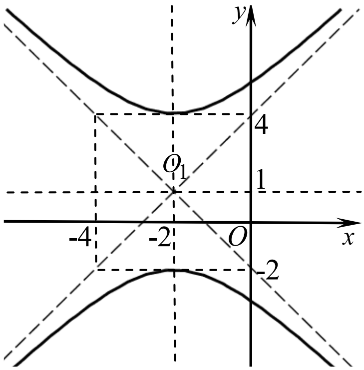


Figure 3.19

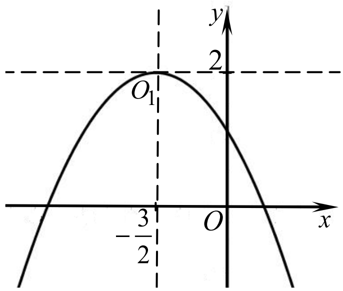


Figure 3.20

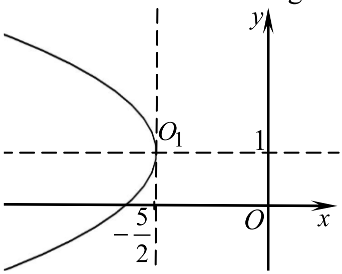


Figure 3.21

3.6. Reduce the equation of the surface to the canonical form. Determine the type of the surface and sketch the drawing.

a) $225x^2 + 36y^2 - 100z^2 - 900x - 216y - 200z + 224 = 0$;

b) $4x^2 + 12y^2 - 3z^2 + 8x - 48y - 18z + 37 = 0$;

c) $x^2 + 2z^2 + 4x - 4z - 16y + 14 = 0$;

d) $25y^2 + 9z^2 - 50y + 54z - 119 = 0$;

e) $x^2 + 2x + 2y - 5 = 0$;

f) $12x^2 + 4y^2 + 9z^2 + 72x - 16y - 18z + 97 = 0$.

Answer: **a)** hyperboloid of one sheet $\frac{X^2}{4} + \frac{Y^2}{25} - \frac{Z^2}{9} = 1$, fig. 3.22; **b)** hyperboloid of two sheets $\frac{X^2}{3} + \frac{Y^2}{1} - \frac{Z^2}{4} = -1$, fig. 3.23; **c)** elliptic paraboloid $\frac{X^2}{8} + \frac{Z^2}{4} = 2Y$, fig. 3.24; **d)** elliptic cylinder $\frac{Y^2}{9} + \frac{Z^2}{25} = 1$, fig. 3.25; **e)** parabolic cylinder $X^2 = -2Y$, fig. 3.26; **f)** parabolic cylinder $\frac{X^2}{3} + \frac{Y^2}{9} + \frac{Z^2}{4} = 1$, fig. 3.27.

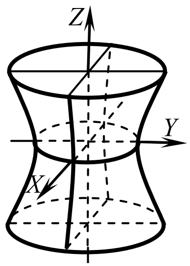


Figure 3.22

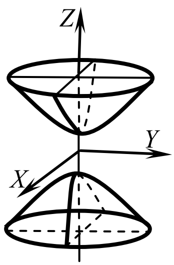


Figure 3.23

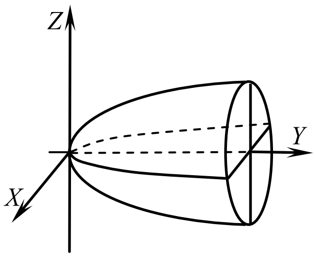


Figure 3.24

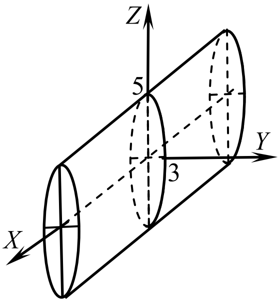


Figure 3.25

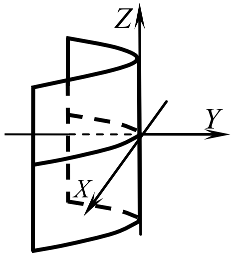


Figure 3.26

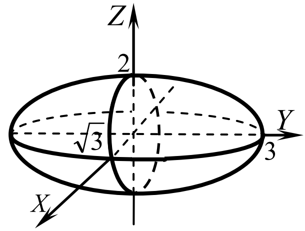


Figure 3.27

4 LIMITS AND CONTINIUTY OF FUNCTIONS OF A SINGLE VARIABLE

Limit of a function of a single variable

A real number A is called the ***limit*** of a function $f(x)$ as x tends to a $(x \to a)$ if for any positive real number ε can be found a positive real number $\delta = \delta(\varepsilon)$ such that if $0 < |x - a| < \delta$ then $|f(x) - A| < \varepsilon$:

$$\lim_{x \to a} f(x) = A .$$

Let's also consider a few partial cases:

1) $\lim\limits_{x \to \infty} f(x) = A \Leftrightarrow$ if for any positive real number ε can be found a positive real number $M = M(\varepsilon)$ such that if $|x| > M$ then $|f(x) - A| < \varepsilon$;

2) $\lim\limits_{x \to a} f(x) = \infty \Leftrightarrow$ if for any positive real number ε can be found a positive real number $\delta = \delta(\varepsilon)$ such that if $0 < |x - a| < \delta$ then $|f(x)| > \varepsilon$.

One-sided limits

A real number A_1 (A_2) is called the ***right -sided*** (***left -sided***) ***limit*** of a function $f(x)$ as $x \to a$ from the right (left) if for any positive real number ε can be found a positive real number $\delta = \delta(\varepsilon)$ such that if $a - \delta < x < a$ $(a < x < a + \delta)$ then $|f(x) - A_1| < \varepsilon$ $(|f(x) - A_2| < \varepsilon)$:

$$\lim_{x \to a-0} f(x) = A_1 \left(\lim_{x \to a+0} f(x) = A_2 \right).$$

Left-sided or right-sided limit is generally called the ***one-sided limit***. Let's note that one-sided limits can also be denoted:

$$A_1 = \lim_{x \to a-0} f(x) = \lim_{x \to a-} f(x) = f(a - 0);$$

$$A_2 = \lim_{x \to a+0} f(x) = \lim_{x \to a+} f(x) = f(a + 0).$$

Theorem 4.1. A function $f(x)$ has the limit as $x \to a$ if and only if there exist one-sided limits at the same point, which are equal to each other $\lim\limits_{x\to a-0} f(x)=\lim\limits_{x\to a+0} f(x)=A \Leftrightarrow \lim\limits_{x\to a} f(x)=A$.

Properties of the limits

If $\lim\limits_{x\to a} f(x)=A$, $\lim\limits_{x\to a} g(x)=B$ and C is a constant, then:

1) $\lim\limits_{x\to a} C=C$;

2) $\lim\limits_{x\to a} [Cf(x)]=CA$;

3) $\lim\limits_{x\to a} [f(x)\pm g(x)]=\lim\limits_{x\to a} f(x)\pm \lim\limits_{x\to a} g(x)=A\pm B$;

4) $\lim\limits_{x\to a} [f(x)g(x)]=\lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x)=AB$;

5) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\dfrac{A}{B}$, if $B\neq 0$.

Remark. The given properties also work for the one-sided limits.

Problem 4.1. Using the definition prove the given limits:

a) $\lim\limits_{x\to 1}(13x-7)=6$; **b)** $\lim\limits_{x\to\infty}\dfrac{2}{x+1}=0$.

Solution.

a) $\lim\limits_{x\to 1}(13x-7)=6$.

Let ε be an arbitrary positive real number, then we have to find $\delta=\delta(\varepsilon)>0$ such that the following condition is satisfied:

$$0<|x-1|<\delta \Rightarrow |(13x-7)-6|<\varepsilon .$$

Using the given inequality, we can obtain:

$$|(13x-7)-6|<\varepsilon \Rightarrow |13x-13|<\varepsilon \Rightarrow 13|x-1|<\varepsilon \Rightarrow |x-1|<\frac{\varepsilon}{13} .$$

Since we have to obtain inequalities in the terms of $(x-1)$, therefore we can take:

$$\delta=\frac{\varepsilon}{13}>0 .$$

We obtained that for any positive real number ε, we can find a positive real number $\delta=\dfrac{\varepsilon}{13}>0$ such that if $0<|x-1|<\delta$ then

$|(13x-7)-6|<\varepsilon$. Therefore, the given limit was proven by the definition.

b) $\lim\limits_{x\to\infty}\dfrac{2}{x+1}=0$.

Let ε be an arbitrary positive real number, then we have to find a positive real number $M=M(\varepsilon)$ such that the following condition is satisfied:

$$|x|>M \Rightarrow \left|\frac{2}{x+1}\right|<\varepsilon.$$

Let's simplify the last inequality:

$$\left|\frac{2}{x+1}\right|<\varepsilon \Rightarrow \left|\frac{x+1}{2}\right|>\frac{1}{\varepsilon} \Rightarrow |x+1|>\frac{2}{\varepsilon} \Rightarrow \left[\begin{array}{l} x<-\dfrac{2}{\varepsilon}-1; \\ x>\dfrac{2}{\varepsilon}-1. \end{array}\right.$$

Finally, we can obtain the expression for M:

$$M=\max\left\{\left|-\frac{2}{\varepsilon}-1\right|,\left|\frac{2}{\varepsilon}-1\right|\right\}=\max\left\{\frac{2}{\varepsilon}+1,\left|\frac{2}{\varepsilon}-1\right|\right\}=\frac{2}{\varepsilon}+1>0.$$

We obtained that for any positive real number ε, we can find a positive real number $M=\dfrac{2}{\varepsilon}+1>0$ such that if $|x|>M$ then $\left|\dfrac{2}{x+1}\right|<\varepsilon$. Therefore, the given limit was proven by the definition.

Indeterminate forms

As the first step in the process of evaluating limits, we should substitute x (or any other variable) with the expression to which the variable tends. Quite often one of the following expressions can be obtained:

$$\left[\frac{0}{0}\right],\ \left[\frac{\infty}{\infty}\right],\ [0\cdot\infty],\ [\infty-\infty],\ \left[1^{\infty}\right],\ \left[0^{0}\right],\ \left[\infty^{0}\right]$$

which are called ***indeterminate forms***.

There are some methods to evaluate indeterminate forms. In the simplest cases, we can transform the initial expression using algebra. Then it should be easier to apply different approaches to calculate the limits.

Problem 4.2. Calculate the limits:

a) $\lim\limits_{x\to -2}\dfrac{3x^2-3x-18}{2x^2+18x+28}$; **e)** $\lim\limits_{x\to -7}\dfrac{\sqrt{x+11}-2}{x^2+6x-7}$;

b) $\lim\limits_{x\to\infty}\dfrac{2x^5+4x^4-x^2+18}{2x^3-13x+100}$; **f)** $\lim\limits_{x\to -2}\dfrac{x^3+3x^2+6x+8}{\sqrt[3]{17-5x}-3}$;

c) $\lim\limits_{x\to\infty}\dfrac{3x^2+7x-12}{5x^2+4x-1}$; **g)** $\lim\limits_{x\to\infty}\left(\dfrac{x^3+3}{x^2+5x+10}-x+\dfrac{1}{2}\right)$.

d) $\lim\limits_{x\to\infty}\dfrac{x^2-4x+110}{3x^4+x^3-x^2+8x-1}$;

Solution.

a) $\lim\limits_{x\to -2}\dfrac{3x^2-3x-18}{2x^2+18x+28}=\left[\dfrac{3(-2)^2-3(-2)-18}{2(-2)^2+18(-2)+28}=\dfrac{0}{0}\right]$.

We obtained the indeterminate form $\left[\dfrac{0}{0}\right]$. This means that both the numerator and the denominator have a common root $x=-2$. Therefore, we can evaluate this limit by cancelling the factor $(x+2)$. So we have to find the roots of both polynomials and factor them:

$$3x^2-3x-18=0\Rightarrow x_1=-2,\ x_2=3\Rightarrow 3x^2-3x-18=3(x+2)(x-3).$$

$$2x^2+18x+28=0\Rightarrow x_1=-2,\ x_2=-7\Rightarrow 2x^2+18x+28=2(x+2)(x+7).$$

Finally:

$$\lim_{x\to -2}\frac{3x^2-3x-18}{2x^2+18x+28}=\lim_{x\to -2}\frac{3(x+2)(x-3)}{2(x+2)(x+7)}=$$

$$\lim_{x\to -2}\frac{3(x-3)}{2(x+7)}=\frac{3(-2-3)}{2(-2+7)}=-\frac{3}{2}.$$

b) $\lim\limits_{x\to\infty}\dfrac{2x^5+4x^4-x^2+18}{2x^3-13x+100}=\left[\dfrac{\infty}{\infty}\right]$.

We obtained the indeterminate form $\left[\dfrac{\infty}{\infty}\right]$, which can be evaluated by factoring out the highest power of the numerator and the denominator and using the limit $\lim\limits_{x\to\infty}\dfrac{1}{x}=0$.

$$\lim_{x\to\infty}\frac{2x^5+4x^4-x^2+18}{2x^3-13x+100}=\lim_{x\to\infty}\frac{x^5\left(2+\frac{4}{x}-\frac{1}{x^3}+\frac{18}{x^5}\right)}{x^3\left(2-\frac{13}{x^2}+\frac{100}{x^3}\right)}=$$

$$=\lim_{x\to\infty}\frac{x^2\left(2+\frac{4}{x}-\frac{1}{x^3}+\frac{18}{x^5}\right)}{2-\frac{13}{x^2}+\frac{100}{x^3}}=\left[\frac{\infty\left(2+\frac{4}{\infty}-\frac{1}{\infty}+\frac{18}{\infty}\right)}{2-\frac{13}{\infty}+\frac{100}{\infty}}=\right.$$

$$\left.=\frac{\infty(2+0-0+0)}{2-0+0}\right]=\infty\,.$$

The same approach can be applied to issue (c) and (d).

c) $$\lim_{x\to\infty}\frac{3x^2+7x-12}{5x^2+4x-1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{x^2\left(3+\frac{7}{x}-\frac{12}{x^2}\right)}{x^2\left(5+\frac{4}{x}-\frac{1}{x^2}\right)}=$$

$$=\lim_{x\to\infty}\frac{3+\frac{7}{x}-\frac{12}{x^2}}{5+\frac{4}{x}-\frac{1}{x^2}}=\left[\frac{3+\frac{7}{\infty}-\frac{12}{\infty}}{5+\frac{4}{\infty}-\frac{1}{\infty}}=\frac{3+0-0}{5+0-0}\right]=\frac{3}{5}\,.$$

d) $$\lim_{x\to\infty}\frac{x^2-4x+110}{3x^4+x^3-x^2+8x-1}=\left[\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{x^2\left(1-\frac{4}{x}+\frac{110}{x^2}\right)}{x^4\left(3+\frac{1}{x}-\frac{1}{x^2}+\frac{8}{x^3}-\frac{1}{x^4}\right)}=\lim_{x\to\infty}\frac{1-\frac{4}{x}+\frac{110}{x^2}}{x^2\left(3+\frac{1}{x}-\frac{1}{x^2}+\frac{8}{x^3}-\frac{1}{x^4}\right)}=$$

$$=\left[\frac{1-\frac{4}{\infty}+\frac{110}{\infty}}{\infty\left(3+\frac{1}{\infty}-\frac{1}{\infty}+\frac{8}{\infty}-\frac{1}{\infty}\right)}=\frac{1-0+0}{\infty(3+0-0+0-0)}\right]=0\,.$$

e) $$\lim_{x\to-7}\frac{\sqrt{x+11}-2}{x^2+6x-7}=\left[\frac{\sqrt{-7+11}-2}{(-7)^2+6(-7)-7}=\frac{0}{0}\right].$$

We obtained the indeterminate form $\left[\frac{0}{0}\right]$. This means that both the numerator and the denominator have a common root $x=-7$. Therefore, we can evaluate this limit by cancelling the factor $(x+7)$. So we have to find the roots of the polynomial in the denominator and factor it. Then we have to rationalize the numerator by multiplying both the numerator and the denominator by the conjugate of the numerator.

$$x^2+6x-7=0 \Rightarrow x_1=-7,\ x_2=1 \Rightarrow x^2+6x-7=(x+7)(x-1).$$

$$\frac{\sqrt{x+11}-2}{(x+7)(x-1)}=\frac{\left(\sqrt{x+11}-2\right)\left(\sqrt{x+11}+2\right)}{(x+7)(x-1)\left(\sqrt{x+11}+2\right)}=\frac{x+11-4}{(x+7)(x-1)\left(\sqrt{x+11}+2\right)}=$$

$$=\frac{x+7}{(x+7)(x-1)\left(\sqrt{x+11}+2\right)}=\frac{1}{(x-1)\left(\sqrt{x+11}+2\right)}.$$

Finally:

$$\lim_{x\to-7}\frac{\sqrt{x+11}-2}{x^2+6x-7}=\lim_{x\to-7}\frac{1}{(x-1)\left(\sqrt{x+11}+2\right)}=$$

$$=\frac{1}{(-7-1)\left(\sqrt{-7+11}+2\right)}=-\frac{1}{32}.$$

f) $$\lim_{x\to-2}\frac{x^3+3x^2+6x+8}{\sqrt[3]{17-5x}-3}=\left[\frac{(-2)^3+3(-2)^2+6(-2)+8}{\sqrt[3]{17-5(-2)}-3}=\frac{0}{0}\right].$$

We obtained the indeterminate form $\left[\frac{0}{0}\right]$. This means that both the numerator and the denominator have a common root $x=-2$. Therefore, we can evaluate this limit by cancelling the factor $(x+2)$. So we have to factor the polynomial in the numerator. Taking into account that it is a cubic polynomial function and we already know one of its roots, the quickest way to factor it is to use long division. Then we have to rationalize the denominator by multiplying both the numerator and the denominator by the conjugate of the denominator. The conjugate of the denominator can be obtained by using the formula of the difference of cubes.

$$\begin{array}{r|l} -\begin{array}{l} x^3+3x^2+6x+8 \\ x^3+2x^2 \end{array} & \begin{array}{l} x+2 \\ \hline x^2+x+4 \end{array} \\ \hline -\begin{array}{l} x^2+6x+8 \\ x^2+2x \end{array} & \\ \hline -\begin{array}{l} 4x+8 \\ 4x+8 \end{array} & \\ \hline 0 & \end{array}$$

$$x^3+3x^2+6x+8=(x+2)(x^2+x+4).$$

$$\frac{(x+2)(x^2+x+4)}{\sqrt[3]{17-5x}-3}=\frac{(x+2)(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+3^2\right)}{\left(\sqrt[3]{17-5x}-3\right)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+3^2\right)}=$$

$$=\frac{(x+2)(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+9\right)}{\left(\sqrt[3]{17-5x}\right)^3-3^3}=$$

$$=\frac{(x+2)(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+9\right)}{17-5x-27}=$$

$$=\frac{(x+2)(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+9\right)}{-5(x+2)}=$$

$$=\frac{(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+9\right)}{-5}.$$

Finally:

$$\lim_{x\to-2}\frac{x^3+3x^2+6x+8}{\sqrt[3]{17-5x}-3}=\lim_{x\to-2}\frac{(x^2+x+4)\left(\left(\sqrt[3]{17-5x}\right)^2+3\sqrt[3]{17-5x}+9\right)}{-5}=$$

$$=\frac{\left((-2)^2+(-2)+4\right)\left(\left(\sqrt[3]{17-5(-2)}\right)^2+3\sqrt[3]{17-5(-2)}+9\right)}{-5}=$$

$$= \frac{6(3^2+3\cdot 3+9)}{-5} = \frac{6\cdot 27}{-5} = -\frac{162}{5}.$$

g) $\lim\limits_{x\to\infty}\left(\dfrac{x^3+3}{x^2+5x+10} - x + \dfrac{1}{2}\right) = [\infty-\infty].$

In this limit, the result for the first term was obtained by using the same approach that was used in issues (b), (c), and (d). As the result we obtained indeterminate in the form $[\infty-\infty]$, which can be evaluated by reducing the difference under the limit to a fraction and then using approach analogously to issue (b).

$$\frac{x^3+3}{x^2+5x+10} - x + \frac{1}{2} = \frac{2x^3+6-2x^3-10x^2-20x+x^2+5x+10}{2(x^2+5x+10)} =$$

$$= \frac{-9x^2-15x+16}{2(x^2+5x+10)}.$$

Finally:

$$\lim_{x\to\infty}\left(\frac{x^3+3}{x^2+5x+10} - x + \frac{1}{2}\right) = \lim_{x\to\infty}\frac{-9x^2-15x+16}{2(x^2+5x+10)} =$$

$$= \lim_{x\to\infty}\frac{x^2\left(-9-\frac{15}{x}+\frac{16}{x^2}\right)}{2x^2\left(1+\frac{5}{x}+\frac{10}{x^2}\right)} = \frac{-9-0+0}{2(1+0+0)} = -\frac{9}{2}.$$

Answer: **a)** $-\frac{3}{2}$; **b)** ∞; **c)** $\frac{3}{5}$; **d)** 0; **e)** $-\frac{1}{32}$; **f)** $-\frac{162}{5}$; **g)** $-\frac{9}{2}$.

The most important limits

The first important limit:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

The second important limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Some corollaries:

1) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$; **2)** $\lim\limits_{x\to\infty}\left(1+\dfrac{\alpha}{x}\right)^{x}=e^{\alpha}$; **3)** $\lim\limits_{x\to 0}(1+\alpha x)^{\frac{1}{x}}=e^{\alpha}$.

Equivalent infinitesimals

A function $f(x)$ is called an ***infinitesimal*** as $x\to a\ (a\in R)$ if $\lim\limits_{x\to a} f(x)=0$.

If $\alpha(x)$ is an infinitesimal as $x\to a\ (a\in R)$:

1) $\sin(\alpha(x))\sim\alpha(x)$; **2)** $\tan(\alpha(x))\sim\alpha(x)$; **3)** $\arcsin(\alpha(x))\sim\alpha(x)$;

4) $\arctan(\alpha(x))\sim\alpha(x)$; **5)** $\ln(1+\alpha(x))\sim\alpha(x)$; **6)** $e^{\alpha(x)}-1\sim\alpha(x)$;

7) $a^{\alpha(x)}-1\sim\alpha(x)\ln a$; **8)** $(1+\alpha(x))^{n}-1\sim n\alpha(x)\ (n>0)$;

9) $1-\cos(\alpha(x))\sim\dfrac{\alpha^{2}(x)}{2}$.

Theorem 4.2. If $\alpha(x)\sim\beta(x)$ as $x\to a$, then:

$$\lim_{x\to a}\alpha(x)\gamma(x)=\lim_{x\to a}\beta(x)\gamma(x),\quad \lim_{x\to a}\frac{\alpha(x)}{\gamma(x)}=\lim_{x\to a}\frac{\beta(x)}{\gamma(x)},$$

$$\lim_{x\to a}\frac{\gamma(x)}{\alpha(x)}=\lim_{x\to a}\frac{\gamma(x)}{\beta(x)}.$$

The most important limits and corresponding corollaries are mostly used for evaluating some indeterminate in form $\left[\dfrac{0}{0}\right]$ or $\left[1^{\infty}\right]$.

Problem 4.3. Use the most important limits to calculate the limits:

a) $\lim\limits_{x\to 0}\dfrac{1-\cos 7x}{\cos 4x-\cos 6x}$; **d)** $\lim\limits_{x\to\infty}\left(\dfrac{2x^{2}}{2x^{2}+6x-3}\right)^{3x-1}$;

b) $\lim\limits_{x\to 0}\dfrac{2xe^{5x}-2x}{\ln^{2}(x+1)}$; **e)** $\lim\limits_{x\to 1}(5x-4)^{\frac{1}{3(x-1)}}$.

c) $\lim\limits_{x\to\infty}\left(\dfrac{4x+5}{4x-1}\right)^{x+6}$;

Solution.

a) $\lim\limits_{x\to 0}\dfrac{1-\cos 7x}{\cos 4x-\cos 6x}=\left[\dfrac{1-\cos(7\cdot 0)}{\cos(4\cdot 0)-\cos(6\cdot 0)}=\dfrac{0}{0}\right]=$

$$=\left\{\begin{array}{l}1-\cos 7x\sim\dfrac{(7x)^2}{2}=\dfrac{49x^2}{2},\ \text{as}\ x\to 0.\\ \cos 4x-\cos 6x=-2\sin\dfrac{4x+6x}{2}\sin\dfrac{4x-6x}{2}=-2\sin 5x\sin(-x)\sim\\ \sim-2\cdot 5x\cdot(-x)=10x^2,\ \text{as}\ x\to 0.\end{array}\right\}=$$

$$=\lim_{x\to 0}\frac{\frac{49x^2}{2}}{10x^2}=\lim_{x\to 0}\frac{49x^2}{20x^2}=\frac{49}{20}.$$

b) $\lim\limits_{x\to 0}\dfrac{2xe^{5x}-2x}{\ln^2(3x+1)}=\left[\dfrac{2\cdot 0\cdot e^{5\cdot 0}-2\cdot 0}{\ln^2(3\cdot 0+1)}=\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{2x\left(e^{5x}-1\right)}{\ln^2(3x+1)}=$

$$=\left\{\begin{array}{l}e^{5x}-1\sim 5x,\ \text{as}\ x\to 0.\\ \ln^2(3x+1)=(\ln(1+3x))^2\sim(3x)^2=9x^2,\ \text{as}\ x\to 0.\end{array}\right\}=$$

$$=\lim_{x\to 0}\frac{2x\cdot 5x}{9x^2}=\lim_{x\to 0}\frac{10x^2}{9x^2}=\frac{10}{9}.$$

c) $\lim\limits_{x\to\infty}\left(\dfrac{4x+5}{4x-1}\right)^{x+6}=\left[1^\infty\right]=\left\{\dfrac{4x+5}{4x-1}=\dfrac{4x-1+1+5}{4x-1}=\right.$

$$\left.=\frac{4x-1}{4x-1}+\frac{6}{4x-1}=1+\frac{6}{4x-1}\right\}=\lim_{x\to\infty}\left(1+\frac{6}{4x-1}\right)^{x+6}=$$

$$=\lim_{x\to\infty}\left(1+\frac{6}{4x-1}\right)^{\frac{4x-1}{6}\frac{6}{4x-1}(x+6)}=\lim_{x\to\infty}\left[\left(1+\frac{6}{4x-1}\right)^{\frac{4x-1}{6}}\right]^{\frac{6}{4x-1}(x+6)}=$$

$$=\lim_{x\to\infty}e^{\frac{6(x+6)}{4x-1}}=e^{\lim\limits_{x\to\infty}\frac{6x+36}{4x-1}}=$$

$$=\left\{\lim_{x\to\infty}\frac{6x+36}{4x-1}=\lim_{x\to\infty}\frac{x\left(6+\frac{36}{x}\right)}{x\left(4-\frac{1}{x}\right)}=\frac{3}{2}\right\}=e^{\frac{3}{2}}=\sqrt{e^3}.$$

d) $$\lim_{x\to\infty}\left(\frac{2x^2}{2x^2+6x-3}\right)^{3x-1}=\left[1^\infty\right]=\left\{\frac{2x^2}{2x^2+6x-3}=\right.$$

$$\left.=\frac{2x^2+6x-3-(6x-3)}{2x^2+6x-3}=\frac{2x^2+6x-3}{2x^2+6x-3}-\frac{6x-3}{2x^2+6x-3}=1-\frac{6x-3}{2x^2+6x-3}\right\}=$$

$$=\lim_{x\to\infty}\left(1-\frac{6x-3}{2x^2+6x-3}\right)^{3x-1}=\lim_{x\to\infty}\left(1-\frac{6x-3}{2x^2+6x-3}\right)^{\frac{2x^2+6x-3}{6x-3}\cdot\frac{6x-3}{2x^2+6x-3}(3x-1)}=$$

$$=\lim_{x\to\infty}\left[\left(1-\frac{6x-3}{2x^2+6x-3}\right)^{\frac{2x^2+6x-3}{6x-3}}\right]^{\frac{6x-3}{2x^2+6x-3}(3x-1)}=\lim_{x\to\infty}e^{-\frac{(6x-3)(3x-1)}{2x^2+6x-3}}=$$

$$=e^{-\lim\limits_{x\to\infty}\frac{18x^2-15x+3}{2x^2+6x-3}}=\left\{\lim_{x\to\infty}\frac{18x^2-15x+3}{2x^2+6x-3}=\lim_{x\to\infty}\frac{x^2\left(18-\frac{15}{x}+\frac{3}{x^2}\right)}{x^2\left(2+\frac{6}{x}-\frac{3}{x^2}\right)}=\frac{18}{2}=9\right\}=$$

$$=e^{-9}.$$

e) $$\lim_{x\to1}(5x-4)^{\frac{1}{3(x-1)}}=\left[1^{\frac{1}{0}}=1^\infty\right]=\{5x-4=1+5x-5=1+5(x-1)\}=$$

$$=\lim_{x\to1}(1+5(x-1))^{\frac{1}{3(x-1)}}=\{t=x-1;\ x\to1\Rightarrow t\to0\}=\lim_{t\to0}(1+5t)^{\frac{1}{3t}}=$$

$$=\lim_{t\to0}\left((1+5t)^{\frac{1}{t}}\right)^{\frac{1}{3}}=\left(e^5\right)^{\frac{1}{3}}=e^{\frac{5}{3}}.$$

Answer**: a)** $\frac{49}{20}$; **b)** $\frac{10}{9}$; **c)** $\sqrt{e^3}$; **d)** e^{-9}; **e)** $e^{\frac{5}{3}}$.

Continuity of a function

A function $f(x)$ is said to be ***continuous*** at a point $x = x_0$ if:

1) $f(x)$ is defined at the point $x = x_0$;
2) $\lim\limits_{x \to x_0} f(x)$ exists;
3) $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Using one-sided limits the last definition can be rewritten in the form:
a function $f(x)$ is said to be continuous at a point $x = x_0$ if:

1) $f(x)$ is defined at point $x = x_0$;
2) $f(x_0 - 0) = \lim\limits_{x \to x_0 - 0} f(x)$ and $f(x_0 + 0) = \lim\limits_{x \to x_0 + 0} f(x)$ exist;
3) $f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.

Let Δx be an ***increment*** of the variable x at a point $x = x_0$, then $\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$ is the respective ***increment*** of a function $f(x)$ at the point $x = x_0$.

The definition of continuity of a function at a point can be rewritten in terms of the increment of a function at a point:
a function $f(x)$ is continuous at the point $x = x_0$, if:

$$\lim_{\Delta x \to 0} \Delta f(x_0) = \lim_{\Delta x \to 0} \left(f(x_0 + \Delta x) - f(x_0)\right) = 0.$$

Based on the definition of the continuity of a function and properties of the limits, the following properties of continuous functions can be obtained:
if functions $f(x)$ and $g(x)$ are continuous at a point $x = x_0$, then:

1) $f(x) \pm g(x)$ are continuous at the point $x = x_0$;
2) $f(x) \cdot g(x)$ is continuous at the point $x = x_0$;
3) $\dfrac{f(x)}{g(x)}$ is continuous at the point $x = x_0$, provided $g(x_0) \neq 0$.

Theorem 4.3. If a function $y = f(x)$ is continuous at the point $x = x_0$ and a function $z = g(y)$ is continuous at the point $y_0 = f(x_0)$, then the composite function $z = h(x) = g(f(x))$ is continuous at the point $x = x_0$.

A function $f(x)$, which is continuous at every point of some interval (open or closed), is said to be continuous on that interval.

Theorem 4.4. Any basic function is continuous at each point of its domain of definition.

Some properties of functions being continuous on an interval

1) If a function $f(x)$ is continuous on the interval $[a, b]$, then $f(x)$ is ***bounded*** on $[a, b]$, which means: there exists a positive real number L, such that $|f(x)| < L$ for every $x \in [a, b]$.

2) If a function $f(x)$ is continuous on the closed interval $[a, b]$, then $f(x)$ takes on the least value m and the greatest value M on this interval.

3) If a function $f(x)$ is continuous on the interval $[a, b]$, and $f(a) \cdot f(b) < 0$, then the equation $f(x) = 0$ has at least one root on the considered interval.

Types of discontinuity

A function $f(x)$ is called ***discontinuous*** at a point $x = x_0$ it fails to satisfy at least one of the conditions of continuity.

There are three types of the points of discontinuity of a function:

1) If there exist both of the one-sided limits of a function $f(x)$ as $x \to x_0$, however $f(x_0 - 0) = f(x_0 + 0) \neq f(x_0)$, then the function $f(x)$ has a ***removable discontinuity*** at the point $x = x_0$.

2) If there exist both of the one-sided limits of a function $f(x)$ as $x \to x_0$, however $f(x_0 - 0) \neq f(x_0 + 0)$, then the function $f(x)$ has a ***discontinuity of the first type*** at the point $x = x_0$. Such type of discontinuity is also called ***jump discontinuity***, and the number $|f(x_0 - 0) - f(x_0 + 0)|$ is called a ***jump*** of the function $f(x)$.

3) If at least one of the one-sided limits of a function $f(x)$ as $x \to x_0$ doesn't exist or equals infinity, then the function $f(x)$ has a ***discontinuity of the second type*** at the point $x = x_0$. Such type of discontinuity is also called ***essential discontinuity***.

Problem 4.4. Investigate the following functions for continuity. If they have points of discontinuity, determine their type. Sketch the schematic graph of the functions:

a) $f(x)=\begin{cases} x+4, \ x<-2, \\ 1, \ x=-2, \\ \dfrac{x^2}{4}+1, \ -2<x<0, \\ \sin\dfrac{\pi x}{2}+1, \ 0\le x<1, \\ 5-2x, \ x\ge 1; \end{cases}$ **b)** $f(x)=\left(\sqrt{2}\right)^{\frac{1}{3+x}}$.

Solution.

a) Lets notice that every expression in the right-hand side of the function is a basic function. Moreover, they are defined at every point. Therefore, according to theorem 4.4, they separately are continuous at each point. However they connect at points $x_1=-2$, $x_2=0$ and $x_3=1$, so we have to check them by using the definition of continuity of a function at the point.

Let's start with the point $x_1=-2$:

$$f(x_1-0)=\lim_{x\to -2-0} f(x)=\lim_{x\to -2-0}(x+4)=[-2-0+4]=2,$$

$$f(x_1+0)=\lim_{x\to -2+0} f(x)=\lim_{x\to -2+0}\left(\frac{x^2}{4}+1\right)=\left[\frac{(-2+0)^2}{4}+1\right]=2.$$

Since $f(x_1-0)=f(x_1+0)=2\neq 1=f(x_1)$ then at the point $x_1=-2$ the function has a removable discontinuity.

Let's consider the point $x_2=0$:

$$f(x_2-0)=\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\left(\frac{x^2}{4}+1\right)=\left[\frac{(0-0)^2}{4}+1\right]=1,$$

$$f(x_2+0)=\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}\left(\sin\frac{\pi x}{2}+1\right)=\left[\sin\frac{(0+0)\pi}{2}+1\right]=1.$$

Since $f(x_2-0)=f(x_2+0)=1=f(x_2)$ then the function is continuous at the point $x_2=0$.

Let's consider the last point $x_3=1$:

$$f(x_3+0)=\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}\left(\sin\frac{\pi x}{2}+1\right)=\left[\sin\frac{(1-0)\pi}{2}+1\right]=2,$$

$$f(x_3+0)=\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}(5-2x)=[5-2(1+0)]=3.$$

Since $f(x_3-0)=2\neq 3=f(x_3+0)$ then the function has a discontinuity of the first type (jump discontinuity) at the point $x_3=1$. Let's calculate the jump of the function at the current point:

$$|f(x_3-0)-f(x_3+0)|=|2-3|=1.$$

Finally, we can conclude:
if $x\in(-\infty,-2)\cup(-2,1)\cup(1,+\infty)$, then the function is continuous;
at the point $x=-2$ the function has a removable discontinuity;
at the point $x=1$ the function has a discontinuity of the first type (jump discontinuity).

Fig. 4.1 shows the schematic graph of the function $y=f(x)$.

b) Let's notice that the function $y=h(x)=\left(\sqrt{2}\right)^x$ is continuous at each point as a basic function. The function $g(x)=\frac{1}{3+x}$ is also continuous at each point except $x=-3$ (because the denominator must be nonzero). Using theorem 4.3, we can conclude that the initial function $f(x)=h(g(x))=\left(\sqrt{2}\right)^{\frac{1}{3+x}}$ is continuous if $x\in(-\infty,-3)\cup(-3,+\infty)$ as a composite function. Therefore, we have only one point $x_0=-3$ to be checked by using the definition of continuity of a function at a point.

$$f(x_0-0)=\lim_{x\to -3-0}\left(\sqrt{2}\right)^{\frac{1}{3+x}}=\left[\left(\sqrt{2}\right)^{\frac{1}{3-3-0}}=\left(\sqrt{2}\right)^{\frac{1}{-0}}=\left(\sqrt{2}\right)^{-\infty}\right]=0.$$

$$f(x_0+0)=\lim_{x\to -3+0}\left(\sqrt{2}\right)^{\frac{1}{3+x}}=\left[\left(\sqrt{2}\right)^{\frac{1}{3-3+0}}=\left(\sqrt{2}\right)^{\frac{1}{+0}}=\left(\sqrt{2}\right)^{+\infty}\right]=+\infty.$$

Since the right-sided limit equals infinity then the function has a discontinuity of the second type (essential discontinuity) at the point $x_0=-3$.

Finally, we can conclude:

if $x \in (-\infty, -3) \cup (-3, +\infty)$, then the function is continuous;
at the point $x = -3$ the function has a discontinuity of the second type (essential discontinuity).

Fig. 4.2 shows the schematic graph of the function $y = f(x)$.

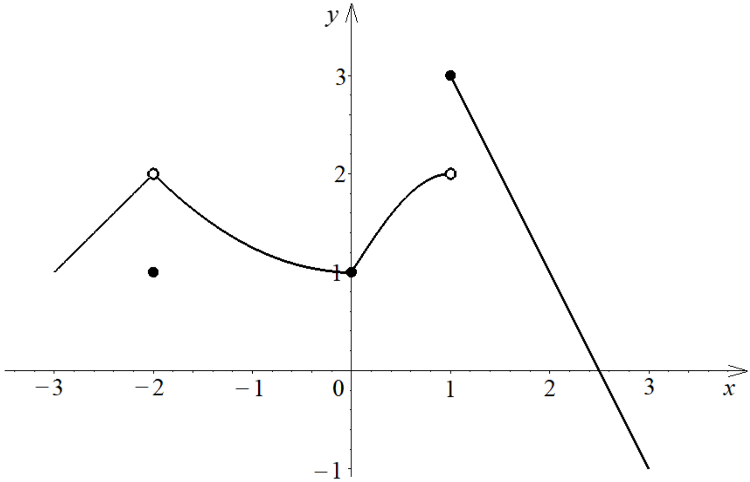


Figure 4.1

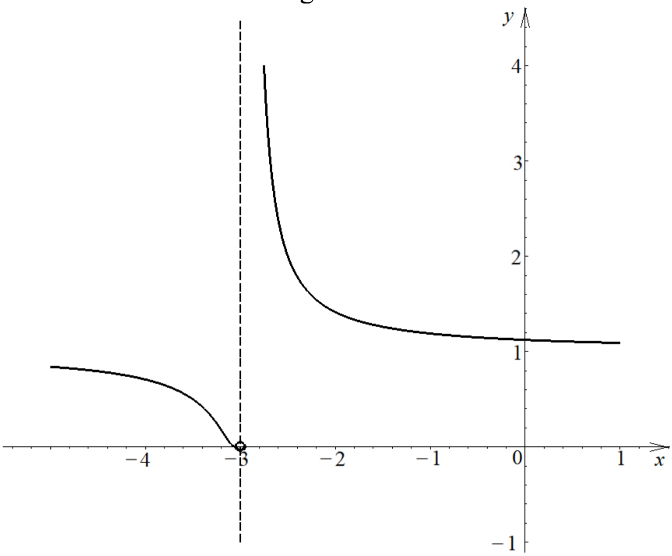


Figure 4.2

Answer: **a)** if $x \in (-\infty, -2) \cup (-2, 1) \cup (1, +\infty)$, then the function is continuous;
at the point $x = -2$ the function has a removable discontinuity;
at the point $x = 1$ the function has a discontinuity of the first type (jump discontinuity);
b) if $x \in (-\infty, -3) \cup (-3, +\infty)$, then the function is continuous;
at the point $x = -3$ the function has a discontinuity of the second type (essential discontinuity).

Exercises

4.1. Using the definition prove the given limits:

a) $\lim\limits_{x \to 3} (4x + 1) = 13$; **b)** $\lim\limits_{x \to \infty} \dfrac{-1}{x - 2} = 0$.

4.2. Calculate the limits:

a) $\lim\limits_{x \to 1} \dfrac{3x^2 + 18x - 21}{2x^2 - 12x + 10}$; **e)** $\lim\limits_{x \to 5} \dfrac{x^2 + 2x - 35}{\sqrt{x + 4} - 3}$;

b) $\lim\limits_{x \to \infty} \dfrac{-x^4 + 3x^3 + 12x - 5}{x^3 - 10x^2 - 20}$; **f)** $\lim\limits_{x \to -2} \dfrac{x^3 + 3x^2 + 9x + 14}{\sqrt[3]{2x + 12} - 2}$;

c) $\lim\limits_{x \to \infty} \dfrac{7x^2 + 2x - 2}{-x^2 - x - 1}$; **g)** $\lim\limits_{x \to \infty} \left(\dfrac{2x^2 + 1}{x - 10} - 2x + 10 \right)$.

d) $\lim\limits_{x \to \infty} \dfrac{3x^3 + 2x^2 - x + 1}{3x^4 + x^3 - 4x + 5}$;

Answer: **a)** -3; **b)** $-\infty$; **c)** -7; **d)** 0; **e)** 72; **f)** 54; **g)** 30.

4.3. Use the most important limits to calculate the limits:

a) $\lim\limits_{x \to 0} \dfrac{\arctan^2(4x)}{6x \sin(2x)}$; **d)** $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x^2 + 3x - 1} \right)^{x+1}$;

b) $\lim\limits_{x \to 0} \dfrac{\ln(1 - 2x)\ln(1 + 3x)}{\left(e^{3x} - 1\right)^2}$; **e)** $\lim\limits_{x \to 2} (2x - 3)^{\frac{5}{2(x-2)}}$.

c) $\lim\limits_{x \to \infty} \left(\dfrac{2x + 1}{2x - 3} \right)^{x+1}$;

Answer: **a)** $\frac{4}{3}$; **b)** $-\frac{2}{3}$; **c)** e^2; **d)** e^{-3}; **e)** e^5.

4.4. Investigate the following functions for continuity. If they have points of discontinuity, determine their type. Sketch the schematic graph of the functions:

a) $f(x)=\begin{cases} 2-x^2, \ x<0, \\ 2-\ln(x+1), \ 0\le x<2, \\ 5-x, \ x\ge 2; \end{cases}$ **b)** $f(x)=-3^{\frac{1}{1-x}}$.

Answer: **a)** if $x\in(-\infty, 2)\cup(2, +\infty)$, then the function is continuous;
at the point $x=2$ the function has a discontinuity of the first type (jump discontinuity);
fig. 4.3 shows the schematic graph of the function $y=f(x)$;

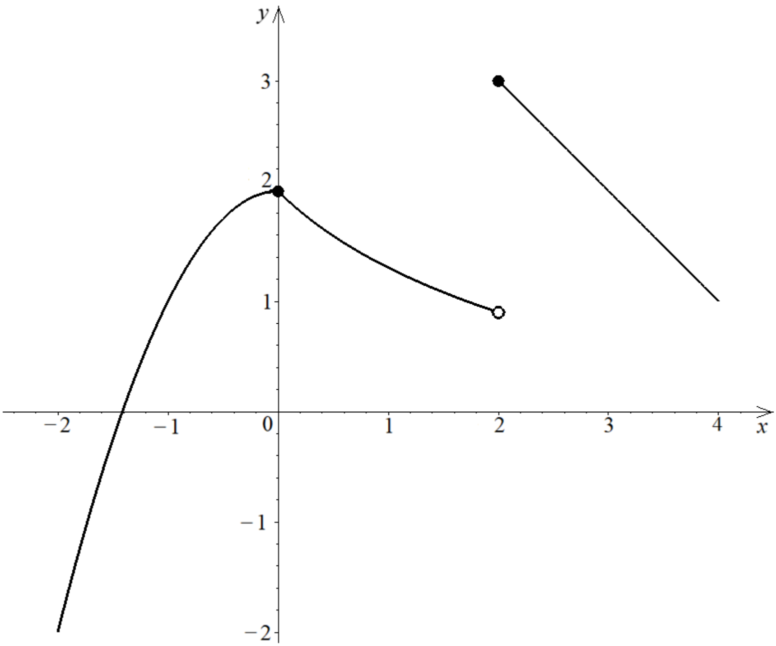


Figure 4.3

b) if $x \in (-\infty, 1) \cup (1, +\infty)$, then the function is continuous;
at the point $x = 1$ the function has a discontinuity of the second type (essential discontinuity);
fig. 4.4 shows the schematic graph of the function $y = f(x)$.

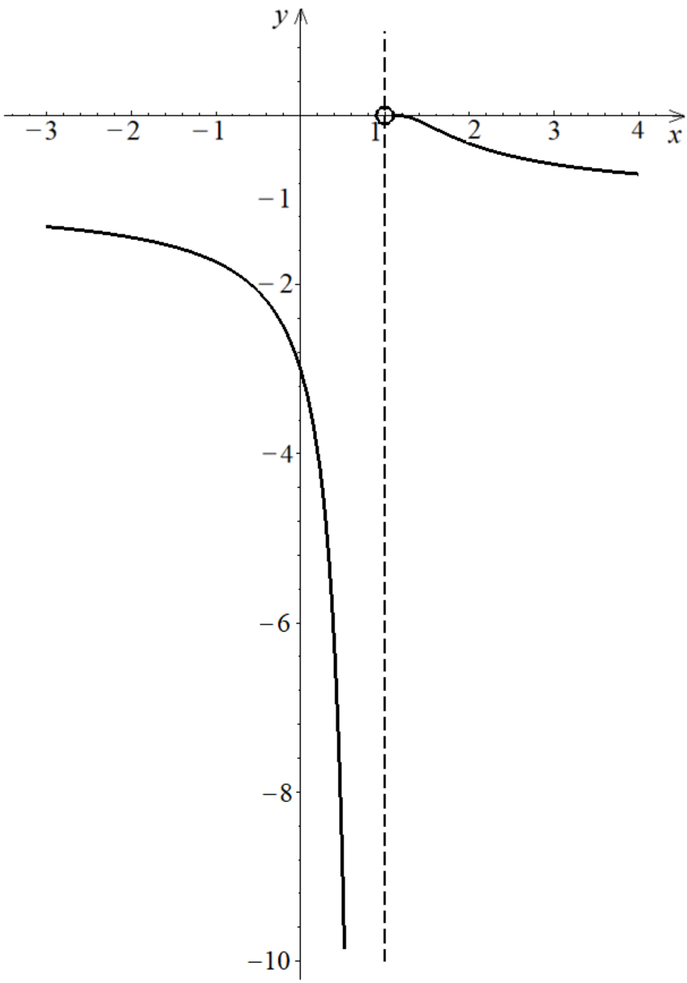


Figure 4.4

5 DIFFERENTIAL CALCULUS OF FUNCTIONS OF A SINGLE VARIABLE

Derivative of a function of a single variable

The ***derivative*** of a function $y = f(x)$ with respect to x at a point $x = x_0$ is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

provided the given limit exists.

The derivative is usually indicated by the symbols y', $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx} f(x)$ or $D_x y$.

Differentiation rules

Let $u = u(x)$ and $v = v(x)$ be differentiable functions, then

1) $C' = 0$, C is a constant;

2) $(Cu)' = Cu'$, C is a constant;

2) $(u \pm v)' = u' \pm v'$;

3) $(uv)' = u'v + uv'$ (***product rule***);

4) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ (***quotient rule***);

5) if $y = f(u)$ has the derivative with respect to u and $u = g(x)$ has the derivative with respect to x, then function $y = f(g(x))$ has the derivative with respect to x and can be found as follows:

$$y' = y'_u u' \text{ (chain rule).}$$

Logarithmic derivative:

$$f'(x) = f(x)[\ln(f(x))]'.$$

The **table of derivatives** is given in Appendix A.

Derivatives of higher order

The derivative $f'(x)$ of a function $y = f(x)$ is called ***the first derivative*** of the function. However, if $f'(x)$ also has the derivative, it is called the ***second derivative*** of a function $y = f(x)$ and is designated as y'', $\frac{d^2 y}{dx^2}$, $f''(x)$, $\frac{d^2}{dx^2}f(x)$ or $D_x^2 y$.

In general, if the $(n-1)$th-order derivative of a function has the derivative, it's called the n ***th-order derivative*** of the same function and is defined as:

$$y^{(n)} = \left(y^{(n-1)}\right)'.$$

It also can be designated as $\frac{d^n y}{dx^n}$, $f^{(n)}(x)$, $\frac{d^n}{dx^n}f(x)$ or $D_x^n y$.

Derivative of an implicit function

The derivative of an ***implicit function*** $(F(x, y) = 0)$ may be obtained by the following procedure. Thinking of y as a function of x, differentiate the function $F(x, y)$ with respect to x and solve the derived relation for y'.

The table of the derivatives is given in Appendix C.

Differentiable functions

A function $y = f(x)$ is ***differentiable*** at a point $x = x_0$ if its increment Δy at the current point can be represented in the form:

$$\Delta y = A\Delta x + o(\Delta x)$$

where A is independent of Δx, however, in general, may depend on x.

***Theorem* 5.1** A function $y = f(x)$ is differentiable at a point $x = x_0$ if and only if it has the derivative at the current point and $A = f'(x)$.

Differential of a function

The expression:

$$d\,y = f'(x)d\,x$$

is called the ***differential*** of a function.

The differential of a function satisfies rules, which are similar to the differential rules.

The ***differential of the*** n ***th-order*** is defined as:

$$d^n y = d\left(d^{n-1} y\right) = f^{(n)}(x)dx^n.$$

Problem 5.1. Differentiate the following functions:

a) $y = 14\sqrt[7]{x^5} - 2x^3 + \dfrac{2}{3x^6} - \arctan(e+5)$;

b) $y = e^{x^2+5x-17}\ln(\cos x) + \dfrac{\arctan(5x+1)}{\sqrt{3-7x}}$;

c) $y = \left(\ln\left(x^2+4\right)\right)^{\tan 5x}$;

d) $y^4 + x^3 - 7x^2y^2 - y^2 + 14x - 7 = 0$.

Solution.

a) $y' = \left(14\sqrt[7]{x^5} - 2x^3 + \dfrac{2}{3x^6} - \arctan(e+5)\right)' = \left(14\sqrt[7]{x^5}\right)' - \left(2x^3\right)' +$

$$+\left(\frac{2}{3x^6}\right)' - \left(\arctan(e+5)\right)' = 14\left(x^{\frac{5}{7}}\right)' - 2\left(x^3\right)' + \frac{2}{3}\left(x^{-6}\right)' - 0 =$$

$$= 14\cdot\frac{5}{7}\cdot x^{-\frac{2}{7}} - 6x^2 + \frac{2}{3}\cdot(-6)x^{-7} = \frac{10}{\sqrt[7]{x^2}} - 6x^2 - \frac{4}{x^7}.$$

b) $y' = \left(e^{x^2+5x-17}\ln(\cos x) + \dfrac{\arctan(5x+1)}{\sqrt{3-7x}}\right)' =$

$$= \left(e^{x^2+5x-17}\ln(\cos x)\right)' + \left(\frac{\arctan(5x+1)}{\sqrt{3-7x}}\right)'.$$

At first, let's find the derivative of the first term using the product rule. Then find the derivative of the second term using the quotient rule (additionally we should use the chain rule):

$$\left(e^{x^2+5x-17}\ln(\cos x)\right)' = \left(e^{x^2+5x-17}\right)'\ln(\cos x) + e^{x^2+5x-17}\left(\ln(\cos x)\right)' =$$

$$= e^{x^2+5x-17}\left(x^2+5x-17\right)'\ln(\cos x) + e^{x^2+5x-17}\frac{1}{\cos x}(\cos x)' =$$

$$= e^{x^2+5x-17}(2x+5)\ln(\cos x) + e^{x^2+5x-17}\frac{1}{\cos x}(-\sin x) =$$

$$= e^{x^2+5x-17}(2x+5)\ln(\cos x) - e^{x^2+5x-17}\tan x,$$

$$\left(\frac{\arctan(5x+1)}{\sqrt{3-7x}}\right)' = \frac{(\arctan(5x+1))'\sqrt{3-7x} - \arctan(5x+1)\left(\sqrt{3-7x}\right)'}{\left(\sqrt{3-7x}\right)^2} =$$

$$= \frac{\frac{1}{1+(5x+1)^2}(5x+1)'\sqrt{3-7x} - \arctan(5x+1)\frac{1}{2\sqrt{3-7x}}(3-7x)'}{3-7x} =$$

$$= \frac{\frac{5\sqrt{3-7x}}{1+(5x+1)^2} - \frac{\arctan(5x+1)(-7)}{2\sqrt{3-7x}}}{3-7x} = \frac{5\sqrt{3-7x}}{\left(1+(5x+1)^2\right)(3-7x)} +$$

$$+ \frac{7\arctan(5x+1)}{2\sqrt{3-7x}(3-7x)} = \frac{5}{\left(1+(5x+1)^2\right)\sqrt{3-7x}} + \frac{7\arctan(5x+1)}{2\sqrt{(3-7x)^3}}.$$

And finally:

$$y' = e^{x^2+5x-17}(2x+5)\ln(\cos x) - e^{x^2+5x-17}\tan x +$$

$$+ \frac{5}{\left(1+(5x+1)^2\right)\sqrt{3-7x}} + \frac{7\arctan(5x+1)}{2\sqrt{(3-7x)^3}}.$$

c) $y = \left(\ln\left(x^2+4\right)\right)^{\tan 5x}$.

Here, we obtain the result by using the logarithmic derivative:

$$y' = y[\ln y]' = \left(\ln\left(x^2+4\right)\right)^{\tan 5x}\left[\ln\left(\ln\left(x^2+4\right)\right)^{\tan 5x}\right]'$$

Let's find the derivative of the function in the brackets:

$$\left[\ln\left(\ln\left(x^2+4\right)\right)^{\tan 5x}\right]' = \left[\tan 5x \cdot \ln\left(\ln\left(x^2+4\right)\right)\right]' = (\tan 5x)' \ln\left(\ln\left(x^2+4\right)\right) +$$

$$+ \tan 5x\left(\ln\left(\ln\left(x^2+4\right)\right)\right)' = \frac{5}{\cos^2 5x}\ln\left(\ln\left(x^2+4\right)\right) + \tan 5x \frac{1}{\ln\left(x^2+4\right)}\frac{1}{x^2+4}2x =$$

$$= \frac{5\ln\left(\ln\left(x^2+4\right)\right)}{\cos^2 5x} + \frac{2x\tan 5x}{\left(x^2+4\right)\ln\left(x^2+4\right)}.$$

Therefore:

$$y' = \left(\ln\left(x^2+4\right)\right)^{\tan 5x}\left[\frac{5\ln\left(\ln\left(x^2+4\right)\right)}{\cos^2 5x} + \frac{2x\tan 5x}{\left(x^2+4\right)\ln\left(x^2+4\right)}\right].$$

d) $y^4 + x^3 - 7x^2y^2 - y^2 + 14x - 7 = 0$.

In the current case we assume that $y = y(x)$. Let's find the derivative of the left-hand side of the equation with respect to x:

$$\left(y^4 + x^3 - 7x^2y^2 - y^2 + 14x - 7\right)' =$$

$$= 4y^3y' + 3x^2 - 14xy^2 - 14x^2yy' - 2yy' + 14 = 0 .$$

In order to find y' we have to solve the last equation for y':

$$4y^3y' + 3x^2 - 14xy^2 - 14x^2yy' - 2yy' + 14 = 0 ;$$

$$4y^3y' - 14x^2yy' - 2yy' = -3x^2 + 14xy^2 - 14 ;$$

$$y'\left(4y^3 - 14x^2y - 2y\right) = -3x^2 + 14xy^2 - 14 \Rightarrow y' = \frac{-3x^2 + 14xy^2 - 14}{4y^3 - 14x^2y - 2y} .$$

Answer: **a)** $y' = \frac{10}{\sqrt[7]{x^2}} - 6x^2 - \frac{4}{x^7}$;

b) $y' = e^{x^2+5x-17}(2x+5)\ln(\cos x) - e^{x^2+5x-17}\tan x +$

$+ \frac{5}{\left(1+(5x+1)^2\right)\sqrt{3-7x}} + \frac{7\arctan(5x+1)}{2\sqrt{(3-7x)^3}}$;

c) $y' = \left(\ln\left(x^2+4\right)\right)^{\tan 5x}\left[\frac{5\ln\left(\ln\left(x^2+4\right)\right)}{\cos^2 5x} + \frac{2x\tan 5x}{\left(x^2+4\right)\ln\left(x^2+4\right)}\right]$;

d) $y' = \frac{-3x^2 + 14xy^2 - 14}{4y^3 - 14x^2y - 2y}$.

Differentiation of functions represented parametrically

If the functional dependence between y and x is given by means of a third variable or ***parameter*** t: $x = \varphi(t)$, $y = \psi(t)$, then the ***first*** and ***second derivatives*** can be written as follows:

$$y'_x = \frac{\psi'(t)}{\varphi'(t)}, \quad (5.1)$$

$$y''_{xx} = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{\left(\varphi'(t)\right)^3} . \quad (5.2)$$

Problem 5.2. Find the first and second derivatives of the parametric functions:

a) $\begin{cases} x = \frac{4}{3}t^3 + t, \\ y = \operatorname{arctan} 2t, \end{cases}$ **b)** $\begin{cases} x = 2\cos^3 t, \\ y = 2\sin^3 t. \end{cases}$

Solution.

a) In the current case $\varphi(t) = \frac{4}{3}t^3 + t$, $\psi(t) = \operatorname{arctan} 2t$. By (5.1) and (5.2), we obtain:

$$y'_x = \frac{\psi'(t)}{\varphi'(t)} = \frac{(\operatorname{arctan} 2t)'}{\left(\frac{4}{3}t^3 + t\right)'} = \frac{\frac{2}{1+4t^2}}{4t^2+1} = \frac{2}{\left(4t^2+1\right)^2},$$

$$y'_{xx} = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{(\varphi'(t))^3} = \frac{\left(\frac{2}{1+4t^2}\right)'\left(1+4t^2\right) - \frac{2}{1+4t^2}\left(1+4t^2\right)'}{\left(1+4t^2\right)^3} =$$

$$= \frac{\frac{-16t}{\left(1+4t^2\right)^2}\left(1+4t^2\right) - \frac{16t}{1+4t^2}}{\left(1+4t^2\right)^3} = \frac{\frac{-16t}{1+4t^2} - \frac{16t}{1+4t^2}}{\left(1+4t^2\right)^3} = -\frac{32t}{\left(1+4t^2\right)^4}.$$

b) Analogous to the previous issue: $\varphi(t) = 2\cos^3 t$, $\psi(t) = 2\sin^3 t$. By (5.1) and (5.2), we obtain:

$$y'_x = \frac{\psi'(t)}{\varphi'(t)} = \frac{\left(2\sin^3 t\right)'}{\left(2\cos^3 t\right)'} = \frac{2\cdot 3\sin^2 t \cdot \cos t}{-2\cdot 3\cos^2 t \cdot \sin t} = -\frac{\sin t}{\cos t} = -\tan t,$$

$$y''_{xx} = \frac{\psi''(t)\varphi'(t) - \psi'(t)\varphi''(t)}{(\varphi'(t))^3} =$$

$$= \frac{\left(6\sin^2 t \cdot \cos t\right)'\left(-6\cos^2 t \cdot \sin t\right) - \left(6\sin^2 t \cdot \cos t\right)\left(-6\cos^2 t \cdot \sin t\right)'}{\left(-6\cos^2 t \cdot \sin t\right)^3} =$$

$$=\frac{\left(12\sin t\cdot\cos^2 t-6\sin^3 t\right)\left(-6\cos^2 t\cdot\sin t\right)}{-216\cos^6 t\cdot\sin^3 t}-$$

$$-\frac{\left(6\sin^2 t\cdot\cos t\right)\left(12\cos t\cdot\sin^2 t-6\cos^3 t\right)}{-216\cos^6 t\cdot\sin^3 t}=$$

$$=\frac{-36\sin^2 t\cdot\cos^2 t\left(2\cos^2 t-\sin^2 t+2\sin^2 t-\cos^2 t\right)}{-216\cos^6 t\cdot\sin^3 t}=$$

$$=\frac{\cos^2 t+\sin^2 t}{6\cos^4 t\cdot\sin t}=\frac{1}{6\cos^4 t\cdot\sin t}.$$

Answer: **a)** $y'_x=\dfrac{2}{\left(4t^2+1\right)^2}$, $y''_{xx}=-\dfrac{32t}{\left(1+4t^2\right)^4}$; **b)** $y'_x=-\tan t$; $y''_{xx}=\dfrac{1}{6\cos^4 t\cdot\sin t}$.

L'Hopital's rule for evaluating indeterminate forms

Let both functions $f(x)$ and $g(x)$ be differentiable. If $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$ or $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=\infty$, then:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\left[\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right]=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

under condition that there exists a limit of the ratio of the derivatives.

Remark. The indeterminate form $[0\cdot\infty]$ can be evaluated by applying L'Hopital's rule using one of the following ways:

$$\lim_{x\to a} f(x)g(x)=[0\cdot\infty]=\lim_{x\to a}\frac{f(x)}{1/g(x)}=\left[\frac{0}{0}\right]=\lim_{x\to a}\frac{f'(x)}{(1/g(x))'};$$

$$\lim_{x\to a} f(x)g(x)=[0\cdot\infty]=\lim_{x\to a}\frac{g(x)}{1/f(x)}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to a}\frac{g'(x)}{(1/f(x))'}.$$

Problem 5.3. Applying L'Hopital's rule find the following limits:

a) $\lim\limits_{x\to\frac{1}{2}}\dfrac{1-\sin^2(\pi x)}{2x^2-3x+1}$; **c)** $\lim\limits_{x\to-3}\left(x^2+2x-3\right)^{\frac{1}{x^2-9}}$.

b) $\lim\limits_{x\to\infty}\dfrac{(x+5)e^x}{x\ln(1+x)}$;

Solution.

a) $$\lim_{x\to\frac{1}{2}}\frac{1-\sin^2(\pi x)}{2x^2-3x+1}=\left[\frac{1-\sin^2\left(\frac{\pi}{2}\right)}{\frac{1}{2}-\frac{3}{2}+1}=\frac{0}{0}\right]=\lim_{x\to\frac{1}{2}}\frac{\left(1-\sin^2(\pi x)\right)'}{\left(2x^2-3x+1\right)'}=$$

$$=\lim_{x\to\frac{1}{2}}\frac{-2\pi\sin(\pi x)\cos(\pi x)}{4x-3}=\lim_{x\to\frac{1}{2}}\frac{-\pi\sin(2\pi x)}{4x-3}=\frac{-\pi\sin(\pi)}{2-3}=0.$$

b) $$\lim_{x\to\infty}\frac{(x+5)e^x}{x\ln(1+x)}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left((x+5)e^x\right)'}{\left(x\ln(1+x)\right)'}=$$

$$=\lim_{x\to\infty}\frac{e^x+(x+5)e^x}{\ln(1+x)+\frac{x}{1+x}}=\lim_{x\to\infty}\frac{e^x(x+6)}{\ln(1+x)+\frac{x}{1+x}}=\left[\frac{\infty}{\infty+1}=\frac{\infty}{\infty}\right]=$$

$$=\lim_{x\to\infty}\frac{\left(e^x(x+6)\right)'}{\left(\ln(1+x)+\frac{x}{1+x}\right)'}=\lim_{x\to\infty}\frac{e^x(x+6)+e^x}{\frac{1}{1+x}+\frac{1+x-x}{(1+x)^2}}=\lim_{x\to\infty}\frac{e^x(x+7)}{\frac{2+x}{(1+x)^2}}=$$

$$\lim_{x\to\infty}\frac{e^x(x+7)(1+x)^2}{2+x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\left(e^x(x+7)(1+x)^2\right)'}{(2+x)'}=$$

$$=\lim_{x\to\infty}\frac{e^x(x+7)(1+x)^2+e^x\left((x+7)(1+x)^2\right)'}{1}=$$

$$=\lim_{x\to\infty}e^x\left((x+7)(1+x)^2+(1+x)^2+2(x+7)(1+x)\right)=$$

$$=\lim_{x\to\infty}e^x(1+x)\left((x+7)(1+x)+1+x+2(x+7)\right)=$$

$$=\lim_{x\to\infty}e^x(1+x)\left(x^2+11x+22\right)=\infty.$$

c) $$\lim_{x\to-3}\left(x^2+2x-3\right)e^{\frac{1}{x^2-9}}=\left[(9+(-6)-3)e^{\frac{1}{9-9}}=0e^{\frac{1}{0}}=0e^{\infty}=0\cdot\infty\right].$$

In the current issue we can't apply L'Hopital's rule, however the obtained indeterminate in form can be reduced to the indeterminate form $\left[\frac{\infty}{\infty}\right]$ as it was mentioned in the remark:

$$\left(x^2+2x-3\right)e^{\frac{1}{x^2-9}}=\frac{e^{\frac{1}{x^2-9}}}{\frac{1}{x^2+2x-3}};\quad \frac{e^{\frac{1}{x^2-9}}}{\frac{1}{x^2+2x-3}}\to\frac{\infty}{\frac{1}{0}}=\frac{\infty}{\infty},\quad \text{as } x\to-3.$$

Finally, we can apply L'Hopital's rule and then we can factor the polynomial function in both numerator and denominator as it has been done in problem 4.2 issue (a):

$$\lim_{x\to-3}\frac{e^{\frac{1}{x^2-9}}}{\frac{1}{x^2+2x-3}}=\lim_{x\to-3}\frac{\left(e^{\frac{1}{x^2-9}}\right)'}{\left(\frac{1}{x^2+2x-3}\right)'}=\lim_{x\to-3}\frac{\frac{2x}{\left(x^2-9\right)^2}e^{\frac{1}{x^2-9}}}{\frac{2x+2}{\left(x^2+2x-3\right)^2}}=$$

$$=\lim_{x\to-3}\frac{2x\left(x^2+2x-3\right)^2e^{\frac{1}{x^2-9}}}{2(x+1)\left(x^2-9\right)^2}=\lim_{x\to-3}\frac{x(x-1)^2(x+3)^2e^{\frac{1}{x^2-9}}}{(x+1)(x-3)^2(x+3)^2}=$$

$$=\lim_{x\to-3}\frac{x(x-1)^2e^{\frac{1}{x^2-9}}}{(x+1)(x-3)^2}=\left[\frac{-3(-3-1)^2e^{\frac{1}{9-9}}}{(-3+1)(-3-3)^2}=\frac{-48e^{\frac{1}{0}}}{-72}=\frac{2\cdot\infty}{3}=\infty\right]=\infty.$$

Answer: **a)** 0; **b)** ∞; **c)** ∞.

Investigating of functions

Even and odd functions

A function $f(x)$ is called ***even*** (***odd***) if for every x from the domain of definition of the function $f(-x)=f(x)$ $\left(f(-x)=-f(x)\right)$.

If a function satisfies the condition neither for even functions nor for odd functions it is called ***neither odd nor even***.

Sufficient condition of monotonicity. Let a function $f(x)$ be differentiable on (a,b), then:

a) if $f'(x) \geq 0$ $(f'(x) > 0)$ when $x \in (a, b) \Rightarrow$ $f(x)$ is a ***non-decreasing*** (***steadily increasing***) function on (a, b);
b) if $f'(x) \leq 0$ $(f'(x) < 0)$ when $x \in (a, b) \Rightarrow$ $f(x)$ is a ***non-increasing*** (***steadily decreasing***) function on (a, b).

Extremum of a function

The values of x at which $f'(x) = 0$ are called ***stationary points*** of the function $f(x)$. Stationary points and the points at which $f'(x)$ doesn't exist are called ***critical points*** for $f(x)$.

Sufficient condition for a local extremum. Let $x = x_0$ be a critical point of a function $f(x)$. Then:
a) if $f'(x)$ changes from "+" to "–" as x increases through $x = x_0$, $f(x)$ has a ***maximum*** at $x = x_0$;
b) if $f'(x)$ changes from "–" to "+"as x increases through $x = x_0$, $f(x)$ has a ***minimum*** at $x = x_0$;
c) if the sign of $f'(x)$ remains unchanged as x increases through $x = x_0$, $f(x)$ doesn't have an extremum at $x = x_0$.

Maximum and minimum of a function on a segment

Let a function $f(x)$ be differentiable on a segment $[a, b]$. Let the points x_1, x_2, … x_n be the critical points for the current function, which belong to the segment $[a, b]$. Then we have to evaluate $n + 2$ numbers (calculate values of the function at all critical points and endpoints of the segment):

$$f(x_1),\ f(x_2),\ \ldots\ f(x_n),\ f(a),\ f(b).$$

The greatest and least numbers among the obtained ones are the maximum and the minimum of the function $f(x)$ on the segment $[a, b]$.

Direction of bending, points of inflection

Sufficient condition for the convexity (concavity). If $f''(x) < 0$ $(f''(x) > 0)$ on (a, b), then the curve $y = f(x)$ is ***concave downwards*** (***concave upwards***).

A point $x = x_0$ is called a ***critical point of the 2-nd derivative*** if $f''(x_0) = 0$ or $f''(x_0)$ doesn't exist.

Sufficient condition for a point of inflection. Let $x = x_0$ be a critical point of the 2-nd derivative for a function $f(x)$. Then the curve $y = f(x)$ has the point $(x_0, f(x_0))$ as the ***point of inflection*** if $f''(x)$ changes its sign as x increases through $x = x_0$.

Asymptotes

A straight line $x = x_1$ is a ***vertical asymptote*** of the curve $y = f(x)$ if at least one of the limits $\lim\limits_{x \to x_1 - 0} f(x)$ or $\lim\limits_{x \to x_1 + 0} f(x)$ equals $+\infty$ or $-\infty$.

An inclined (oblique) asymptote is the line, which is described by the equation

$$y = kx + b, \tag{5.3}$$

where $k = \lim\limits_{\substack{x \to +\infty \\ (x \to -\infty)}} \dfrac{f(x)}{x}$, $b = \lim\limits_{\substack{x \to +\infty \\ (x \to -\infty)}} [f(x) - kx]$.

If $k = 0$, the inclined asymptote is called a ***horizontal asymptote***. Its equation is $y = b$, with $b = \lim\limits_{\substack{x \to +\infty \\ (x \to -\infty)}} f(x)$.

General scheme of curve sketching

When constructing the graph of a function, it is recommended:
1) to find the domain of definition;
2) to find the points of intersection of the graph and the coordinate axes (intercepts);
3) to test the function for evenness, oddness and periodicity;
4) to find the intervals of monotonicity, the points of maxima and minima, the maximum and minimum values of the function;
5) to examine the function for direction of bending and the points of inflection;
6) to find the asymptotes;
7) to sketch the graph of the function. It is also advisable to compute values of the function at some points.

Problem 5.4. Find the maximum and minimum of the functions on the given segment:

a) $y=(x+1)^3,\ [0,\ 2];$ **c)** $y=\dfrac{1}{x^2-2x+2},\ [0,\ 3].$

b) $y=(3+2x)e^{-\frac{x^2}{2}},\ [-4,\ 2];$

Solution.

a) According to the described scheme at first we have to find critical points of the considered function, which belong to the given segment. Let's find critical points:

$$y'=\left((x+1)^3\right)'=3(x+1)^2;$$

$$y'=3(x+1)^2=0 \Leftrightarrow x_1=-1.$$

Since the obtained critical point doesn't belong to the considered segment $[0,\ 2]$, we have to evaluate values of the function only at the endpoints of the segment:

$$y(0)=(0+1)^3=1;\ y(2)=(2+1)^3=27.$$

Therefore, we can conclude: the given function has the maximum $y_{\max}=27$ at the point $x=2$ and the minimum $y_{\min}=1$ at the point $x=0$.

b) Analogous to the previous issue, as the first step let's find the critical points of the function:

$$y'=\left((3+2x)e^{-\frac{x^2}{2}}\right)'=2e^{-\frac{x^2}{2}}+(3+2x)\left(-xe^{-\frac{x^2}{2}}\right)=e^{-\frac{x^2}{2}}\left(-2x^2-3x+2\right);$$

$$y'=e^{-\frac{x^2}{2}}\left(-2x^2-3x+2\right)=0 \Leftrightarrow -2x^2-3x+2=0 \Leftrightarrow \left[\begin{array}{l} x_1=-2; \\ x_2=\dfrac{1}{2}. \end{array}\right.$$

Since both critical points belong to the given segment $[-4,\ 2]$, it means we have to test values of the given function at four points (including endpoints of the segment) for being maximum and minimum:

$$y(-4)=(3+2\cdot(-4))e^{-\frac{(-4)^2}{2}}=-5e^{-8}=-\frac{5}{e^8}\approx -0.0017\,;$$

$$y(x_1)=(3+2\cdot(-2))e^{-\frac{(-2)^2}{2}}=-e^{-2}=-\frac{1}{e^2}\approx -0.1353\,;$$

$$y(x_2)=\left(3+2\cdot\frac{1}{2}\right)e^{-\frac{\left(\frac{1}{2}\right)^2}{2}}=4e^{-\frac{1}{8}}=\frac{4}{\sqrt[8]{e}}\approx 3.5300\,;$$

$$y(2)=(3+2\cdot 2)e^{-\frac{2^2}{2}}=7e^{-2}=\frac{7}{e^2}\approx 0.9473\,.$$

As a result we obtain: the given function has the maximum $y_{\max}=\frac{4}{\sqrt[8]{e}}$ at the point $x=\frac{1}{2}$ and the minimum $y_{\min}=-\frac{1}{e^2}$ at the point $x=-2$.

c) Since the function to be considered is a fraction let's analyze the discriminant:

$$D=(-2)^2-4\cdot 2=-4<0\Rightarrow x^2-2x+2\neq 0,\ x\in R\,.$$

Therefore, the function $y=\frac{1}{x^2-2x+2}$ is continuous for all x, so we can obtain the result as we did in the previous issue. Let's start with finding critical points of the function:

$$y'=\left(\frac{1}{x^2-2x+2}\right)'=-\frac{2x-2}{\left(x^2-2x+2\right)^2}\,;$$

$$y'=-\frac{2x-2}{\left(x^2-2x+2\right)^2}=0\Leftrightarrow 2x-2=0\Leftrightarrow x_1=1\,.$$

Taking into account, that the critical point belongs to the given segment, let's test values of the function at it and at the endpoints of the segment for being maximum and minimum:

$$y(0)=\frac{1}{0^2-2\cdot 0+2}=\frac{1}{2};\ y(x_1)=\frac{1}{(1)^2-2\cdot 1+2}=1\,;$$

$$y(3)=\frac{1}{3^2-2\cdot 3+2}=\frac{1}{5}.$$

Therefore, we can conclude that the given function has the maximum $y_{\max}=1$ at the point $x=1$ and the minimum $y_{\min}=\frac{1}{5}$ at the point $x=3$.

Answer: **a)** $y_{\max}=27$, $y_{\min}=1$; **b)** $y_{\max}=\frac{4}{\sqrt[8]{e}}$, $y_{\min}=-\frac{1}{e^2}$; **c)** $y_{\max}=1$, $y_{\min}=\frac{1}{5}$.

Problem 5.5. Sketch the graphs of the functions:

a) $y=\frac{x^4}{(x+1)^3}$; **b)** $y=\frac{x-1}{1+x^2}$.

Solution.

a) Let's use the general scheme of curve sketching.

1) $x+1\neq 0 \Rightarrow x\neq -1 \Rightarrow D(y)=(-\infty,\,-1)\cup(-1,\,+\infty)$.

2) Points of intersection of the graph with the x-axis $(y=0)$:

$$y=\frac{x^4}{(x+1)^3}=0\Rightarrow\begin{cases}x^4=0,\\(x+1)^3\neq 0,\end{cases}\Rightarrow\begin{cases}x=0,\\x\neq -1.\end{cases}\Rightarrow(0,\,0).$$

The point of intersection of the graph and the y-axis $(x=0)$ is the same: $(0,\,0)$.

3) Let's check a function for evenness or oddness:

$$y(-x)=\frac{(-x)^4}{(-x+1)^3}=\frac{x^4}{(-x+1)^3},$$

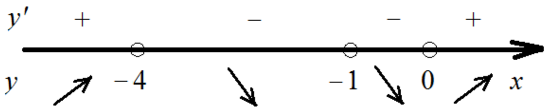
so $y(-x)\neq y(x)$ and $y(-x)\neq -y(x)$. It means the function is neither odd nor even.

4) $$y'=\left(\frac{x^4}{(x+1)^3}\right)'=\frac{\left(x^4\right)'(x+1)^3-x^4\left((x+1)^3\right)'}{(x+1)^6}=$$

$$=\frac{4x^3(x+1)^3-x^4 3(x+1)^2}{(x+1)^6}=\frac{x^3(x+1)^2(4(x+1)-3x)}{(x+1)^6}=\frac{x^3(x+4)}{(x+1)^4}.$$

$$y'=0\Rightarrow\begin{cases}x^3(x+4)=0,\\(x+1)^4\neq 0,\end{cases}\Rightarrow\begin{cases}\left[\begin{array}{l}x^3=0,\\x+4=0,\end{array}\right.\\x\neq -1,\end{cases}\Rightarrow\begin{cases}\left[\begin{array}{l}x=0,\\x=-4,\end{array}\right.\\x\neq -1,\end{cases}$$

therefore $x=0,\ x=-4$ and $x=-1$ are the critical points of the function.



When $x\in(-\infty,\,-4)\cup(0,\,+\infty)$ $y'>0$ and y increases.
When $x\in(-4,\,-1)\cup(-1,\,0)$ $y'<0$ and y decreases.
As x increases through $x=-4$, y' changes its sign from + to –; hence at $x=-4$ y has a maximum value $y(-4)=\dfrac{(-4)^4}{(-4+1)^3}=-\dfrac{256}{27}$.
As x increases through $x=0$, y' changes its sign from – to +; hence at $x=0$ y has a minimum value $y(0)=\dfrac{0^4}{(0+1)^3}=0$.

5) $$y''=\left(\frac{x^3(x+4)}{(x+1)^4}\right)'=\frac{\left(x^3(x+4)\right)'(x+1)^4-x^3(x+4)\left((x+1)^4\right)'}{(x+1)^8}=$$

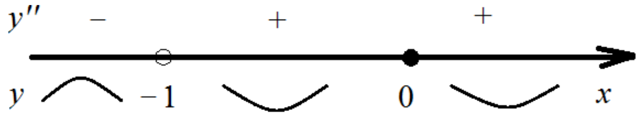
$$=\frac{\left(3x^2(x+4)+x^3\right)(x+1)^4-x^3(x+4)4(x+1)^3}{(x+1)^8}=$$

$$=\frac{x^2(x+1)^3\left((3(x+4)+x)(x+1)-4x(x+4)\right)}{(x+1)^8}=\frac{4x^2\left((x+3)(x+1)-x(x+4)\right)}{(x+1)^5}=$$

$$=\frac{4x^2\left(x^2+4x+3-x^2-4x\right)}{(x+1)^5}=\frac{12x^2}{(x+1)^5}.$$

$$y''=0\Rightarrow\frac{12x^2}{(x+1)^5}=0\Rightarrow\begin{cases}12x^2=0,\\(x+1)^5\neq 0,\end{cases}\Rightarrow\begin{cases}x=0,\\x\neq -1,\end{cases}$$

therefore $x=0, x=-1$ are the critical points of the 2-nd derivative of the function.



When $x \in (-1, 0)$ $y'' > 0$ and the curve is concave upwards.
When $x \in (-\infty, -1) \cup (0, +\infty)$ $y'' < 0$ and the curve is concave downwards.

6) $\lim\limits_{x \to -1-0} y(x) = -\infty$ and $\lim\limits_{x \to -1+0} y(x) = +\infty$. Thus, the line $x=-1$ is a vertical asymptote of the curve. By (5.3):

$$k = \lim_{x \to \pm\infty} \frac{y(x)}{x} = \lim_{x \to \pm\infty} \frac{x^4}{x(x+1)^3} = \lim_{x \to \pm\infty} \frac{x^3}{x^3\left(1+\frac{1}{x}\right)^3} = \lim_{x \to \pm\infty} \frac{1}{(1+0)^3} = 1.$$

$$b = \lim_{x \to \pm\infty} [y(x) - kx] = \lim_{x \to \pm\infty} \left[\frac{x^4}{(x+1)^3} - x \right] = \lim_{x \to \pm\infty} \frac{x^4 - x(x+1)^3}{(x+1)^3} =$$

$$= \lim_{x \to \pm\infty} \frac{x^4 - x^4 - 3x^3 - 3x^2 - x}{(x+1)^3} = \lim_{x \to \pm\infty} \frac{-3x^3 - 3x^2 - x}{(x+1)^3} =$$

$$= \lim_{x \to \pm\infty} \frac{x^3\left(-3 - \frac{3}{x} - \frac{1}{x^2}\right)}{x^3\left(1+\frac{1}{x}\right)^3} = \lim_{x \to \pm\infty} \frac{-3-0-0}{(1+0)^3} = -3.$$

Hence the line $y = x-3$ is the inclined asymptote as $x \to -\infty$ and as $x \to +\infty$.

7) Fig. 5.1 shows the graph of the function.

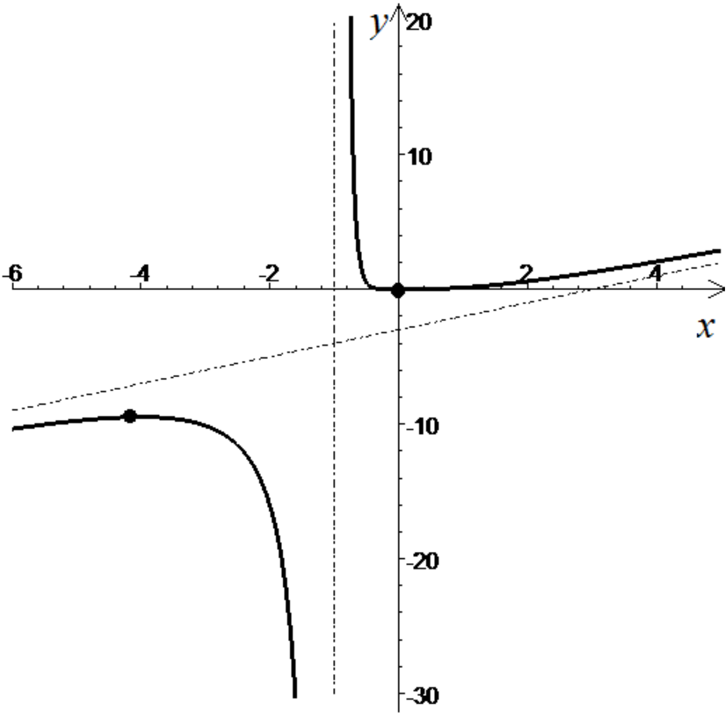


Figure 5.1

b) Let's use the general scheme of curve sketching.

1) $x^2+1 \neq 0 \Leftrightarrow x \in R \Rightarrow D(y)=(-\infty, +\infty)$.

2) Points of intersection of the graph and the x-axis $(y=0)$:

$$y=\frac{x-1}{1+x^2}=0 \Rightarrow \begin{cases} x-1=0, \\ 1+x^2 \neq 0. \end{cases} \Rightarrow x=1 \Rightarrow (1, 0).$$

The point of intersection of the graph and the y-axis $(x=0)$:

$$y(0)=\frac{0-1}{1+0^2}=-1 \Rightarrow (0, -1).$$

3) Let's check a function for evenness or oddness:

$$y(-x)=\frac{-x-1}{1+(-x)^2}=-\frac{x+1}{1+x^2},$$

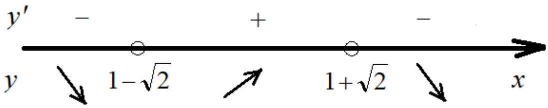
so $y(-x)\neq y(x)$ and $y(-x)\neq -y(x)$. It means the function is neither odd nor even.

$$4)\ y'=\left(\frac{x-1}{1+x^2}\right)'=\frac{(x-1)'\left(1+x^2\right)-(x-1)\left(1+x^2\right)'}{\left(1+x^2\right)^2}=$$

$$=\frac{1+x^2-(x-1)2x}{\left(1+x^2\right)^2}=\frac{-x^2+2x+1}{\left(1+x^2\right)^2}.$$

$$y'=0\Rightarrow\begin{cases}-x^2+2x+1=0,\\ \left(1+x^2\right)^2\neq 0,\end{cases}\Rightarrow\left[\begin{array}{l}x=1+\sqrt{2},\\ x=1-\sqrt{2},\end{array}\right.$$

therefore $x=1-\sqrt{2}$ and $x=1+\sqrt{2}$ are the critical points of the function.



When $x\in\left(1-\sqrt{2},\,1+\sqrt{2}\right)$ $y'>0$ and y increases.

When $x\in\left(-\infty,\,1-\sqrt{2}\right)\cup\left(1+\sqrt{2},\,+\infty\right)$ $y'<0$ and y decreases.

As x increases through $x=1-\sqrt{2}$, y' changes its sign from – to +; hence at $x=1-\sqrt{2}$ y has a minimum value

$$y\left(1-\sqrt{2}\right)=\frac{1-\sqrt{2}-1}{1+\left(1-\sqrt{2}\right)^2}=\frac{-\sqrt{2}}{4-2\sqrt{2}}=\frac{-\sqrt{2}\left(2+\sqrt{2}\right)}{2\left(2-\sqrt{2}\right)\left(2+\sqrt{2}\right)}=-\frac{2\sqrt{2}+2}{4}=$$

$$=-\frac{\sqrt{2}+1}{2}\approx -1.21.$$

As x increases through $x=1+\sqrt{2}$, y' changes its sign from + to –; hence at $x=1+\sqrt{2}$ y has a maximum value

$$y\left(1+\sqrt{2}\right)=\frac{1+\sqrt{2}-1}{1+\left(1+\sqrt{2}\right)^2}=\frac{\sqrt{2}}{4+2\sqrt{2}}=\frac{\sqrt{2}\left(2-\sqrt{2}\right)}{2\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)}=\frac{2\sqrt{2}-2}{4}=$$

$$=\frac{\sqrt{2}-1}{2}\approx 0.21\,.$$

5) $$y''=\left(\frac{-x^2+2x+1}{\left(1+x^2\right)^2}\right)'=$$

$$=\frac{\left(-x^2+2x+1\right)'\left(1+x^2\right)^2-\left(-x^2+2x+1\right)\left(\left(1+x^2\right)^2\right)'}{\left(1+x^2\right)^4}=$$

$$=\frac{(-2x+2)\left(1+x^2\right)^2-\left(-x^2+2x+1\right)\left(2\left(1+x^2\right)2x\right)}{\left(1+x^2\right)^4}=$$

$$=\frac{2\left(1+x^2\right)\left((1-x)\left(1+x^2\right)+2x\left(x^2-2x-1\right)\right)}{\left(1+x^2\right)^4}=$$

$$=\frac{2\left(1-x+x^2-x^3+2x^3-4x^2-2x\right)}{\left(1+x^2\right)^3}=\frac{2\left(x^3-3x^2-3x+1\right)}{\left(1+x^2\right)^3}\,.$$

$$y''=0\Rightarrow\frac{2\left(x^3-3x^2-3x+1\right)}{\left(1+x^2\right)^3}=0\Rightarrow\begin{cases}x^3-3x^2-3x+1=0,\\ \left(1+x^2\right)^3\neq 0,\end{cases}\Rightarrow$$

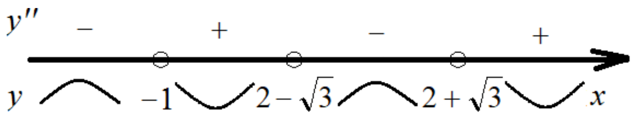
$$\Rightarrow x^3-3x^2-3x+1=0\,.$$

It's quite obvious, that one of the roots of the last equation is $x=-1$. It means that the left-hand side of the last equation has a factor $(x+1)$, therefore we can factor the whole equation by using long division (analogous to problem 4.2 issue (f)):

$$
\begin{array}{r|l}
-\begin{array}{r} x^3-3x^2-3x+1 \\ x^3+x^2 \end{array} & \dfrac{x+1}{x^2-4x+1} \\
\hline
-\begin{array}{r} -4x^2-3x+1 \\ -4x^2-4x \end{array} & \\
\hline
-\begin{array}{r} x+1 \\ x+1 \end{array} & \\
\hline
0 &
\end{array}
$$

$$x^3-3x^2-3x+1=(x+1)(x^2-4x+1)=0 \Leftrightarrow \left[\begin{array}{l} x=-1, \\ x=2-\sqrt{3}, \\ x=2+\sqrt{3}, \end{array}\right.$$

therefore $x=-1$, $x=2-\sqrt{3}$ and $x=2+\sqrt{3}$ are the critical points of the 2-nd derivative of the function.



When $x\in(-1,\, 2-\sqrt{3})\cup(2+\sqrt{3},\, +\infty)$ $y''>0$ and the curve is concave upwards.

When $x\in(-\infty,\, -1)\cup(2-\sqrt{3},\, 2+\sqrt{3})$ $y''<0$ and the curve is concave downwards.

As x increases through $x=-1$, y'' changes its sign; hence y has a point of inflection with coordinates $y(-1)=\dfrac{-1-1}{1+(-1)^2}=-1 \Rightarrow (-1,\, -1)$.

As x increases through $x=2-\sqrt{3}$, y'' changes its sign; hence y has a point of inflection with coordinates $y(2-\sqrt{3})=\dfrac{2-\sqrt{3}-1}{1+(2-\sqrt{3})^2}=\dfrac{1-\sqrt{3}}{8-4\sqrt{3}}=-\dfrac{1+\sqrt{3}}{4} \Rightarrow \left(2-\sqrt{3},\, -\dfrac{1+\sqrt{3}}{4}\right)$.

As x increases through $x = 2+\sqrt{3}$, y'' changes its sign; hence y has a point of inflection with coordinates $y\left(2+\sqrt{3}\right) = \frac{2+\sqrt{3}-1}{1+\left(2+\sqrt{3}\right)^2} = \frac{1+\sqrt{3}}{8+4\sqrt{3}} = \frac{\sqrt{3}-1}{4} \Rightarrow \left(2+\sqrt{3},\ \frac{\sqrt{3}-1}{4}\right)$.

6) Since the given function is a ratio of two polynomial functions and the denominator is a positive number for any real value of x, therefore the equation $\lim\limits_{x\to x_0} y(x) = \infty$ has no solution for x_0. Thus, the graph of the given function has no vertical asymptotes.

Let's check whether the graph of the current function has incline asymptotes. By (5.3):

$$k = \lim_{x\to\pm\infty} \frac{y(x)}{x} = \lim_{x\to\pm\infty} \frac{x-1}{x\left(1+x^2\right)} = \lim_{x\to\pm\infty} \frac{x\left(1-\frac{1}{x}\right)}{x\left(1+x^2\right)} = \lim_{x\to\pm\infty} \frac{1-0}{1+x^2} = 0.$$

$$b = \lim_{x\to\pm\infty} \left[y(x) - kx\right] = \lim_{x\to\pm\infty} \left[\frac{x-1}{1+x^2} - 0\right] = \lim_{x\to\pm\infty} \frac{x-1}{1+x^2} =$$

$$= \lim_{x\to\pm\infty} \frac{x\left(1-\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{1-0}{x(1+0)} = \lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

Hence the line $y = 0$ is the horizontal asymptote as $x \to -\infty$ and as $x \to +\infty$.

7) Fig. 5.2 shows the graph of the function. The marked points correspond to maximum and minimum values of the function and the points of inflection.

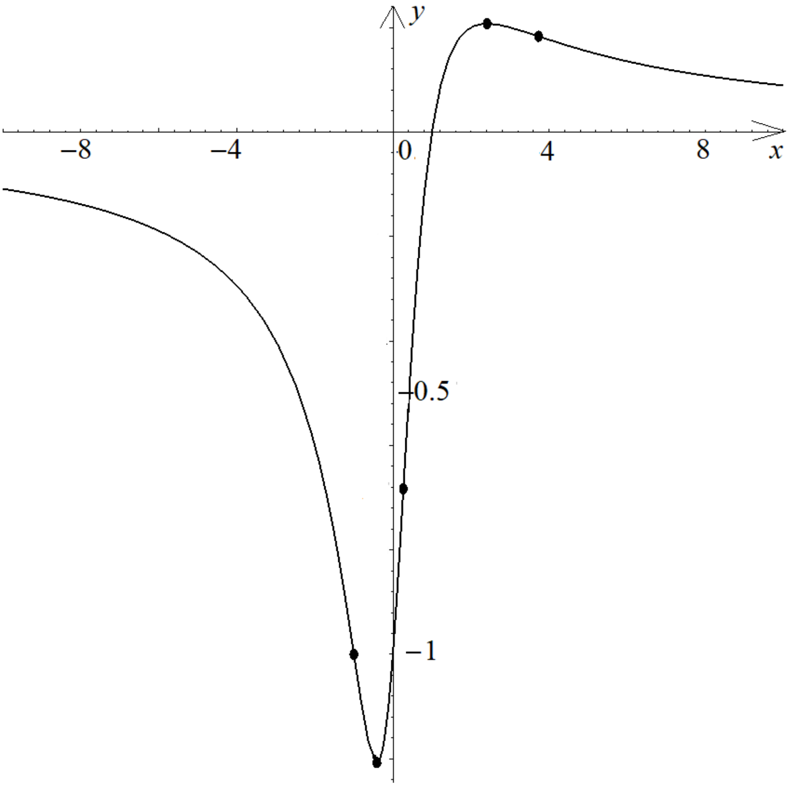


Figure 5.2

Exercises

5.1. Differentiate the following functions:

a) $y = 15\sqrt[5]{x^4} - x^2 + \dfrac{1}{4x^8} - \sin(\pi - 1)$;

b) $y = e^{2x^2-5x+1}\arcsin(2x) + \dfrac{\ln(1-3x)}{\sqrt{3x+5}}$;

c) $y = (\arctan(2x+1))^{\sin 2x}$;

d) $y^3 + x^4 - 2x^4y^2 + 3y + 10x^2 - 1 = 0$.

Answer: **a)** $y' = \frac{12}{\sqrt[5]{x}} - 2x - \frac{2}{x^9}$; **b)** $y' = e^{2x^2-5x+1}(4x-5)\arcsin(2x) +$

$$+ e^{2x^2-5x+1}\frac{2}{\sqrt{1-4x^2}} - \frac{3}{(1-3x)\sqrt{3x+5}} - \frac{3\ln(1-3x)}{2\sqrt{(3x+5)^3}};$$

c) $y = (\arctan(2x+1))^{\sin 2x}\Big[2\cos(2x)\ln(\arctan(2x+1)) +$

$$+ \frac{2\sin 2x}{\arctan(2x+1)\left(1+(2x+1)^2\right)}\Big];$$ **d)** $y' = -\frac{4x^3 - 8x^3y^2 + 20x}{3y^2 - 4x^4y + 3}$.

5.2. Find the first and the second derivatives of the parametric functions:

a) $\begin{cases} x = \ln t, \\ y = t + \frac{1}{t}; \end{cases}$ **b)** $\begin{cases} x = 1 + \cos\frac{t}{2}, \\ y = t - \sin t. \end{cases}$

Answer: **a)** $y'_x = \frac{1-t^2}{t}$, $y''_{xx} = \frac{t^2+1}{t}$; **b)** $y'_x = -4\sin\frac{t}{2}$, $y''_{xx} = 4\cot\frac{t}{2}$.

5.3. Applying L'Hopital's rule find the following limits:

a) $\lim\limits_{x\to 0}\frac{e^x - e^{\frac{x}{2}}}{\cos\frac{x}{2}\sin\frac{x}{2}}$; **c)** $\lim\limits_{x\to 0}\ln\left(1+\sin\frac{x}{2}\right)\cot x$.

b) $\lim\limits_{x\to\infty}\frac{xe^{2x}}{x+e^{2x}}$;

Answer: **a)** 1; **b)** ∞; **c)** $\frac{1}{2}$.

5.4. Find the maximum and minimum of the functions on the given segment:

a) $y = x^3 - 3x + 3$, $[-2, 3]$; **c)** $y = 2\sin(2x) + \cos(4x)$, $[0, \pi]$.

b) $y = (3-x)e^{-\frac{x}{4}}$, $[0, 4]$;

Answer: **a)** $y_{max}=21$, $y_{min}=1$; **b)** $y_{max}=3$, $y_{min}=-\frac{1}{e}$; **c)** $y_{max}=\frac{3}{2}$, $y_{min}=-3$.

5.5. Sketch the graph of the functions:

a) $y=\frac{x^3}{x^2-1}$; **b)** $y=\frac{x^2}{x^2+2}$.

Answer: **a)** Fig. 5.3 shows the graph of the function; **b)** Fig. 5.4 shows the graph of the function.

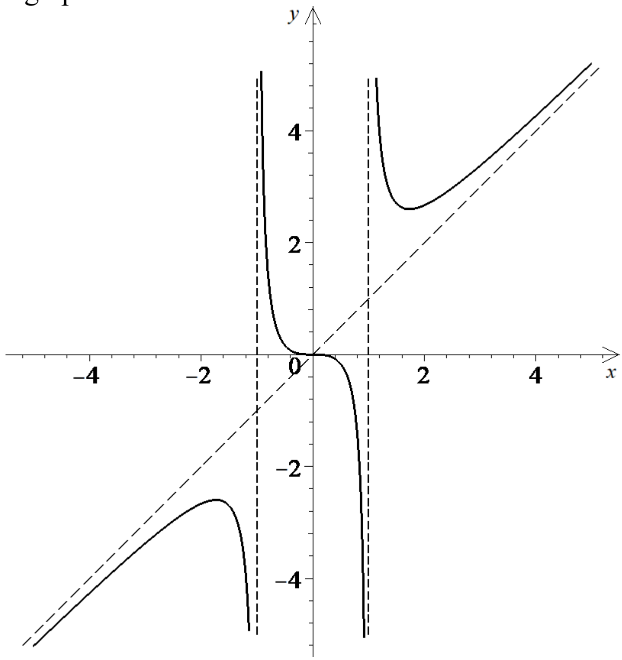


Figure 5.3

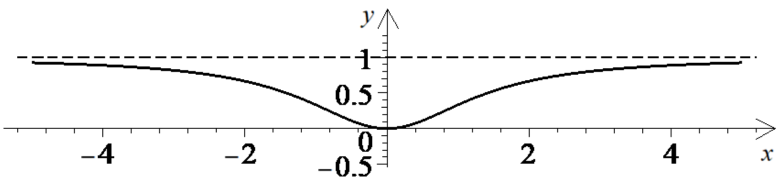


Figure 5.4

6 DIFFERENTIAL CALCULUS OF FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES

Domain of definition of a function

Let $D(f)$ be the set of points on the xy-plain, where a function $f(x, y)$ is defined. $D(f)$ is called ***the domain of definition*** of the function $f(x, y)$.

The set of points (x, y) such that $\sqrt{(x-x_0)^2+(y-y_0)^2}<\varepsilon$ is called the ε ***-neighborhood*** of the point (x_0, y_0).

A set D is called the ***open set*** or ***domain*** if every point of D has neighborhood consisting entirely of points that belonging to D.

A ***boundary point*** of a set D is a point every neighborhood of which contains both points that belong to D and points that do not belong to D.

A set D is called ***closed*** if it is contains all its boundary points.

A ***region*** is a set consisting of a domain plus, perhaps, some or all of its boundary points.

Problem 6.1. Find and sketch the domain of definition of the function of two variables:

$$z=\ln\left(16-x^2-y^2\right)+\frac{\arcsin(y+1)}{x-y+1}.$$

Solution.

The function z assumes real values under the conditions:

$$\begin{cases}16-x^2-y^2>0,\\ |y+1|\le 1,\\ x-y+1\ne 0,\end{cases}\Rightarrow\begin{cases}x^2+y^2<16,\\ -1\le y+1\le 1,\\ y\ne x+1,\end{cases}\Rightarrow\begin{cases}x^2+y^2<16,\\ -2\le y\le 0,\\ y\ne x+1.\end{cases}$$

Fig. 6.1 shows the domain of definition $D(z)$ of the given function.

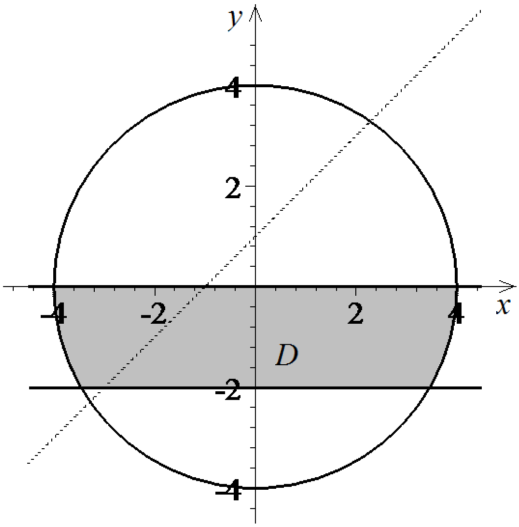


Figure 6.1

Answer: $D(z):\begin{cases} x^2+y^2<16, \\ -2\le y\le 0, \\ y\ne x+1. \end{cases}$

Limit and continuity of a function of two variables

A real number A is called the ***limit*** of a function $f(x, y)$ in as $M(x, y)$ tends $M_0(x_0, y_0)$ $(M(x, y)\to M_0(x_0, y_0))$ if for any positive real number ε can be found a positive real number $\delta=\delta(\varepsilon)$, such that if $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ then $|f(x, y)-A|<\varepsilon$:

$$\lim_{\substack{x\to x_0 \\ y\to y_0}} f(x, y)=A.$$

If $M(x, y)$ is some arbitrary point in δ-neighborhood of the point $M_0(x_0, y_0)$, then the last formula can be rewritten in the form:

$$\lim_{M\to M_0} f(M)=A,$$

where $f(M)=f(x, y)$.

A function $z = f(x, y)$ is said to be ***continuous*** at a point $M_0(x_0, y_0)$ provided $f(M_0)$ is defined and $\lim\limits_{M \to M_0} f(M) = f(M_0)$.

A function $z = f(x, y)$ is said to be ***continuous in a domain*** if it is continuous at each point of this domain.

Partial derivatives

Let $z = f(x, y)$ be a function of the independent variables x and y.

The ***partial derivative*** of $z = f(x, y)$ with respect to x, written $\frac{\partial z}{\partial x}$, z'_x, f'_x, is defined as:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta_x f(x, y)}{\Delta x}.$$

Here $\Delta_x z = \Delta_x f(x, y) = f(x + \Delta x, y) - f(x, y)$ is the ***partial increment*** of z with respect to x.

The ***partial derivative*** of $z = f(x, y)$ with respect to y, written $\frac{\partial z}{\partial y}$, z'_y, f'_y, is defined as:

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta_y f(x, y)}{\Delta y}.$$

Here $\Delta_y z = \Delta_y f(x, y) = f(x, y + \Delta y) - f(x, y)$ is the ***partial increment*** of z with respect to y.

Partial derivatives of higher orders

The ***second-order partial derivatives*** of the function $z = f(x, y)$ are defined as the partial derivatives of its first-order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f''_{xx}; \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f''_{xy};$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f''_{yx}; \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f''_{yy}.$$

Analogously, partial derivatives of any order can be defined.

Theorem 6.1. If $z = f(x, y)$ and its partial derivatives f'_x, f'_y, f''_{xy} and f''_{yx} are continuous in a neighborhood of a point $M(x, y)$, then $f''_{xy} = f''_{yx}$ at this point.

Total differential

The ***total increment*** of a function $z = f(x, y)$ at the point $M(x, y)$ is the difference:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

The function $z = f(x, y)$ is called ***differentiable*** at the point $M(x, y)$ if its total increment at this point can be represented in the form

$$\Delta z = A\Delta x + B\Delta y + \alpha\Delta x + \beta\Delta y,$$

where $\alpha = \alpha(\Delta x, \Delta y) \to 0$, $\beta = \beta(\Delta x, \Delta y) \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$, A and B are quantities independent of Δx and Δy.

Total differential of the function $z = f(x, y)$ is the principal part of the total increment Δz which is linear with respect to the increment of the arguments Δx and Δy, namely,

$$\Delta z = A\Delta x + B\Delta y.$$

Theorem 6.2. If $z = f(x, y)$ has continuous first partial derivatives at a point $M(x, y)$, it is differentiable at the point and has the total differential, which can be written in the form:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{6.1}$$

Theorem 6.3. If $z = f(x, y)$ is differentiable at the point $M(x, y)$, then the function is continuous at this point.

Problem 6.2. The function $z = \arcsin\left(x^2 + y^2\right) + e^{5xy - x^2}$ is given. Find dz.

Solution. By (6.1), we get:

$$\frac{\partial f}{\partial x} = \frac{2x}{\sqrt{1 - \left(x^2 + y^2\right)^2}} + (5y - 2x)e^{5xy - x^2};$$

$$\frac{\partial f}{\partial y} = \frac{2y}{\sqrt{1-\left(x^2+y^2\right)^2}} + 5xe^{5xy-x^2} \; ;$$

$$dz = \left(\frac{2x}{\sqrt{1-\left(x^2+y^2\right)^2}} + (5y-2x)e^{5xy-x^2} \right) dx +$$

$$+ \left(\frac{2x}{\sqrt{1-\left(x^2+y^2\right)^2}} + 5xe^{5xy-x^2} \right) dy \; .$$

Derivatives of composite functions
If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, then:
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \; ; \; \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \; . \qquad (6.2)$$
If $z = f(x, y)$ and $x = x(t)$, $y = y(t)$, then:
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \; . \qquad (6.3)$$
If $z = f(x, y)$ and $y = y(x)$, then:
$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \; . \qquad (6.4)$$

Problem 6.3. Find the derivatives of the composite functions:
a) $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$; b) $\frac{dz}{dt}$; c) $\frac{dz}{dx}$.

a) $z = \sin\left(2x + y^2\right)$, $x = u^3\sqrt{u-v}$, $y = 2^{2u+3v}$;

b) $z = \arctan(xy)$, $x = 2\sin 2t$, $y = e^{\sqrt{t}}$;

c) $z = \ln\left(y - e^x\right)$, $y = \tan\left(5 + x^2\right)$.

***Solution*.** **a)** By (6.2) we obtain:

$$\frac{\partial z}{\partial x} = 2\cos\left(2x + y^2\right); \; \frac{\partial z}{\partial y} = 2y\cos\left(2x + y^2\right);$$

$$\frac{\partial x}{\partial u} = 3u^2\sqrt{u-v} + \frac{u^3}{2\sqrt{u-v}} \; ; \; \frac{\partial x}{\partial v} = -\frac{u^3}{2\sqrt{u-v}} \; ;$$

$$\frac{\partial y}{\partial u} = 2^{2u+3v} \cdot 2\ln 2 = 2^{2u+3v+1}\ln 2\ ;\ \frac{\partial y}{\partial v} = 2^{2u+3v} \cdot 3\ln 2\ .$$

$$\frac{\partial z}{\partial u} = 2\cos\left(2x + y^2\right)\left(3u^2\sqrt{u-v} + \frac{u^3}{2\sqrt{u-v}}\right) + 2y\cos\left(2x + y^2\right)2^{2u+3v+1}\ln 2 =$$

$$= 2\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right)\left(3u^2\sqrt{u-v} + \frac{u^3}{2\sqrt{u-v}}\right) +$$

$$+ 2^{2u+3v+1}\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right) \cdot 2^{2u+3v+1}\ln 2 =$$

$$= 2\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right)\left(3u^2\sqrt{u-v} + \frac{u^3}{2\sqrt{u-v}} + 2^{4u+6v+1}\ln 2\right).$$

$$\frac{\partial z}{\partial v} = 2\cos\left(2x + y^2\right) \cdot \left(-\frac{u^3}{2\sqrt{u-v}}\right) + 2y\cos\left(2x + y^2\right) \cdot 2^{2u+3v} \cdot 3\ln 2 =$$

$$= -\frac{u^3\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right)}{\sqrt{u-v}} +$$

$$+ 2 \cdot 2^{2u+3v}\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right)2^{2u+3v} \cdot 3\ln 2 =$$

$$= -\cos\left(2u^3\sqrt{u-v} + 2^{4u+6v}\right)\left(\frac{u^3}{\sqrt{u-v}} - 2^{4u+6v+1}3\ln 2\right).$$

b) By (6.3) we get:

$$\frac{\partial z}{\partial x} = \frac{y}{1+x^2y^2}\ ;\ \frac{\partial z}{\partial y} = \frac{x}{1+x^2y^2}\ ;\ \frac{dx}{dt} = 4\cos 2t\ ;\ \frac{dy}{dt} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}\ .$$

$$\frac{dz}{dt} = \frac{y}{1+x^2y^2}4\cos 2t + \frac{x}{1+x^2y^2}\frac{e^{\sqrt{t}}}{2\sqrt{t}} = \frac{1}{1+x^2y^2}\left(4y\cos 2t + x\frac{e^{\sqrt{t}}}{2\sqrt{t}}\right) =$$

$$= \frac{1}{1+(2\sin 2t)^2\left(e^{\sqrt{t}}\right)^2}\left(4e^{\sqrt{t}}\cos 2t + 2\sin 2t\frac{e^{\sqrt{t}}}{2\sqrt{t}}\right) =$$

$$= \frac{e^{\sqrt{t}}}{1+4e^{2\sqrt{t}}\sin^2 2t}\left(4\cos 2t + \sin 2t\frac{1}{\sqrt{t}}\right).$$

c) By (6.4) we obtain:

$$\frac{\partial z}{\partial x}=\frac{-e^x}{y-e^x};\ \frac{\partial z}{\partial y}=\frac{1}{y-e^x};\ \frac{dy}{dx}=\frac{2x}{\cos^2\left(5+x^2\right)}.$$

$$\frac{dz}{dx}=\frac{-e^x}{y-e^x}+\frac{1}{y-e^x}\frac{2x}{\cos^2\left(5+x^2\right)}=\frac{1}{y-e^x}\left(-e^x+\frac{2x}{\cos^2\left(5+x^2\right)}\right)=$$

$$=\frac{1}{\tan\left(5+x^2\right)-e^x}\left(-e^x+\frac{2x}{\cos^2\left(5+x^2\right)}\right).$$

Answer: **a)** $\frac{\partial z}{\partial u}=2\cos\left(2u^3\sqrt{u-v}+2^{4u+6v}\right)\left(3u^2\sqrt{u-v}+\frac{u^3}{2\sqrt{u-v}}+\right.$

$\left.+2^{4u+6v+1}\ln 2\right)$; $\frac{\partial z}{\partial v}=-\cos\left(2u^3\sqrt{u-v}+2^{4u+6v}\right)\left(\frac{u^3}{\sqrt{u-v}}-2^{4u+6v+1}3\ln 2\right)$;

b) $\frac{dz}{dt}=\frac{e^{\sqrt{t}}}{1+4e^{2\sqrt{t}}\sin^2 2t}\left(4\cos 2t+\sin 2t\frac{1}{\sqrt{t}}\right)$;

c) $\frac{dz}{dx}=\frac{1}{\tan\left(5+x^2\right)-e^x}\left(-e^x+\frac{2x}{\cos^2\left(5+x^2\right)}\right)$.

Extremum of a function of two independent variables

Necessary condition. If $z=f(x,y)$ is differentiable and has an ***extremum*** at point $M(x_0,y_0)$, then:

$$\left.\frac{\partial z}{\partial x}\right|_M=0;\ \left.\frac{\partial z}{\partial y}\right|_M=0. \tag{6.5}$$

The ***stationary points*** of the function are the points at which all its partial derivatives vanish.

Sufficient condition. If $M(x_0,y_0)$ is a stationary point of the function $z=f(x,y)$, and $A=\frac{\partial^2 z}{\partial x^2}$, $B=\frac{\partial^2 z}{\partial x\partial y}$, $C=\frac{\partial^2 z}{\partial y^2}$, $D=B^2-AC$, then:

1) If $D(M)<0$ and $A(M)<0$, then $z=f(x,y)$ has ***maximum*** at $M(x_0,y_0)$;

2) If $D(M)<0$ and $A(M)>0$, then $z=f(x, y)$ has ***minimum*** at $M(x_0, y_0)$;
3) If $D(M)>0$, then $z=f(x, y)$ has no extremum at $M(x_0, y_0)$;
4) If $D(M)=0$, then the nature of the stationary point is undetermined and this case requires further investigation.

Problem 6.4. Examine function $z=x^3+3xy^2-15x-12y$ for maximum and minimum values.

Solution. We can obtain stationary points by using equations (6.5):

$$\frac{\partial z}{\partial x}=3x^2+3y^2-15\,;\ \frac{\partial z}{\partial y}=6xy-12\,.$$

$$\begin{cases}\dfrac{\partial z}{\partial x}=0,\\ \dfrac{\partial z}{\partial y}=0,\end{cases}\Rightarrow\begin{cases}3x^2+3y^2-15=0,\\ 6xy-12=0,\end{cases}\Rightarrow\begin{cases}x^2+y^2-5=0,\\ xy=2,\end{cases}\Rightarrow\begin{cases}x^2+\left(\dfrac{2}{x}\right)^2-5=0,\\ y=\dfrac{2}{x}.\end{cases}$$

$$x^2+\frac{4}{x^2}-5=0\Rightarrow\frac{x^4-5x^2+4}{x^2}=0\Leftrightarrow\begin{cases}x^4-5x^2+4=0,\\ x\neq 0.\end{cases}$$

$$x^4-5x^2+4=0\Rightarrow\left\{t=x^2\right\}\Rightarrow t^2-5t+4=0\Rightarrow\left[\begin{array}{l}t_1=1,\\ t_1=4,\end{array}\right.\Rightarrow\left[\begin{array}{l}x_1=1,\\ x_2=-1,\\ x_3=2,\\ x_4=-2.\end{array}\right.$$

As the result, we've obtained $\left[\begin{array}{l}x_1=1,\\ x_2=-1,\\ x_3=2,\\ x_4=-2,\end{array}\right.$ and $y=\dfrac{2}{x}$, thus we get four stationary points: $M_1(1, 2)$, $M_2(-1, -2)$, $M_3(2, 1)$, $M_4(-2, -1)$.

Then we find $A=\dfrac{\partial^2 z}{\partial x^2}=6x$, $B=\dfrac{\partial^2 z}{\partial x\partial y}=6y$, $C=\dfrac{\partial^2 z}{\partial y^2}=6x$, $D=B^2-AC=36y^2-36x^2=36\left(y^2-x^2\right)$.

$D(M_1) = 36(2^2 - 1^2) = 108 > 0 \Rightarrow$ the given function doesn't have an extremum at the point M_1.

$D(M_2) = 36((-2)^2 - (-1)^2) = 108 > 0 \Rightarrow$ the given function doesn't have an extremum at the point M_2.

$D(M_3) = 36(1^2 - 2^2) = -108 < 0 \Rightarrow$ the given function has an extremum at the point M_3, $A(M_3) = 6 \cdot 2 = 12 > 0 \Rightarrow$ the given function has a minimum at this point: $z_{\min} = 2^3 + 3 \cdot 2 \cdot 1^2 - 15 \cdot 2 - 12 \cdot 1 = -28$.

$D(M_4) = 36((-1)^2 - (-2)^2) = -108 < 0 \Rightarrow$ the given function has an extremum at the point M_4, $A(M_4) = 6 \cdot (-2) = -12 < 0 \Rightarrow$ the given function has a maximum at this point: $z_{\max} = (-2)^3 + 3 \cdot (-2) \cdot (-1)^2 - 15 \cdot (-2) - 12 \cdot (-1) = 28$.

Answer: at both points $M_1(1, 2)$ and $M_2(-1, -2)$ the given function doesn't have an extremum;
at the point $M_3(2, 1)$ the given function has a minimum $z_{\min} = -28$;
at the point $M_4(-2, -1)$ the given function has a maximum $z_{\max} = 28$.

Conditional extremum of a function of two independent variables

The term ***conditional extremum*** is used when we need to find an extremum of a function $z = f(x, y)$ under condition that variables x and y satisfy the equation:

$$\phi(x, y) = 0$$

which is called the ***equation of constraint***.

In the current case we should consider so-called ***Lagrange's function***:

$$L(x, y, \lambda) = f(x, y) + \lambda\phi(x, y). \qquad (6.5)$$

Then the solution of the initial problem is reduced to finding of an extremum of a function $L(x, y, \lambda)$ of three independent variables. In the present case the necessary condition is:

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0, \\ \dfrac{\partial L}{\partial y} = 0, \\ \dfrac{\partial L}{\partial \lambda} = 0. \end{cases} \tag{6.6}$$

We should notice that not every solution of the system (6.6) yields a point of extremum and additional investigation to find the nature of the point is required. However, in solving practical problems, the nature of a stationary point can be often determined from the entity of the problem.

Remark. If the variables x and y satisfy to more than one equation of constraint:

$$\phi_i(x, y) = 0\,,\ i = 1,\dots,n\,,$$

then Lagrange's function can be considered in the form:

$$L(x, y, \lambda_1,\dots,\lambda_n) = f(x, y) + \lambda_1\phi_1(x, y) + \dots + \lambda_n\phi_n(x, y)$$

and system (6.6) can be written in the form:

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0, \\ \dfrac{\partial L}{\partial y} = 0, \\ \dfrac{\partial L}{\partial \lambda_1} = 0, \\ \dots \\ \dfrac{\partial L}{\partial \lambda_n} = 0. \end{cases}$$

Problem 6.5. Given a region bounded by the x- and y-axes and the parabola $y + x^2 - 3 = 0$ $\left(0 \le x \le \sqrt{3}\right)$. Find the area S of the largest rectangle with sides along the coordinate axes which can be inscribe in this region (Fig. 6.2).

Solution. Since the length and width of the rectangle are x and y respectively, then the area of the rectangle is $S = xy$. Taking into account, that the region, in which the rectangle is inscribed, is bounded in particular by the parabola, x and y have to satisfy to its equation:

$$y + x^2 - 3 = 0 .$$

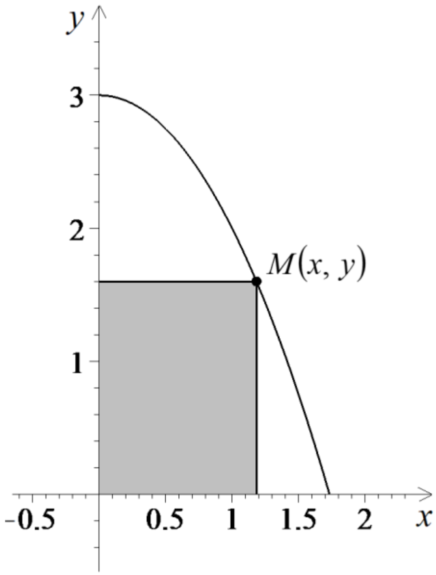


Figure 6.2

It means we have to find the maximum of the function $S = xy$ under the condition that variables x and y satisfy the equation of constraint:

$$y + x^2 - 3 = 0 .$$

Let's consider the corresponding Lagrange function (6.5):

$$L(x,\ y,\ \lambda) = xy + \lambda\left(y + x^2 - 3\right)$$

and system (6.6):

$$\begin{cases} \dfrac{\partial L}{\partial x} = y + 2x\lambda = 0, \\ \dfrac{\partial L}{\partial y} = x + \lambda = 0, \\ \dfrac{\partial L}{\partial \lambda} = y + x^2 - 3 = 0. \end{cases}$$

From the first and second equations of the last system we can obtain x and y in terms of λ and plug in the results in the third one:

$$x + \lambda = 0 \Rightarrow x = -\lambda\ ;\ \ y + 2x\lambda = 0 \Rightarrow y = -2x\lambda = 2\lambda^2\ ;$$

$$y + x^2 - 3 = 0 \Rightarrow 2\lambda^2 + \lambda^2 - 3 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda_{1,2} = \pm 1 .$$

Since, for the considered region $x \ge 0$, then $\lambda_1 = 1$ is not suitable, because the corresponding value of $x = -\lambda_1 = -1$ does not belong to the region. Therefore, we obtain the solution:

$$\lambda_2 = -1 \Rightarrow x = 1;\ y = 2 \Rightarrow S_{\max} = 2 .$$

Answer: $S_{\max} = 2$.

Extreme values of a function in a closed domain

***Theorem* 6.4** If a function $z = f(x, y)$ is continuous in a closed domain D, then:

1) the function is bounded in the domain D;

2) the function takes its least and greatest values in the domain D.

Let a function $z = f(x, y)$ be differentiable in the closed domain. Then the greatest and least values of the function in the closed domain can be found by using the following approach:

1) to find stationary points of the function which belong to the domain;

2) to find stationary points on the boundaries:

– boundary lines can be considered as the equation of constraint and then Lagrange's function can be used to find stationary points on each boundary line (the first method);

– to express one variable in terms of another one from the equation of the boundary line (if it's possible). Then, the obtained expression has to be plugged in the equation of the function. After we get a function of a single variable from which stationary points can be found (the second method);

the first or second method should be used for each boundary line separately;

3) to calculate values of the function at the obtained points and points of intersection between the boundary lines:

4) to select the greatest and least values among the obtained ones.

Problem 6.6. Find the greatest and least values of the function $z = \frac{x^2}{2} - xy + y$ in the domain D bounded by the parabola $y = \frac{x^2}{3}$ and the line $y = 3$ (Fig. 6.3).

Solution. Let's use the described approach.

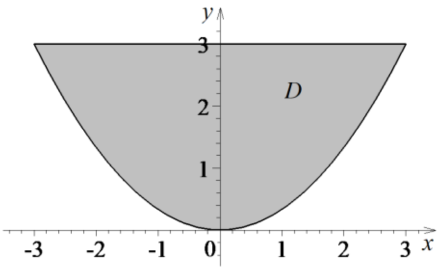


Figure 6.3

1) Let's find stationary points of the function in the domain D:

$$\begin{cases} z'_x = x - y = 0, \\ z'_y = -x + 1 = 0; \end{cases} \Rightarrow \begin{cases} y = 1, \\ x = 1. \end{cases}$$

The obtained point $M_1(1, 1)$ definitely belongs to the considered domain.

2) Let's find all the stationary points of the function on the boundary lines using both methods described in issue 2.

The first method. Let's start with the parabola $y = \frac{x^2}{3} \Rightarrow y - \frac{x^2}{3} = 0$, $x \in [-3, 3]$. Then the corresponding Lagrange function is

$$L(x, y, \lambda) = \frac{x^2}{2} - xy + y + \lambda\left(y - \frac{x^2}{3}\right).$$

System (6.6) has the following form:

$$\begin{cases} \frac{\partial L}{\partial x} = x - y - \frac{2}{3}\lambda x = 0, \\ \frac{\partial L}{\partial y} = -x + 1 + \lambda = 0, \\ \frac{\partial L}{\partial \lambda} = y - \frac{x^2}{3} = 0; \end{cases} \Rightarrow \begin{cases} y = x - \frac{2}{3}\lambda x = 1 + \frac{1}{3}\lambda - \frac{2}{3}\lambda^2, \\ x = 1 + \lambda, \\ 3\lambda^2 + \lambda - 2 = 0. \end{cases}$$

From the last equation $\lambda_1 = -1$, $\lambda_2 = \frac{2}{3}$. Therefore, we obtain two additional points:

$$\left(1+\lambda_1,\ 1+\frac{1}{3}\lambda_1-\frac{2}{3}\lambda_1^2\right)=M_2(0,\ 0);$$

$$\left(1+\lambda_2,\ 1+\frac{1}{3}\lambda_2-\frac{2}{3}\lambda_2^2\right)=M_3\left(\frac{5}{3},\ \frac{25}{27}\right).$$

Now let's do the same, but using the second method. From the equation of the parabola we have expression for y in terms of x:

$$y=\frac{x^2}{3}.$$

After substitution y with its expression in terms of x in the equation of the function, we get:

$$z=\frac{x^2}{2}-x\frac{x^2}{3}+\frac{x^2}{3}=-\frac{x^3}{3}+\frac{5x^2}{6},\ x\in[-3,\ 3].$$

Therefore, we obtained the function of one independent variable, which stationary points are:

$$z'=-x^2+\frac{5x}{3}=0\Leftrightarrow x_1=0,\ x_2=\frac{5}{3}.$$

$$\left(x_1,\ \frac{x_1^2}{3}\right)=M_2(0,\ 0);\ \left(x_2,\ \frac{x_2^2}{3}\right)=M_3\left(\frac{5}{3},\ \frac{25}{27}\right).$$

We obtained the same pair of points.

The second boundary line has the equation $y=3$, so we can substitute y with 3 in the equation of the function and reduce to a function of one variable:

$$z=\frac{x^2}{2}-3x+3.$$

Let's find the stationary point of the obtained function is:

$$z'=x-3=0\Rightarrow x=3\Rightarrow M_4(3,\ 3).$$

3) Let's calculate values of the function at the obtained points:

$$z_1=z(M_1)=\frac{1}{2};\ z_2=z(M_2)=0;\ z_3=z(M_3)=\frac{125}{162};\ z_4=z(M_4)=-\frac{3}{2}.$$

Points of intersection between the boundary lines are:

$$\begin{cases} y=\frac{x^2}{3}, \\ y=3, \end{cases}\Rightarrow\begin{cases} x^2=9, \\ y=3, \end{cases}\Rightarrow\begin{cases} x=\pm 3, \\ y=3. \end{cases}\Rightarrow M_4(3,\ 3);\ M_5(-3,\ 3).$$

We obtained only one new point and the value of the function at this point is:

$$z_5 = z(M_5) = \frac{33}{2}.$$

4) The greatest and least values of the function in the domain D are:

$$z_{max} = z_5 = \frac{33}{2}; \; z_{min} = z_4 = -\frac{3}{2}.$$

Answer: $z_{max} = \frac{33}{2}$ at the point $M_5(-3, 3)$; $z_{min} = -\frac{3}{2}$ at the point $M_4(3, 3)$.

Exercises

6.1. Find and sketch the domain of definition of the function of two variables:

$$z = \frac{\arcsin(y - 2x - 1)}{\sqrt{4 - x^2 - y^2}}.$$

Answer: $D(z): \begin{cases} x^2 + y^2 < 4, \\ y \geq 2x, \\ y \leq 2x + 2. \end{cases}$ Fig. 6.4 shows the domain of definition $D(z)$ of the given function.

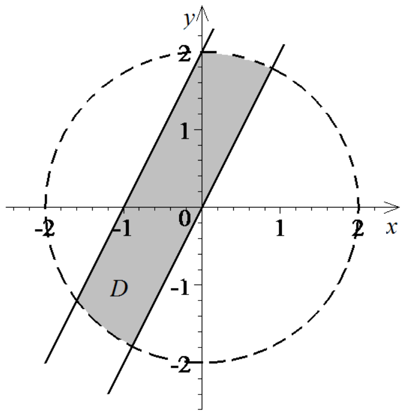


Figure 6.4

6.2. The function $z = e^{-(x^2+y^2)}\left(3x^2 + \frac{y^2}{3}\right)$ is given. Find dz.

Answer: $dz = e^{-(x^2+y^2)}\left(6x - 6x^3 - \frac{y^3}{3}\right)dx +$

$+ e^{-(x^2+y^2)}\left(-6x^2 y + \frac{2y - 2y^3}{3}\right)dy$.

6.3. Find the derivatives of the composite functions:
a) $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$; b) $\frac{dz}{dt}$; c) $\frac{dz}{dx}$.

a) $z = x^3 y^3 + \frac{x^2}{y^2}$, $x = \frac{u}{v}$, $y = u^2 + v^2$;

b) $z = e^{x^2 - 2y}$, $x = \sin 3t$, $y = 2t^3$;

c) $z = \ln\left(\frac{x}{\sqrt{y}}\right)$, $y = x^3 + 2$.

Answer: **a)** $\frac{\partial z}{\partial u} = \frac{3u^2}{v^3}\left(u^2+v^2\right)^2\left(3u^2+v^2\right) + \frac{2u\left(v^2-u^2\right)}{v^2\left(u^2+v^2\right)^3}$;

$\frac{\partial z}{\partial v} = \frac{3u^2}{v^4}\left(u^2+v^2\right)^2\left(v^2-u^2\right) - \frac{2u^2\left(u^2+3v^2\right)}{v^3\left(u^2+v^2\right)^3}$;

b) $\frac{dz}{dt} = 3e^{\sin^2 3t - 4t^3}\left(\sin 6t - 4t^2\right)$; **c)** $\frac{dz}{dx} = \frac{4 - x^3}{2x\left(x^3+2\right)}$.

6.4. Examine the function $z = x^2 + y^3 - 6xy + 18x - 39y + 1$ for maximum and minimum values.

Answer: at the point $M_1(-6, 1)$ the given function doesn't have an extremum; at the point $M_2(5, 6)$ the given function has a minimum $z_{\min} = -105$.

6.5. Examine the function $z = x + 2y$ for maximum and minimum values under the condition $x^2 + y^2 = 5$.

Answer: $z_{\max} = 5$ at the point $M_1(1, 2)$: $z_{\min} = -5$ at the point $M_2(-1, -2)$.

6.6. Find the greatest and least values of the function $z = x^2 + 2y^2 - 2x - 4y + 2$ in the domain bounded by the lines $x = 0$, $y = 0$ and $3x + y = 4$.

Answer: $z_{\max} = 18$ at the point $M_1(0, 4)$; $z_{\min} = -1$ at the point $M_2(1, 1)$.

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Appendix A
Table of Derivatives

($u = u(x)$ is a differentiable function)

1. $(C)' = 0$, $C - const$;	11. $(\tan u)' = \frac{1}{\cos^2 u} \cdot u'$;
2. $(x)' = 1$;	12. $(\cot u)' = -\frac{1}{\sin^2 u} \cdot u'$;
3. $(u^n)' = n \cdot u^{n-1} \cdot u'$;	13. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$;
4. $(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u'$;	14. $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$;
5. $(a^u)' = a^u \cdot \ln a \cdot u', a - const$;	15. $(\arctan u)' = \frac{1}{1+u^2} \cdot u'$;
6. $(e^u)' = e^u \cdot u'$;	16. $(\cot^{-1} u)' = -\frac{1}{1+u^2} \cdot u'$;
7. $(\log_a u)' = \frac{1}{u \cdot \ln a} \cdot u'$;	17. $(\sinh u)' = \cosh u \cdot u'$;
8. $(\ln u)' = \frac{1}{u} \cdot u'$;	18. $(\cosh u)' = \sinh u \cdot u'$;
9. $(\sin u)' = \cos u \cdot u'$;	19. $(\tanh u)' = \frac{1}{\cosh^2 u} \cdot u'$;
10. $(\cos u)' = -\sin u \cdot u'$;	20. $(\coth u)' = -\frac{1}{\sinh^2 u} \cdot u'$.

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Підписано до друку 07.11.2025 Формат 60×84/16. Ум. друк. арк. 7,85.
Тираж 100 прим. Зам. № 988.

Національний університет «Запорізька політехніка»
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Тел.: (061) 769–82–96, 220–12–14

Свідоцтво суб’єкта видавничої справи ДК № 6952 від 22.10.2019.