

**MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
National University Zaporizhzhia Polytechnic**

**LECTURES ON COMPLEX ANALYSIS AND OPERATIONAL
CALCULUS**

for students majoring in
141 Power Engineering, Electrical Engineering
and Electrical Mechanics

2024

Lectures on Complex Analysis and Operational Calculus for students majoring in 141 Power Engineering, Electrical Engineering and Electrical Mechanics / Compilers: Nina Antonenko. – Zaporizhzhia: National University Zaporizhzhia Polytechnic, 2024. – 76 p.

Compiler: Antonenko Nina,
candidate of science (physics and
mathematics),
associate professor

Executive editor: Antonenko Nina,
candidate of science (physics and
mathematics),
associate professor

Reviewer: Fasoliak Anton,
candidate of science (physics and
mathematics),
associate professor

Approved
by the Department of Mathematics
(Minutes № 7 of 18.04.2024)

Recommended
by the Scientific and Methodological
Council of the Faculty of Mechanical
Engineering
(Minutes № 8 of 22.04.2024)

CONTENTS

1 Complex Analysis	4
1.1 Complex Numbers. Operations with Complex Numbers	4
1.2 Sets, Domains, Curves	7
1.3 Functions of a Complex Variable	11
1.4 Differentiation of Functions of a Complex Variable	14
1.5 Integral with Respect to a Complex Variable	18
1.6 Number and Power Series with Complex Terms	26
1.6.1 Number Series with Complex Terms	26
1.6.2 Power Series with Complex Terms	27
1.6.3 Taylor Series	28
1.6.4 Laurent Series	30
1.7 Zeros and Singular Points	33
1.7.1 Zeros	33
1.7.2 Singular Points	35
1.8 Residues of Functions and Their Application	38
1.8.1 Definition of Residue and Its Evaluation	38
1.8.2 Application of Residues	43
2 Operational Calculus	47
2.1 Laplace Transform	47
2.2 Inverse Laplace Transform	55
2.3 Application of Operational Calculus to Solving Linear Differential Equations with Constant Coefficients	60
2.4 Application of Operational Calculus to Solving Systems of Differential Equations with Constant Coefficients	66
References	71
Appendix A. Table of Laplace Transforms	73
Appendix B. Table of Derivatives	74
Appendix C. Table of Basic Integrals	75

1 COMPLEX ANALYSIS

1.1 Complex Numbers. Operations with Complex Numbers

A complex number z is an expression

$$z = x + iy,$$

where x and y are real numbers, i is called the *imaginary unit*, which is defined as $i = \sqrt{-1}$ ($i^2 = -1$).

x is called the *real part* of z and y is called the *imaginary part* of z . The following designations are used: $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

The complex number $\bar{z} = x - iy$ is termed *complex conjugate* of the complex number $z = x + iy$.

Any complex number can be represented as a point $M(x, y)$ or a vector $\overline{OM}$ in the xy -plane (Fig. 1.1). The xy -plane which is used to represent complex numbers is called the *complex plane*, the x -axis is called the *real axis*, and the y -axis is called the *imaginary axis*.

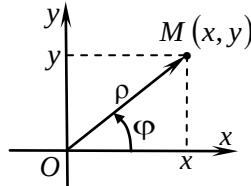


Figure 1.1

The length of the vector $\overline{OM}$ is called the *modulus* or *absolute value* of z and is denoted by $|z|$ or ρ . The angle that the vector $\overline{OM}$ makes with the positive direction of the x -axis is termed the *argument* of z and is denoted by $\operatorname{Arg} z$. For $z = 0$, φ is undefined. If $z \neq 0$, $\operatorname{Arg} z$ is determined up to multiples of 2π . The value of $\operatorname{Arg} z \in (-\pi, \pi]$ is called the *principal value* of the argument. It is denoted by $\arg z$ or φ . From Fig. 1.1, we obtain the formulas:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi,$$

$$\rho = \sqrt{x^2 + y^2}, \quad \tan \varphi = y/x. \quad (1.1)$$

To find $\arg z$, the following formulas are used:

$$\arg z = \begin{cases} \arctan \frac{y}{x}, & x > 0, \\ \pi + \arctan \frac{y}{x}, & x < 0, y \geq 0, \\ -\pi + \arctan \frac{y}{x}, & x < 0, y < 0, \\ \frac{\pi}{2}, & x = 0, y > 0 \quad \text{and} \quad -\frac{\pi}{2}, & x = 0, y < 0. \end{cases} \quad (1.2)$$

Trigonometric or polar form of complex number is

$$z = \rho(\cos \varphi + i \sin \varphi). \quad (1.3)$$

Using Euler's formula,

$$e^{iu} = \cos u + i \sin u, \quad (1.4)$$

where u is a real number, (1.3) can be represented in the following form

$$z = \rho e^{i\varphi}. \quad (1.5)$$

Expression (1.5) is called the *exponential form* of complex number.

Operations with complex numbers.

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are given, then:

- 1) $z_1 = z_2 \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2$;
- 2) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$;
- 3) $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$;
- 4) $\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$.

The *product* and the *quotient* of two numbers $z_1 = \rho_1 e^{i\varphi_1}$ and $z_2 = \rho_2 e^{i\varphi_2}$ given in exponential form are introduced as follows:

$$z_1 z_2 = \rho_1 \rho_2 e^{i(\varphi_1 + \varphi_2)} = \rho_1 \rho_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)],$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1 - \varphi_2)} = \frac{\rho_1}{\rho_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)].$$

The n -th power of z can be found by the formula

$$z^n = \rho^n e^{in\varphi} = \rho^n [\cos n\varphi + i \sin n\varphi], \quad n \in \mathbb{N}. \quad (1.6)$$

The last formula is called *formula of De Moivre*.

Given $z = \rho(\cos \varphi + i \sin \varphi)$. For $z \neq 0$, the n -th root ($n \in \mathbb{N}$) of z has n distinct values which can be found by the formula

$$\sqrt[n]{z} = \sqrt[n]{\rho} \left(\cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right), \quad k = 0, 1, \dots, n-1. \quad (1.7)$$

Example 1.1. Two complex numbers $z_1 = 1 + 2i$ and $z_2 = 1 - i$ are given. Find $z_1 + z_2$, $z_1 - z_2$, $z_1 \cdot z_2$, $\frac{z_1}{z_2}$.

Solution

$$z_1 + z_2 = (1 + 2i) + (1 - i) = (1 + 1) + i(2 - 1) = 2 + i;$$

$$z_1 - z_2 = (1 + 2i) - (1 - i) = (1 - 1) + i(2 - (-1)) = 3i;$$

$$z_1 \cdot z_2 = (1 + 2i)(1 - i) = 1 - i + 2i - 2i^2 = 3 + i;$$

$$\frac{z_1}{z_2} = \frac{1 + 2i}{1 - i} = \frac{1 + 2i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{1 + i + 2i + 2i^2}{1 - i^2} = \frac{-1 + 3i}{2} = -\frac{1}{2} + \frac{3}{2}i. \bullet$$

Example 1.2. The numbers $1, -1, i, -2i$ and $-1 + i$ are given. Find their moduli and principal values of their arguments.

Solution

Using the formulas (1.1) and (1.2), we get:

$$1) \text{ for } z = 1, \quad x = 1, \quad y = 0, \quad \rho = \sqrt{1^2 + 0^2} = 1, \quad \varphi = \arctan \frac{0}{1} = 0;$$

$$2) \text{ for } z = -1, \quad x = -1, \quad y = 0, \quad \rho = \sqrt{(-1)^2 + 0^2} = 1, \quad \varphi = \arctan \frac{0}{-1} = 0;$$

$$3) \text{ for } z = i, \quad x = 0, \quad y = 1, \quad \rho = \sqrt{0^2 + 1^2} = 1, \quad \varphi = \pi/2;$$

$$4) \text{ for } z = -2i, \quad x = 0, \quad y = -2, \quad \rho = \sqrt{0^2 + (-2)^2} = 2, \quad \varphi = -\pi/2;$$

$$5) \text{ for } \quad z = -1 + i, \quad x = -1, \quad y = 1, \quad \rho = \sqrt{(-1)^2 + 1^2} = \sqrt{2},$$

$$\varphi = \pi + \arctan \left(\frac{1}{-1} \right) = \pi + \arctan(-1) = \pi - \arctan(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}. \bullet$$

Example 1.3. Write $z = -3 - i\sqrt{3}$ in the trigonometric and the exponential form.

Solution

For $z = -3 - i\sqrt{3}$ we have $x = -3$, $y = -\sqrt{3}$. By formulas (1.1) and (1.2), we obtain:

$$\rho = \sqrt{(-3)^2 + (-\sqrt{3})^2} = 2\sqrt{3}, \quad \varphi = -\pi + \arctan \frac{-\sqrt{3}}{-3} = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}.$$

Hence, by (1.3) and (1.5), $z = 2\sqrt{3} \left(\cos\left(-\frac{5\pi}{6}\right) + i \sin\left(-\frac{5\pi}{6}\right) \right)$ (the

trigonometric form), $z = 2\sqrt{3}e^{-\frac{5\pi}{6}i}$ (the exponential form). •

Example 1.4. Find $(-1+i)^4$.

Solution

Let $z = -1 + i$. From Example 1.2 $\rho = \sqrt{2}$ and $\varphi = 3\pi/4$. Hence, by (1.6), $z^4 = (-1+i)^4 = \rho^4 (\cos 3\varphi + i \sin 3\varphi) = 4(\cos 3\pi + i \sin 3\pi) = -4$. •

Example 1.5. Find all values of $\sqrt[3]{-8}$.

Solution

We have $z = -8 \Rightarrow \rho = 8, \varphi = \pi$. Hence, by (1.7),

$$\sqrt[3]{z} = \sqrt[3]{8} \left(\cos \frac{\pi + 2k\pi}{3} + i \sin \frac{\pi + 2k\pi}{3} \right), \quad k = 0, 1, 2.$$

Let's find all values of $\sqrt[3]{-8}$:

$$k = 0: w_0 = 2(\cos \pi/3 + i \sin \pi/3) = 1 + i\sqrt{3},$$

$$k = 1: w_1 = 2(\cos \pi + i \sin \pi) = -2,$$

$$k = 2: w_2 = 2(\cos 5\pi/3 + i \sin 5\pi/3) = 1 - i\sqrt{3}. \bullet$$

1.2 Sets, Domains, Curves

ε -neighbourhood of z_0 is interior of a circle of radius ε with center at the point z_0 .

Let D be a set of points in the complex plane. The point $z_0 \in D$ is called an *interior point* of D if there exists an ε -neighbourhood of z_0 which is contained in D .

The set D is called *open* if all its points are interior points of D . A point $z_1 \in L$ is called a *boundary point* of the set D if every neighbourhood of this point contains points belonging and not belonging to D . The set of all boundary points of the set D is called the *boundary* of D .

An open set D is called *connected* if every two points can be connected with a polygonal line having a finite number of links entirely lying in D . An open connected set is called a *domain*.

The domain D together with its boundary L is called the *closed domain* and denoted by $\bar{D} = D \cup L$.

A closed curve C without points of self-intersection is called a *closed contour* or *contour*.

The *orientation of the closed contour* C is said to be *positive* if the direction in which it is run is such that the domain D remains on the left. The opposite direction is called *negative*. C or $C^+ (C^-)$ is denoted positively (negatively) oriented contours.

A domain D is called *simply connected* if every closed contour C in D can be continuously shrunk to any point in D without leaving D (Fig. 1.2). Otherwise the domain is termed *multiply connected* (Fig. 1.3).

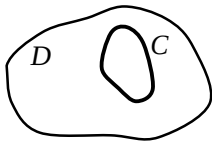


Figure 1.2

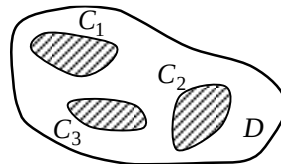


Figure 1.3

Let $x = x(t)$ and $y = y(t)$ be real continuous functions of variable t , $\alpha \leq t \leq \beta$. Then the equation

$$z = z(t) = x(t) + iy(t), \quad \alpha \leq t \leq \beta,$$

gives a *parametric representation* of a continuous curve l on the complex plane.

A plane curve l is called *smooth* if the functions $x = x(t)$ and $y = y(t)$, defining this curve, and their derivatives are continuous and $[x'(t)]^2 + [y'(t)]^2 \neq 0$ at all points of this curve.

A continuous curve composed of a finite number of smooth pieces is called a *piecewise smooth curve*.

Sometimes the equation $F(z) = 0$ is defined the curve at the complex plane and the inequality $F(z) \leq 0$ ($F(z) < 0$) or $F(z) \geq 0$ ($F(z) > 0$) is defined the set of the points at the complex plane which satisfies this inequality. Substituting expressions

$$z = x + iy, \operatorname{Re} z = x, \operatorname{Im} z = y, z^2 = x^2 - y^2 + 2xyi,$$

$$z \cdot \bar{z} = x^2 + y^2, |z| = \sqrt{x^2 + y^2}, |z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}. \quad (1.8)$$

into the equation $F(z) = 0$, we get the equation $F(x, y) = 0$, which represents the curve on the xy -plane. Substituting expressions (1.8) into inequality $F(z) \leq 0$ ($F(z) < 0$) or $F(z) \geq 0$ ($F(z) > 0$), we get some region on the xy -plane.

Example 1.6. Sketch the following sets in the complex plane:

- a)** $\{z \in C : |z - 3 + 2i| = 2\}$; **b)** $\{z \in C : |z - 3 + 2i| < 2\}$;
c) $\left\{z \in C : \operatorname{Re}\left(\frac{1}{\bar{z}}\right) \leq \frac{1}{4}\right\}$; **d)** $\{z \in C : \operatorname{Im}(z^2 + \bar{z}) = 2 + \operatorname{Im}(z)\}$;
e) $\left\{z \in C : 1 \leq |z + 2 - 2i| < 3, \frac{\pi}{2} \leq \arg z < \frac{3\pi}{4}\right\}.$

Solution

a) The set $\{z \in C : |z - 3 + 2i| = 2\}$ represents the circle of radius 2 centered at the point $z_0 = 3 - 2i$ (Fig. 1.4).

b) The set $\{z \in C : |z - 3 + 2i| < 2\}$ represents the disk of radius 2 centered at the point $z_0 = 3 - 2i$ without its boundary (Fig. 1.5).

c) Using expressions $z = x + iy$, $\bar{z} = x - iy$, we have

$$\frac{1}{\bar{z}} = \frac{1}{x - iy} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{y}{x^2 + y^2},$$

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2}, \frac{x}{x^2 + y^2} \leq \frac{1}{4}, x^2 + y^2 \geq 4x, (x^2 - 4x + 4) + y^2 \geq 4,$$

$(x-2)^2 + y^2 \geq 4$. The set $\left\{z \in \mathbb{C} : \operatorname{Re}\left(\frac{1}{z}\right) \leq \frac{1}{4}\right\}$ is shown on Fig. 1.6.

d) Using expressions $z = x + iy$, $\bar{z} = x - iy$, we have $z^2 + \bar{z} = x^2 + x - y^2 + i(2xy - y)$, $\overline{z^2 + \bar{z}} = x^2 + x - y^2 - i(2xy - y)$, $\operatorname{Im}(\overline{z^2 + \bar{z}}) = y - 2xy$, $\operatorname{Im}(z) = y$, $y - 2xy = 2 + y$, $xy = -1$. The set $\left\{z \in \mathbb{C} : \operatorname{Im}(\overline{z^2 + \bar{z}}) = 2 + \operatorname{Im}(z)\right\}$ represents hyperbola (Fig. 1.7).

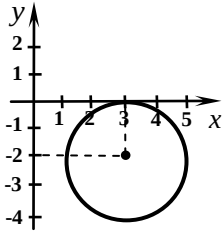


Figure 1.4

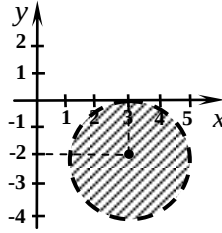


Figure 1.5

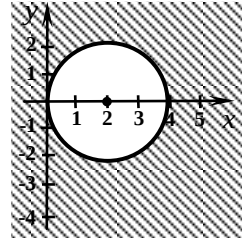


Figure 1.6

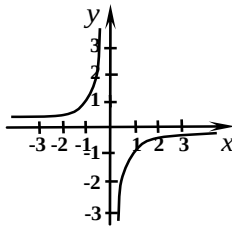


Figure 1.7

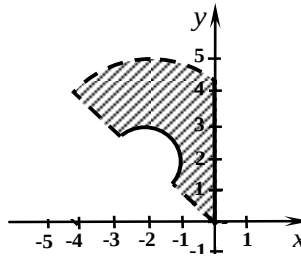


Figure 1.8

e) The set $\{z \in \mathbb{C} : 1 \leq |z + 2 - 2i| < 3\}$ represents an annulus with centre at the point $-2 + 2i$, its inner and outer radii equals 1 and 3,

respectively. The set $\left\{z \in \mathbb{C} : \frac{\pi}{2} \leq \arg z < \frac{3\pi}{4}\right\}$ represents the angle. It is sketched on fig. 1.8. •

1.3 Functions of a Complex Variable

The quantity w is called a *function of a complex variable* z in a domain D if the law is given according to which each value of z ($z \in D$) is assigned one or several values of w .

This law is written in the form $w = f(z)$. The variable z is called the *independent variable* or *argument*, and w is called the *dependent variable* or *value of the function*.

For each value of $z \in D$ w is a complex number, therefore it can be written in the following form

$$w = f(z) = u(x, y) + iv(x, y),$$

where $\operatorname{Re} f(z) = u(x, y)$, $\operatorname{Im} f(z) = v(x, y)$.

If for each value of $z \in D$ corresponds only one (several) value (values) of w , then the function $w = f(z)$ is called *one-valued* or *single-valued* (*many-valued*).

The domain D is called the *domain of definition*.

The set of values of $f(z)$ corresponding to all values of $z \in D$ is called the *domain of values of the function* $f(z)$ and is denoted by E .

Elementary Functions

1. A *linear function* – $w = az + b$, where a, b are complex numbers.

2. A *linear-fractional function* – $w = \frac{az + b}{cz + d}$, where a, b, c, d are complex numbers.

3. A *general rational function* – $w = \frac{a_0 z^n + a_1 z^{n-1} + \dots + a_n}{b_0 z^m + b_1 z^{m-1} + \dots + b_m}$, where a_k, b_k are complex numbers, $k = 0, 1, 2, \dots, n$.

4. A *power function* – $w = z^n$, $n \in \mathbb{N}$.

5. An *exponential function* $w = e^z$ is defined by the following equality

$$w = e^z = e^{x+iy} = e^x (\cos y + i \sin y).$$

The function e^z has a pure imaginary period $2\pi i$.

6. The *logarithmic function* $w = \text{Ln } z$ is defined as the inverse of the exponential function.

$$w = \text{Ln } z = \ln|z| + i \text{Arg} z = \ln|z| + i(\arg z + 2k\pi), \quad k = 0, \pm 1, \pm 2, \dots$$

This function is many-valued. The expression $\ln z = \ln|z| + i \arg z$ is called the *principal value of the natural logarithm of a number z*.

7. The *generalized power function* is defined by the following equality

$$w = z^\alpha = e^{\alpha \text{Ln } z} \quad \text{for all } z \neq 0,$$

where α and z are complex numbers.

This function is many-valued. The *principal value of z^α* is defined by $z^\alpha = e^{\alpha \ln z}$.

8. Let $a \neq 0$ be a complex number. The *generalized exponential function* is defined as

$$w = a^z = e^{z \text{Ln } a}.$$

This function is multiple-valued. Its *principal value* is defined by $a^z = e^{z \ln a}$.

9. *Trigonometric functions* are defined by the following equalities:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

The modulus of the functions $\sin z$ and $\cos z$ can be greater than unit. For the functions $\sin z$, $\cos z$, $\tan z$ and $\cot z$ the formulas of trigonometry of a real variable hold true.

10. *Hyperbolic functions* are defined by the following equalities:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

For these functions the relationships between the hyperbolic functions of a real variable hold true.

The relationships between trigonometric and hyperbolic functions:

$$\begin{array}{ll} \sin z = -i \sinh iz, & \sinh z = -i \sin iz, \\ \cos z = \cosh iz, & \cosh z = \cos iz, \\ \tan z = -i \tanh iz, & \tanh z = -i \tan iz, \\ \cot z = i \coth iz, & \coth z = i \cot iz. \end{array}$$

11. The functions inverse to the trigonometric functions $\text{Arccos } z$, $\text{Arcsin } z$, $\text{Arc tan } z$, $\text{Arc cot } z$ are defined as the inverse to the functions $\sin z$, $\cos z$, $\tan z$ and $\cot z$, respectively.

$$\text{Arccos } z = -i \text{Ln} \left(z + \sqrt{z^2 - 1} \right), \quad \text{Arcsin } z = -i \text{Ln} \left(iz + \sqrt{1 - z^2} \right),$$

$$\text{Arctan } z = -\frac{i}{2} \text{Ln} \frac{1+iz}{1-iz}, \quad \text{Arccot } z = -\frac{i}{2} \text{Ln} \frac{z+i}{z-i}.$$

All of these functions are many-valued. Their principle values we will get if we replace the logarithmic functions with their principle values.

Example 1.7. Find the principle values of the following functions:

$$\begin{array}{ll} \text{a)} \left(\frac{-\sqrt{3}+i}{2} \right)^{-i}; & \text{b)} (-1)^{\sqrt{2}}; \\ \text{c)} \sin(5-3i); & \text{d)} \cosh(7+i). \end{array}$$

Solution

a) Using the definition of the exponential function with base a , we get

$$\left(\frac{-\sqrt{3}+i}{2} \right)^{-i} = e^{-i \text{Ln} \left(\frac{-\sqrt{3}+i}{2} \right)}.$$

Let's find $\text{Ln} \left(\frac{-\sqrt{3}+i}{2} \right)$:

$$z = \frac{-\sqrt{3}+i}{2} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad |z|=1, \quad \arg z = \pi + \arctan \frac{1/2}{-\sqrt{3}/2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6},$$

$$\text{Ln} \left(\frac{-\sqrt{3}+i}{2} \right) = \ln|z| + i \frac{5\pi}{6} + 2\pi k i = i \left(\frac{5\pi}{6} + 2\pi k \right), \quad k \in \mathbb{Z}.$$

$$\left(\frac{-\sqrt{3}+i}{2} \right)^{-i} = e^{-i \cdot i \left(\frac{5\pi}{6} + 2\pi k \right)} = e^{\frac{5\pi}{6} + 2\pi k}, \quad k \in \mathbb{Z}.$$

The principal value of the given function is

$$\left(\frac{-\sqrt{3}+i}{2}\right)^{-i} = e^{-i \ln\left(\frac{-\sqrt{3}+i}{2}\right)} = e^{-i \cdot i \frac{5\pi}{6}} = e^{\frac{5\pi}{6}}.$$

b) We find $\text{Ln}(-1)$:

$$z = -1, |z| = 1, \arg z = \pi,$$

$$\text{Ln}(-1) = \ln |1| + i\pi + 2\pi ki = (\pi + 2\pi k)i.$$

$$\text{Hence, } (-1)^{\sqrt{2}} = e^{\sqrt{2}(\pi + 2\pi k)i}.$$

The principle value of the given function is
 $(-1)^{\sqrt{2}} = e^{\sqrt{2}\text{Ln}(-1)} = e^{\sqrt{2}\pi i} = \cos\sqrt{2}\pi + i\sin\sqrt{2}\pi.$

c) Using the trigonometric formula
 $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$ and formulas $\sin z = -i \sinh iz$,
 $\cos z = \cosh iz$, we get:

$$\begin{aligned} \sin(5 - 3i) &= \sin 5 \cos(3i) - \cos 5 \cdot \sin(3i) = \sin 5 \cosh 3 - \\ &- \cos 5 \cdot (-i \sinh(i \cdot 3)) = \sin 5 \cosh 3 + i \cos 5 \sinh(-3) = \\ &= \sin 5 \cosh 3 - i \cos 5 \sinh 3. \end{aligned}$$

d) Using the formula $\cosh z = \frac{e^z + e^{-z}}{2}$, we obtain:

$$\begin{aligned} \cosh(7 + i) &= \frac{e^{7+i} + e^{-7-i}}{2} = \frac{e^7 \cdot e^i + e^{-7} \cdot e^{-i}}{2} = \\ &= \frac{e^7 \cdot (\cos 1 + i \sin 1) + e^{-7} \cdot (\cos(-1) + i \sin(-1))}{2} = \\ &= \frac{e^7 \cdot (\cos 1 + i \sin 1) + e^{-7} \cdot (\cos 1 - i \sin 1)}{2} = \cos 1 \cdot \frac{e^7 + e^{-7}}{2} + i \sin 1 \cdot \frac{e^7 - e^{-7}}{2} = \\ &= \cos 1 \cdot \cosh 7 + i \sin 1 \cdot \sinh 7. \bullet \end{aligned}$$

1.4 Differentiation of Functions of a Complex Variable

The notions of limit and continuity of a function of a complex variable introduce the same as for a function of a real variable.

The number A is called the *limit of the function* $f(z)$ at the point z_0 (or as $z \rightarrow z_0$), written $\lim_{z \rightarrow z_0} f(z) = A$, if $\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0$ such that for

all $z \neq z_0$ satisfying the inequality $|z - z_0| < \delta$, the inequality $|f(z) - A| < \varepsilon$ is fulfilled.

A necessary and sufficient conditions for $\lim_{z \rightarrow z_0} f(z) = A$, where $f(z) = u(x, y) + iv(x, y)$ and $A = u_0 + iv_0$, is that $\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} u(x, y) = u_0$ and

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} v(x, y) = v_0.$$

A function $f(z)$ is called *continuous at a point* z_0 if it is defined at this point and $\lim_{z \rightarrow z_0} f(z) = f(z_0)$.

If $f(z) = u + iv$ and $z = x + iy$, $u = u(x, y)$ and $v = v(x, y)$ are real-valued functions on the complex plane, then the continuity of $f(z)$ at $z_0 = x_0 + iy_0$ is equivalent to the continuity of u and v at the point (x_0, y_0) .

Let us consider the expression

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \quad z, z + \Delta z \in D.$$

If at the point z this limit exists and is independent of the way in which Δz tends to zero, then it is called the *derivative of the function* $f(z)$ at the point z and is denoted by $f'(z)$ or $\frac{df(z)}{dz}$.

The function $f(z)$ is called *differentiable at the point* z if at the point z a function $f(z)$ has a derivative $f'(z)$.

If function $f(z)$ is differentiable at every point of a neighborhood of z_0 , then it is called an *analytic at* z_0 . A function $f(z)$, differentiable in every point of the domain D and having a continuous derivative $f'(z)$, is called *analytic in the domain* D .

For the function $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain D , it necessary and sufficient that there exist in this domain continuous partial derivatives of the functions $u(x, y)$ and $v(x, y)$ satisfying the *Cauchy-Reimann conditions*:

$$\begin{cases} \frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y}, \\ \frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x}. \end{cases} \quad (1.9)$$

If the Cauchy-Riemann conditions are fulfilled, then $f'(z)$ can be written in one of the following forms:

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}.$$

Example 1.8. Prove that the function $f(z) = \cos z$ is analytic in the entire plane.

Solution

We have:

$$f(z) = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,$$

$$u(x, y) = \cos x \cosh y, \quad v(x, y) = -\sin x \sinh y,$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial v}{\partial x} = -\cos x \sinh y,$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y.$$

The Cauchy-Riemann conditions are fulfilled and the partial derivatives of the functions $u(x, y)$ and $v(x, y)$ are continuous at any point (x, y) . Hence, the given function is analytic in the entire plane. •

Example 1.9. At what points is the function $f(z) = z\bar{z} + 2$ analytic?

Solution

We have $f(z) = z\bar{z} + 2 = x^2 + y^2 + 2$, $u(x, y) = x^2 + y^2 + 2$, $v(x, y) \equiv 0$. The Cauchy-Riemann conditions have the following form:

$$\begin{cases} 2x = 0, \\ 2y = 0. \end{cases}$$

The Cauchy-Riemann conditions are fulfilled only at the point $(0, 0)$. The function is not analytic at any point of the complex plane. •

Example 1.10. Find the analytic function $f(z)$ having $u(x, y) = 5e^x \cos y - 2x$ as its real part.

Solution

From the first condition (1.9) we get:

$$\begin{aligned} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &\Rightarrow v = \int \frac{\partial u}{\partial x} dy + \phi(x) = \int (5e^x \cos y - 2) dy + \phi(x) = \\ &= 5e^x \sin y - 2y + \phi(x), \end{aligned}$$

where $\phi(x)$ is an arbitrary function of x .

Using the second condition in (1.9), we obtain:

$$\begin{aligned} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\Rightarrow \frac{\partial}{\partial y}(5e^x \cos y - 2x) = -\frac{\partial}{\partial x}(5e^x \sin y - 2y + \phi(x)), \\ -5e^x \sin y &= -5e^x \sin y - \phi'(x), \quad \phi'(x) = 0, \quad \phi(x) = C, \end{aligned}$$

where C is a real constant of integration. Therefore, we deduce $v = 5e^x \sin y - 2y + C$.

The corresponding analytic function is

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) = 5e^x \cos y - 2x + i(5e^x \sin y - 2y + C) = \\ &= 5e^x (\cos y + i \sin y) - 2(x + iy) + iC = 5e^x \cdot e^{iy} - 2(x + iy) + iC = \\ &= 5e^{x+iy} - 2(x + iy) + iC = 5e^z - 2z + iC. \bullet \end{aligned}$$

Example 1.11. Find the analytic function $f(z)$ having $v(x, y) = 3x^2y + 2x - y^3$ as its imaginary part.

Solution

From the first condition (1.9), we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow u = \int \frac{\partial v}{\partial y} dx + \psi(y) = \int (3x^2 - 3y^2) dx + \psi(y) = x^3 - 3xy^2 + \psi(y),$$

where $\psi(y)$ is an arbitrary function of y .

Using the second condition (1.9), we have:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -6xy + \psi'(y) = -6xy - 2, \quad \psi'(y) = -2, \quad \psi(y) = -2y + C,$$

where C is a real constant of integration. Therefore, we deduce $u = x^3 - 3xy^2 - 2y + C$. The corresponding analytic function is

$$\begin{aligned}
 f(z) &= u(x, y) + iv(x, y) = x^3 - 3xy^2 - 2y + C + i(3x^2y + 2x - y^3) = \\
 &= (x^3 + 3x^2yi - 3xy^2 - iy^3) + 2(xi - y) + C = (x + iy)^3 + 2i(x + iy) + C = \\
 &= z^3 + 2zi + C. \bullet
 \end{aligned}$$

Harmonic Functions. A real-valued function $\phi(x, y)$ of real variables is called *harmonic in a domain D* if its first and second derivatives are continuous and satisfy the *Laplace equation* $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ for all $(x, y) \in D$.

If the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then functions $u(x, y)$ and $v(x, y)$ are harmonic functions in this domain. The converse statement is not true.

1.5 Integral with Respect to a Complex Variable

If $f(z)$ is the continuous and one-valued function on the piecewise smooth curve C , then

$$\int_C f(z) dz = \int_C u dx - v dy + i \int_C u dy + v dx. \quad (1.10)$$

From (1.10) it is clear that the properties of integral with respect to a complex variable are analogous to those of line integrals of the second kind with respect to a real variable.

Some basic properties of integrals with respect to a complex variable:

$$1) \int_{\gamma^+} f(z) dz = - \int_{\gamma^-} f(z) dz, \text{ where } \gamma^+ \text{ is the contour with a given}$$

positive direction and γ^- is the same contour with an opposite direction;

$$2) \text{ if } \gamma = \gamma_1 \cup \gamma_2, \text{ then } \int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz;$$

3) if $z = \varphi(t)$ is an analytic function establishing one-to-one correspondence between the curve Γ and $\gamma = \varphi(\Gamma)$, then

$$\int_{\gamma} f(z) dz = \int_{\Gamma} f[\varphi(t)] \varphi'(t) dt.$$

If $z = z(t) = x(t) + iy(t)$ is the parametric representation of the curve C , corresponding to the values of the parameter $t \in [\alpha; \beta]$, then

$$\int_C f(z) dz = \int_{\alpha}^{\beta} f(z(t)) z'(t) dt. \quad (1.11)$$

A function $F(z)$ analytic in the domain D is called *the primitive* or *the indefinite integral* of the function $f(z)$ if for all $z \in D$ the relationship $F'(z) = f(z)$ holds true.

If $f(z)$ is the analytic function in the domain D , $z_0, z_1 \in D$ and $F(z)$ is one of the primitives of the function $f(z)$, then the *Newton-Leibniz formula* takes place:

$$\int_{z_0}^{z_1} f(z) dz = F(z) \Big|_{z_0}^{z_1} = F(z_1) - F(z_0). \quad (1.12)$$

Example 1.12. Evaluate the integral $\int_C (2\bar{z} - z) dz$, where C is the line segment joining the points $z_1 = 0$ and $z_2 = 3 + i$.

Solution

On the xy -plane the point $z_1 = 0$ corresponds to the point $(0,0)$, and the point $z_2 = 3 + i$ corresponds to the point $(3,1)$. We find the equation of

the line which passes through these points: $\frac{x-0}{3-0} = \frac{y-0}{1-0}$, $y = \frac{x}{3}$.

$$\begin{aligned} \int_L (2\bar{z} - z) dz &= \int_L (2(x - iy) - (x + iy)) d(x + iy) = \\ &= \int_L x dx + 3y dy + i \int_L x dy - 3y dx = \left| y = \frac{x}{3}, dy = \frac{dx}{3} \right|_{0 \leq x \leq 3} = \end{aligned}$$

$$\begin{aligned}
&= \int_0^3 \left(x dx + 3 \frac{x}{3} \frac{dx}{3} \right) + i \int_0^3 \left(x \frac{dx}{3} - 3 \frac{x}{3} dx \right) = \frac{4}{3} \int_0^3 x dx - i \frac{2}{3} \int_0^3 x dx = \\
&= \frac{4}{3} \int_0^3 x dx - i \frac{2}{3} \int_0^3 x dx = \frac{4}{3} \frac{x^2}{2} \Big|_0^3 - i \frac{2}{3} \frac{x^2}{2} \Big|_0^3 = 6 - 3i . \bullet
\end{aligned}$$

Example 1.13. Evaluate the integral $\int_C \frac{dz}{z - z_0}$, where $C : |z - z_0| = R$.

Solution

Making the substitution $z - z_0 = R e^{it}$ and taking into account that $dz = R i e^{it} dt$, $t \in [0, 2\pi]$, by formula (1.11) we obtain:

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{R i e^{it} dt}{R e^{it}} = i \int_0^{2\pi} dt = 2\pi i . \bullet$$

Example 1.14. Evaluate the integral $\int_C (6z^2 - z \cdot \bar{z}) dz$, where $C : |z| = R, 0 \leq \arg z \leq \pi$.

Solution

Making the substitution $z = R e^{it}$ and taking into account that $\bar{z} = R e^{-it}$, $dz = i R e^{it} dt$, $t \in [0, \pi]$, by formula (1.11) we obtain:

$$\begin{aligned}
&\int_L (6z^2 - z \cdot \bar{z}) dz = \int_0^\pi (6R^2 e^{2it} - R e^{it} \cdot R e^{-it}) i R e^{it} dt = \\
&= i R^3 \int_0^\pi (6e^{3it} - e^{it}) d\varphi = R^3 (2e^{3it} - e^{it}) \Big|_0^\pi = R^3 (2e^{i3\pi} - e^{i\pi} - (2e^{i \cdot 0} - e^{i \cdot 0})) = \\
&= R^3 (2(\cos 3\pi + i \sin 3\pi) - (\cos \pi + i \sin \pi) - 1) = -2R^3 . \bullet
\end{aligned}$$

Example 1.15. Evaluate the integral $\int_{2+i}^{3-i} (2z - 3) dz$.

Solution

Using the Newton-Leibniz formula (1.12), we get:

$$\int_{2+i}^{3-i} (2z-3)dz = \left(z^2 - 3z \right) \Big|_{2+i}^{3-i} = (3-i)^2 - 3(3-i) - (2+i)^2 + 3(2+i) = 2 - 4i . \bullet$$

Example 1.16. Evaluate the integral $\int_0^i (3z-4)\sin zdz$.

Solution

$$\begin{aligned} \int_0^i (3z-4)\sin zdz &= \left| \begin{array}{ll} u = 3z-4 & du = 3dz \\ dv = \sin zdz & v = -\cos z \end{array} \right| = -(3z-4)\cos z \Big|_0^i + \\ &+ 3 \int_0^i \cos zdz = -(3z-4)\cos z \Big|_0^i + 3 \int_0^i \cos zdz = -(3i-4)\cos i - \\ &- 4\cos 0 + 3\sin z \Big|_0^i = -3i\cos i + 4\cos i - 4 + 3\sin i = \\ &= 4\operatorname{ch}1 - 4 + i3(\operatorname{sh}1 - \operatorname{ch}1) . \bullet \end{aligned}$$

Theorem 1.1 (*Cauchy's Integral Theorem*). If the function $f(z)$ is analytic in a simple connected domain D , then for any closed contour $C \subset D$ (Fig. 1.9) we have $\oint_C f(z)dz = 0$.

If $f(z)$ is continuous in the closed domain $\overline{D} = D \cup L$, then $\oint_L f(z)dz = 0$.

Theorem 1.2 (*Cauchy's theorem for a multiply connected domain*). If the function $f(z)$ is analytic in a multiply connected domain D with the boundary L and interior contours $l_1, l_2, \dots, l_n$ (Fig. 1.10), then the following relationship takes place

$$\oint_L f(z)dz = \sum_{k=1}^n \oint_{l_k} f(z)dz ,$$

where all contours are traversed anticlockwise.

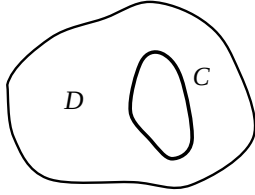


Figure 1.9

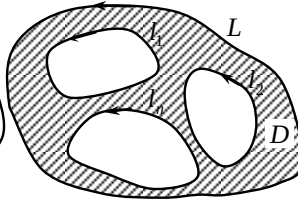


Figure 1.10

Let the function $f(z)$ be analytic in a closed simply-connected domain $\bar{D}$ bounded by the contour L . Then the value of the function $f(z)$ at any point $z_0 \in D$ can be found by *Cauchy's integral formula*

$$f(z_0) = \frac{1}{2\pi i} \oint_L \frac{f(z)}{z - z_0} dz, \quad (1.13)$$

where the orientation of the contour L is positive (i.e. L is transversed anticlockwise).

Formula (1.13) can be written in the form

$$\oint_L \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0), \quad (1.14)$$

which is convenient for computing of line integrals.

If the function $f(z)$ is analytic in the domain D and on its boundary, then $f(z)$ has in this domain derivatives of any order which can be found by the formulas

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n=1, 2, \dots, \quad (1.15)$$

where $z_0 \in D$, $z \in C$ and C is a contour containing the point z_0 and such that the bounded by it domain $D_C \subset D$.

The formulas (1.15) can be written in the form

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0), \quad n=1, 2, \dots, \quad (1.16)$$

which is convenient for computing of line integrals.

Example 1.17. Evaluate $\oint_C \frac{\sinh iz}{z^2 - 4z + 3} dz$ if:

a) $C: |z - 4| = 0,5$; **b)** $C: |z - 4| = 2$; **c)** $C: |z - 4| = 4$.

Solution

a) The integrand $\frac{\sinh iz}{z^2 - 4z + 3} = \frac{\sinh iz}{(z-3)(z-1)}$ is an analytic function on and inside $C: |z - 4| = 0,5$ (Fig. 1.11), so by Cauchy's integral theorem, we get:

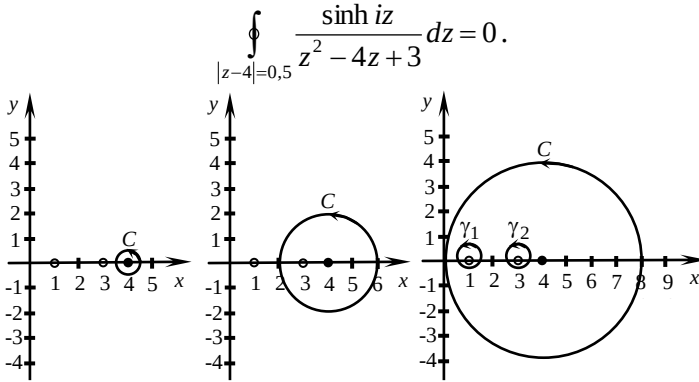


Figure 1.11

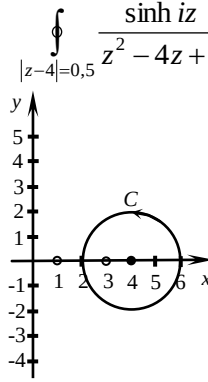


Figure 1.12

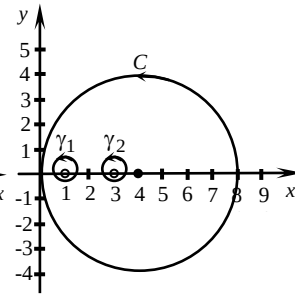


Figure 1.13

b) We can write the integrand as $\frac{\sinh iz}{(z-3)(z-1)} = \frac{\frac{\sinh iz}{z-1}}{z-3} = \frac{f(z)}{z-3}$,

where $f(z) = \frac{\sinh iz}{z-1}$ is analytic on and inside $C: |z - 4| = 2$ (Fig. 1.12), so by formula (1.14), we have:

$$\oint_{|z-4|=2} \frac{\sinh iz}{z^2 - 4z + 3} dz = \oint_{|z-4|=2} \frac{f(z)}{z-3} dz = 2\pi i \frac{\sinh iz}{z-1} \Big|_{z=3} = \pi i \operatorname{sh} 3i = -\pi \sin 3.$$

c) We isolate the points $z_1 = 1$ and $z_2 = 3$ by closed contours γ_1 and γ_2 , respectively, lying inside the contour $C: |z - 4| = 4$ (Fig. 1.13). By Cauchy's theorem for a multiply connected domain, we can express the

given integral as the sum of two integrals, each of them can be evaluated by formula (1.14):

$$\begin{aligned} & \oint_{|z-4|=4} \frac{\sinh iz}{z^2 - 4z + 3} dz = \oint_{\gamma_1} \frac{\sinh iz}{(z-1)(z-3)} dz + \oint_{\gamma_2} \frac{\sinh iz}{(z-1)(z-3)} dz = \\ & = 2\pi i \frac{\sinh iz}{z-3} \Big|_{z=1} + 2\pi i \frac{\sinh iz}{z-1} \Big|_{z=3} = -\pi i \sinh i + \pi i \sinh 3i = \pi(\sin 1 - \sin 3). \bullet \end{aligned}$$

Example 1.18. Evaluate $\oint_{|z+2i|=1} \frac{\cosh \pi z}{(z^2 + 4)^2} dz$.

Solution

We can write the integrand as:

$$\frac{\cosh \pi z}{(z^2 + 4)^2} = \frac{\cosh \pi z}{(z + 2i)^2 (z - 2i)^2} = \frac{\cosh \pi z}{(z - 2i)^2} = \frac{f(z)}{(z + 2i)^2},$$

where $f(z) = \frac{\cosh \pi z}{(z - 2i)^2}$ is an analytic on and inside $C: |z + 2i| = 1$

(Fig. 1.14), so by formula (1.16), we have:

$$\begin{aligned} & \oint_{|z+2i|=1} \frac{\cosh \pi z}{(z^2 + 4)^2} dz = \oint_{|z+2i|=1} \frac{\cosh \pi z}{(z - 2i)^2} dz = \\ & = \frac{2\pi i}{1!} \left(\frac{\cosh \pi z}{(z - 2i)^2} \right)' \Big|_{z=-2i} = 2\pi i \frac{\pi(z - 2i) \sinh \pi z - 2 \cosh \pi z}{(z - 2i)^3} \Big|_{z=-2i} = \\ & = 2\pi i \frac{-4\pi i \sinh(-2\pi i) - 2 \cosh(-2\pi i)}{(-4i)^3} = \\ & = 2\pi i \frac{-4\pi \sin(2\pi) - 2 \cos(2\pi)}{64\pi i} = -\frac{\pi}{16}. \bullet \end{aligned}$$

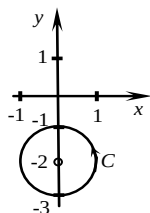


Figure 1.14

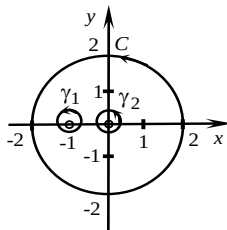


Figure 1.15

Example 1.19. Evaluate $\oint_{|z|=2} \frac{\cos z}{z^4 + z^3} dz$.

Solution

We isolate the points $z_1 = -1$ and $z_2 = 0$ by closed contours γ_1 and γ_2 , respectively, lying inside the domain bounded by the contour $C: |z| = 2$ (Fig. 1.15). By Cauchy's theorem for a multiply connected domain, we can express the integral $\oint_{|z|=2} \frac{\cos z}{z^4 + z^3} dz$ as the sum of two integrals:

$$\oint_{|z|=2} \frac{\cos z}{z^4 + z^3} dz = \oint_{|z|=2} \frac{\cos z}{z^3(z+1)} dz = \oint_{\gamma_1} \frac{\cos z}{z^3(z+1)} dz + \oint_{\gamma_2} \frac{\cos z}{z^3(z+1)} dz.$$

Using formula (1.14), we get:

$$\oint_{\gamma_1} \frac{\cos z}{z^3(z+1)} dz = \oint_{\gamma_1} \frac{\frac{\cos z}{z^3}}{z+1} dz = 2\pi i \frac{\frac{\cos z}{z^3}}{z+1} \bigg|_{z=-1} = -2\pi i \cos 1.$$

By formula (1.16), we obtain:

$$\begin{aligned} \oint_{\gamma_2} \frac{\cos z}{z^3(z+1)} dz &= \oint_{\gamma_1} \frac{\frac{\cos z}{z+1}}{z^3} dz = \frac{2\pi i}{2!} \left(\frac{\cos z}{z+1} \right)'' \bigg|_{z=0} = \\ &= \pi i \frac{(-z \cos z - \cos z)(z+1) + 2(z \sin z + \sin z + \cos z)}{(z+1)^3} \bigg|_{z=0} = \pi i. \\ \oint_{|z|=2} \frac{\cos z}{z^4 + z^3} dz &= \pi i (1 - 2 \cos 1). \bullet \end{aligned}$$

1.6 Number and Power Series with Complex Terms

1.6.1 Number Series with Complex Terms

Compose a series of complex numbers

$$z_1 + z_2 + \dots + z_n + \dots = \sum_{n=1}^{\infty} z_n, \quad (1.17)$$

where $z_n = x_n + iy_n$.

The necessary and sufficient conditions for convergence of series (1.17) are the convergence of the real series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$, and if

$$\sum_{n=1}^{\infty} x_n = S_1 \text{ and } \sum_{n=1}^{\infty} y_n = S_2, \text{ then } S = \sum_{n=1}^{\infty} z_n = S_1 + iS_2.$$

Theorem 1.3. Assume that we are given a series with complex terms

$$z_1 + z_2 + \dots + z_n + \dots = \sum_{n=1}^{\infty} z_n. \quad (1.18)$$

If the series composed of the moduli of its terms

$$|z_1| + |z_2| + \dots + |z_n| + \dots = \sum_{n=1}^{\infty} |z_n| \quad (1.19)$$

is convergent, then series (1.18) is convergent too.

If series (1.19) is convergent, then the series (1.18) is *absolutely convergent*.

Example 1.20. Test the series $\sum_{n=1}^{\infty} \frac{(3-4i)^n}{n!}$ for convergence.

Solution

Applying d'Alembert's test to the series composed of the moduli of

its terms $\sum_{n=1}^{\infty} \frac{|3-4i|^n}{n!} = \sum_{n=1}^{\infty} \frac{5^n}{n!}$, we obtain:

$$\lim_{n \rightarrow \infty} \frac{|z_{n+1}|}{|z_n|} = \lim_{n \rightarrow \infty} \frac{5^{n+1} n!}{(n+1)! 5^n} = \lim_{n \rightarrow \infty} \frac{5}{(n+1)} = 0 < 1.$$

The series $\sum_{n=1}^{\infty} \frac{|3-4i|^n}{n!}$ converges. The considered series is absolutely convergent. •

1.6.2 Power Series with Complex Terms

A series of the form

$$c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots + c_n(z - z_0)^n + \dots = \sum_{n=0}^{\infty} c_n(z - z_0)^n, \quad (1.20)$$

where z is a complex variable, c_n and z_0 are complex numbers, is a *power series*.

We confine ourselves to studying series

$$c_0 + c_1z + c_2z^2 + \dots + c_nz^n + \dots = \sum_{n=0}^{\infty} c_nz^n. \quad (1.21)$$

Since any series of type (1.20) can be reduced to type (1.21) with the aid of the substitution $z - z_0 = t$.

If series (1.21) converges when $|z| < R$ and diverges when $|z| > R$, then R is called *the radius of convergence*. A circle on the complex plane with centre at the origin and radius R is the *circle of convergence of the power series* (1.21). At the boundary of the circle of convergence the question about the convergence of series can be answered, as it was done in the case of real numbers, by means of an additional investigation.

If the power series (1.21) converges only at the point $z = 0$, then $R = 0$. If the series converges for any value of z , then $R = \infty$.

The radius of convergence R of series (1.21) can be found by the formulas:

$$R = \lim_{n \rightarrow \infty} \frac{|c_n|}{|c_{n+1}|}, \quad (1.22)$$

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}}. \quad (1.23)$$

Example 1.21. Find the radius of convergence of the power series:

$$\text{a) } \sum_{n=0}^{\infty} e^{(i-3)n} z^n; \quad \text{b) } \sum_{n=0}^{\infty} \frac{(z-i)^n}{(\sqrt{3}-i)^n}.$$

Solution

a) We have $c_n = e^{(i-3)n}$, $c_{n+1} = e^{(i-3)(n+1)}$. By (1.22), we have:

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n \rightarrow \infty} \frac{|e^{(i-3)n}|}{|e^{(i-3)(n+1)}|} = \lim_{n \rightarrow \infty} \frac{|e^{(i-3)n}|}{|e^{(i-3)(n+1)}|} = \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{e^{i-3}} \right| = |e^{3-i}| = e^3 |\cos 1 - i \sin 1| = e^3. \end{aligned}$$

Consequently, the radius of convergence is $R = e^3$.

$$\text{b) We have } c_n = \frac{1}{(\sqrt{3}-i)^n}, \quad |c_n| = \left| \frac{1}{(\sqrt{3}-i)^n} \right| = \frac{1}{|\sqrt{3}-i|^n} = \frac{1}{2^n}. \quad \text{By}$$

(1.23), we have:

$$R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{1/2^n}} = 2.$$

Consequently, the radius of convergence is $R = 2$. •

1.6.3 Taylor Series

The series

$$f(z) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.$$

is called a *Taylor's series* of the function $f(z)$ in powers of $z-z_0$. If $z_0 = 0$, then this series is called a *Maclaurin's series* of the function $f(z)$.

If a function $f(z)$ is analytic in the circle $|z-z_0| < R$, then it is uniquely represented in it by its Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad |z-z_0| < R,$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{l_r} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi,$$

$$l_r : |z - z_0| < r, \quad r < R.$$

The expansions of the elementary functions in powers of z :

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for all } z,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \text{ for all } z,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \text{ for all } z,$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots + (-1)^{n+1} \frac{z^n}{n} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}, \quad |z| < 1,$$

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots + (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z| < 1,$$

$$\begin{aligned} (1+z)^\alpha &= 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \dots = \\ &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2) \dots (\alpha-n+1)}{n!} z^n, \quad |z| < 1, \end{aligned}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

$$\frac{1}{1+z} = 1 - z + z^2 - \dots + (-1)^n z^n + \dots = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1.$$

Example 1.22. Expand the function $f(z) = \ln(9+z)$ into the Maclaurin series.

Solution

$$f(z) = \ln(9+z) = \ln 9 \left(1 + \frac{z}{9}\right) = \ln 9 + \ln \left(1 + \frac{z}{9}\right) = 2 \ln 3 + \ln \left(1 + \frac{z}{9}\right).$$

Replacing z in the series $\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$ ($|z| < 1$) with $\frac{z}{9}$,

we obtain:

$$f(z) = 2 \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z/9)^n}{n} = 2 \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{9^n \cdot n}.$$

This series is convergent for $\left| \frac{z}{9} \right| < 1$, i.e. $|z| < 9$. •

Example 1.23. Expand the function $f(z) = \frac{1}{5-3z}$ into a series in powers of $(z+2)$.

Solution

$$f(z) = \frac{1}{5-3z} = \frac{1}{5-3(z+2-2)} = \frac{1}{11-3(z+2)} = \frac{1}{11} \frac{1}{1-\frac{3}{11}(z+2)}.$$

Replacing z in the series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ ($|z| < 1$) with $\frac{3}{11}(z+2)$, we get:

$$f(z) = \frac{1}{11} \sum_{n=0}^{\infty} \left[\frac{3}{11}(z+2) \right]^n = \frac{1}{11} \sum_{n=0}^{\infty} \frac{3^n (z+2)^n}{11^n}.$$

This series is convergent for $\left| \frac{3}{11}(z+2) \right| < 1$, i.e. $|z+2| < \frac{11}{3}$. •

1.6.4 Laurent Series

The functional series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=-\infty}^{-1} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_n (z-z_0)^n \quad (1.24)$$

is called the *Laurent series* for $f(z)$ around z_0 .

The first summand $\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$ is called the *principal part of the Laurent series*. The series $\sum_{n=-\infty}^{-1} c_n (z - z_0)^n$ converges to the analytic function when $|z - z_0| > r$.

$$r \text{ can be found by formulas } r = \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|}, \quad r = \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}.$$

The second summand $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ is called the *regular part of the Laurent series*. It converges to the analytic function when $|z - z_0| < R$.

In the circular annulus $r < |z - z_0| < R$ the series (1.24) converges to the analytic function $f(z)$.

The function $f(z)$, analytic in the circular annulus $r < |z - z_0| < R$ is uniquely represented in this annulus by the convergent Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where coefficients c_n are determined by the formulas

$$c_n = \frac{1}{2\pi i} \oint_{l_p} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \quad n = 0, \pm 1, \pm 2, \dots,$$

where l_p is any circle $|z - z_0| = \rho$, $r < \rho < R$.

Example 1.24. Expand the function $f(z) = \frac{1}{z^2 + z - 30}$ into the Laurent series in the following region: **a)** $|z| < 5$; **b)** $5 < |z| < 6$; **c)** $|z| > 6$.

Solution

First, we decompose $f(z)$ into partial fractions:

$$f(z) = \frac{1}{z^2 + z - 30} = \frac{1}{(z+6)(z-5)} = \frac{1}{11} \left(\frac{1}{z-5} - \frac{1}{z+6} \right).$$

a) Taking into account the expansion $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$, we get:

$$\frac{1}{z-5} = -\frac{1}{5} \frac{1}{1-\frac{z}{5}} = -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{5^{n+1}}, \left(\left|\frac{z}{5}\right| < 1, |z| < 5\right),$$

$$\frac{1}{z+6} = \frac{1}{6} \frac{1}{1+\frac{z}{6}} = \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{6}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{6^{n+1}}, \left(\left|\frac{z}{6}\right| < 1, |z| < 6\right).$$

Consequently, for $|z| < 5$ (Fig. 1.16) we have the following expansion:

$$f(z) = \frac{1}{z^2 + z - 30} = -\frac{1}{11} \left(\sum_{n=0}^{\infty} \frac{z^n}{5^{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{6^{n+1}} \right).$$

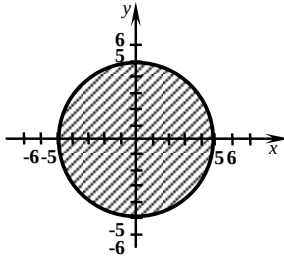


Figure 1.16

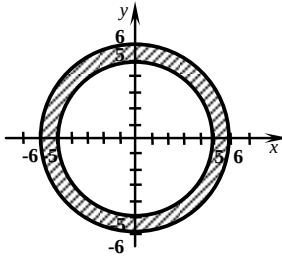


Figure 1.17

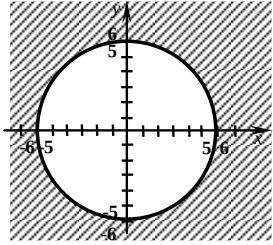


Figure 1.18

b) In the annulus $5 < |z| < 6$ (Fig. 1.17), we have:

$$\frac{1}{z-5} = \frac{1}{z} \frac{1}{1-\frac{5}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{5}{z}\right)^n = \sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}}, \left(\left|\frac{5}{z}\right| < 1, |z| > 5\right),$$

$$\frac{1}{z+6} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{6^{n+1}}, (|z| < 6).$$

Consequently, for $5 < |z| < 6$ we have the following expansion:

$$f(z) = \frac{1}{z^2 + z - 30} = \frac{1}{11} \left(\sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{6^{n+1}} \right).$$

c) In the region $|z| > 6$ (Fig. 1.18), we have:

$$\frac{1}{z-5} = \sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}}, \quad (|z| > 5),$$

$$\frac{1}{z+6} = \frac{1}{z} \frac{1}{1+\frac{6}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{6}{z}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{6^n}{z^{n+1}}, \quad \left(\left|\frac{6}{z}\right| < 1, |z| > 6\right).$$

Consequently, for $|z| > 6$ we have the following expansion:

$$f(z) = \frac{1}{z^2 + z - 30} = \frac{1}{11} \left(\sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n 6^n}{z^{n+1}} \right).$$

1.7 Zeros and Singular Points

1.7.1 Zeros

Let $f(z)$ be analytic function at z_0 . The point z_0 is called a *zero of $f(z)$ of order m* if $f(z_0) = 0$, $f'(z_0) = 0$, ..., $f^{(m-1)}(z_0) = 0$, $f^{(m)}(z_0) \neq 0$.

A zero of order one is called a *simple zero*. A zero of order two is called a *double zero*.

Theorem 1.4. If $f(z)$ is analytic in a neighbourhood of z_0 and has a zero of order m at z_0 , then there is a function $\varphi(z)$ which is analytic in the neighbourhood of z_0 , $\varphi(z_0) \neq 0$ and $f(z) = (z - z_0)^m \varphi(z)$.

Take place the converse theorem.

Theorem 1.5. If $f(z)$ can be expressed as $f(z) = (z - z_0)^m \varphi(z)$ for some analytic function $\varphi(z)$, $\varphi(z_0) \neq 0$, then $f(z)$ has a zero at z_0 of order m .

Example 1.25. Find the zeros and order of the zeros of the following functions:

a) $f(z) = \cos z - 1$; **b)** $f(z) = (z^2 + 4)^3 \sin z$; **c)** $f(z) = \frac{z^9}{1 - \cos z}$.

Solution

a) $\cos z - 1 = 0 \Rightarrow z_n = 2\pi n$ ($n = 0, \pm 1, \pm 2, \dots$) are zeros of the function $f(z) = \cos z - 1$. We have:

$$\begin{aligned} f'(z) &= -\sin z, \quad f'(2\pi n) = -\sin(2\pi n) = 0, \\ f''(z) &= -\cos z, \quad f''(2\pi n) = -\cos(2\pi n) = -1 \neq 0. \end{aligned}$$

We see that the function $f(z) = \cos z - 1$ has the double zeros at the points $z_n = 2\pi n$ ($n = 0, \pm 1, \pm 2, \dots$).

b) $(z^2 + 4)^3 \sin z = 0 \Rightarrow z^2 + 4 = 0$ or $\sin z = 0$. The solutions of these equations are $z = \pm 2i$, $z_k = \pi k$ ($k = 0, \pm 1, \pm 2, \dots$).

If $z = 2i$ then $f(z)$ can be represented in the following form:

$$f(z) = (z - 2i)^3 \varphi(z),$$

where $\varphi(z) = (z + 2i)^3 \sin z$ is analytic in the neighbourhood of $z = 2i$ and $\varphi(2i) = (4i)^3 \sin(2i) = 64 \sinh 2 \neq 0$. By theorem 1.6, the point $z = 2i$ is a zero of order 3.

Analogously, we can prove that the point $z = -2i$ is a zero of order 3.

Consider points $z_k = \pi k$ ($k = 0, \pm 1, \pm 2, \dots$). We have:

$$\begin{aligned} f'(z) &= 6z(z^2 + 4)\sin z + (z^2 + 4)^3 \cos z, \\ f'(\pi k) &= 6\pi k((\pi k)^2 + 4)\sin(\pi k) + ((\pi k)^2 + 4)^3 \cos(\pi k) = \\ &= (-1)^k ((\pi k)^2 + 4)^3 \neq 0, \quad (k = 0, \pm 1, \pm 2, \dots). \end{aligned}$$

The function $f(z) = (z^2 + 4)^3 \sin z$ has the simple zeros at the points $z_k = \pi k$ ($k = 0, \pm 1, \pm 2, \dots$).

c) Using the expansion $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$, we get

$$f(z) = \frac{z^9}{1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right)} = \frac{z^9}{\frac{z^2}{2!} - \frac{z^4}{4!} + \dots} = z^7 \frac{1}{\frac{1}{2!} - \frac{z^2}{4!} + \dots}.$$

Putting $\varphi(z) = \frac{1}{\frac{1}{2!} - \frac{z^2}{4!} + \dots}$, we get $f(z) = z^7 \varphi(z)$, where $\varphi(z)$ is an

analytic function in the neighbourhood of $z = 0$ and $\varphi(0) = 2 \neq 0$. The

solution of equation $f(z) = z^7 \phi(z) = 0$ is $z = 0$. By theorem 1.6, the function $f(z) = \frac{z^9}{1 - \cos z}$ has a zero of order 7 at $z = 0$. •

1.7.2 Singular Points

The points at which $f(z)$ is analytic are called the *regular points* of $f(z)$. The points at which $f(z)$ is not analytic are called the *singular points* of $f(z)$. The point z_0 is called an *isolated singularity* of a function $f(z)$ if there exists neighborhood of z_0 containing no singular points except of z_0 .

There are three types of singular points: removable singularities, poles of order m ($m \geq 1$) and essential singularities.

The point z_0 is called a *removable singular point* (a *removable singularity*) of a function $f(z)$ if the function $f(z)$ is analytic in the neighbourhood of the point z_0 and there exists the finite limit $\lim_{z \rightarrow z_0} f(z) = A$.

The point z_0 is called a *pole of order m* ($m \geq 1$) of a function $f(z)$ if z_0 is a zero of order m of the function $g(z) = 1/f(z)$ and $\lim_{z \rightarrow z_0} f(z) = \infty$.

Theorem 1.6. In order for the function $f(z)$ has a pole of order m at z_0 , it is necessary and sufficient that it can be expressed as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \text{ where } \phi(z) \text{ is an analytic function at the point } z_0, \phi(z_0) \neq 0.$$

The point z_0 is called an *essential singular point* (an *essential singularity*) of a function $f(z)$ if the function $f(z)$ is analytic in the neighbourhood of the point z_0 and $\lim_{z \rightarrow z_0} f(z)$ doesn't exist.

In some cases we can use the following statements for determine the kinds of singular points:

1) if in the expansion of $f(z)$ into the Laurent series in the annulus $0 < |z - z_0| < R$ the principal part is absent then the point z_0 is a *removable singularity* of the function $f(z)$;

2) if the principal part of the expansion of $f(z)$ into the Laurent series in the annulus $0 < |z - z_0| < R$ contains no more than m terms and all $c_n = 0$ for $n \leq -(m+1)$, and $c_{-m} \neq 0$ then the point z_0 is a *pole of order m* ($m \geq 1$) of the function $f(z)$;

3) if the principal part of the expansion of $f(z)$ into the Laurent series in the annulus $0 < |z - z_0| < R$ contains infinite number of terms then the point z_0 is an *essential singular point* of the function $f(z)$.

Example 1.26. Find the singular points of the functions and determine their type:

$$a) f(z) = \frac{5z-6}{(z-3)^4}; \quad b) f(z) = \frac{\sin z}{z^3-3z-2};$$

$$c) f(z) = \frac{1-\sin^4 z}{\cos z}; \quad d) e^{\frac{1}{z+4}}.$$

Solution

a) We find the zeros of the denominator for $f(z)$:

$$(z-3)^4 = 0, \quad z = 3.$$

The point $z = 3$ is the singular point of $f(z)$.

Since $\lim_{z \rightarrow 3} \frac{5z-6}{(z-3)^4} = \infty$ then the point $z = 3$ is the pole.

The point $z = 3$ is the zero of order 4 of the function

$$g(z) = \frac{1}{f(z)} = \frac{(z-3)^4}{5z-6}. \text{ Hence, } z = 3 \text{ is a pole of order 4.}$$

b) We find the zeros of the denominator for $f(z)$:

$$z^3 - 3z - 2 = 0, \quad z(z^2 - 1) - 2(z+1) = 0, \quad (z+1)(z^2 - z - 2) = 0,$$

$$(z+1)^2(z-2) = 0, \quad z_1 = -1, \quad z_2 = 2.$$

The points $z_1 = -1$, $z_2 = 2$ are the singular points of $f(z)$.

The function $f(z) = \frac{\sin z}{z^3 - 3z - 2}$ can be represented in the form

$$f(z) = \frac{\frac{\sin z}{z-2}}{(z+1)^2} = \frac{\varphi(z)}{(z+1)^2},$$

where $\varphi(z) = \frac{\sin z}{z-2}$ is an analytic function in the neighbourhood of the point $z_1 = -1$, $\varphi(-1) = \frac{\sin 1}{3} \neq 0$. By theorem 1.6, the point $z_1 = -1$ is a double pole.

Analogously, the function $f(z)$ can be written in the form

$$f(z) = \frac{\frac{\sin z}{(z+1)^2}}{z-2} = \frac{\varphi(z)}{z-2}, \text{ where } \varphi(z) = \frac{\sin z}{(z+1)^2} \text{ is an analytic function in the}$$

neighbourhood of the point $z_2 = 2$, $\varphi(2) = \frac{\sin 2}{9} \neq 0$. By theorem 1.6, the point $z_2 = 2$ is a simple pole.

c) We find the zeros of the denominator for $f(z)$:

$$\cos z = 0, \quad z_n = \frac{\pi}{2} + \pi n, \quad (n = 0, \pm 1, \pm 2, \dots).$$

$$\begin{aligned} \lim_{z \rightarrow \frac{\pi}{2} + \pi n} \frac{1 - \sin^4 z}{\cos z} &= \left[\frac{0}{0} \right] = \lim_{z \rightarrow \frac{\pi}{2} + \pi n} \frac{(1 - \sin^4 z)'}{(\cos z)'} = \lim_{z \rightarrow \frac{\pi}{2} + \pi n} \frac{-4 \sin^3 z \cdot \cos z}{-\sin z} = \\ &= \lim_{z \rightarrow \frac{\pi}{2} + \pi n} 4 \sin^2 z \cdot \cos z = 0. \end{aligned}$$

The points $z_n = \frac{\pi}{2} + \pi n$, $(n = 0, \pm 1, \pm 2, \dots)$ are the removable singular points.

d) The denominator of the function $\frac{1}{z+4}$ vanishes at the point

$z = -4$. To find the type of this singular point we expand the function $e^{\frac{1}{z+4}}$ into the Laurent series around the point $z = -4$:

$$e^{\frac{1}{z+4}} = 1 + \frac{1}{z+4} + \frac{1}{2!(z+4)^2} + \frac{1}{3!(z+4)^3} + \dots + \frac{1}{n!(z+4)^n} + \dots$$

$z = -4$ is an essential singular point of a function $e^{\frac{1}{z+4}}$, because the principal part of the expansion $e^{\frac{1}{z+4}}$ contains infinite number of terms. •

1.8 Residues of Functions and Their Application

1.8.1 Definition of Residue and Its Evaluation

The *residue* of an analytic function $f(z)$ at an isolated singularity z_0 is defined as a complex number which equals the value of the integral

$$\frac{1}{2\pi i} \oint_C f(z) dz,$$

taken in the positive direction around any closed contour C lying in the domain of analyticity of the function $f(z)$ and containing a unique singularity z_0 . The residue of the function $f(z)$ is denoted by $\text{res}(f(z); z_0)$ or $\text{res } f(z_0)$.

Evaluation of residues:

in general case,

$$\text{res } f(z_0) = c_{-1}; \quad (1.25)$$

if z_0 is a first-order pole of $f(z)$ then

$$\text{res } f(z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z); \quad (1.26)$$

if z_0 is a pole of order $m \geq 2$ of $f(z)$ then

$$\text{res } f(z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]; \quad (1.27)$$

if the point z_0 is a removable singular point of a function $f(z)$ then $\text{res } f(z_0) = 0$;

if the point z_0 is an essential singular point of a function $f(z)$ then $\text{res } f(z_0) = c_{-1}$;

if the function $f(z)$ is represented in the form of a ratio of two analytic functions $f(z) = \frac{\varphi(z)}{\psi(z)}$, where $\varphi(z_0) \neq 0$, $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, then

$$\operatorname{res} f(z_0) = \frac{\varphi(z_0)}{\psi'(z_0)}. \quad (1.28)$$

Example 1.27. Find the residues of the following functions:

a) $f(z) = \frac{e^z}{(z+3)(z-1)^3}$ at the singular points;

b) $f(z) = \frac{\cos \frac{\pi}{2} z}{z^3 + z^2}$ at the singular points;

c) $f(z) = \frac{\sin 2z - 2 \sin z}{(\sin z - z) \sin z}$ at the point $z = 0$;

d) $f(z) = \frac{\sin \frac{1}{z}}{1-z}$ at the singular points.

Solution

a) The function $f(z) = \frac{e^z}{(z+3)(z-1)^3}$ has singular points $z_1 = -3$ and $z_2 = 1$.

The function $f(z)$ can be written in the form $f(z) = \frac{e^z}{(z-1)^3} = \frac{\varphi(z)}{z+3}$,

where $\varphi(z) = \frac{e^z}{(z-1)^3}$ is an analytic function in the neighbourhood of the

point $z_1 = -3$ and $\varphi(-3) = -\frac{e^{-3}}{64} \neq 0$. By theorem 1.6, the point $z_1 = -3$ is a simple pole. Using formula (1.26), we find the residue of $f(z)$ at the point $z_1 = -3$:

$$\operatorname{res} f(-3) = \lim_{z \rightarrow -3} \frac{(z+3)e^z}{(z+3)(z-1)^3} = \lim_{z \rightarrow -3} \frac{e^z}{(z-1)^3} = -\frac{1}{64e^3}.$$

The function $f(z)$ can be written in the form

$$f(z) = \frac{e^z}{(z-1)^3} = \frac{\varphi(z)}{(z-1)^3}, \text{ where } \varphi(z) = \frac{e^z}{z+3} \text{ is an analytic function in the}$$

neighbourhood of the point $z_2=1$ and $\varphi(1) = \frac{e}{4} \neq 0$. By theorem 1.6, the point $z_2=1$ is a pole of order 3. Using formula (1.27), we find the residue of $f(z)$ at the point $z_2=1$:

$$\begin{aligned} \operatorname{res} f(1) &= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left(\frac{(z-1)^3 e^z}{(z+3)(z-1)^3} \right) = \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left(\frac{e^z}{z+3} \right) = \\ &= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{e^z(z^2 + 4z + 5)}{(z+3)^3} = \frac{5e}{64}. \end{aligned}$$

b) The function $f(z) = \frac{\cos \frac{\pi}{2} z}{z^3 + z^2} = \frac{\cos \frac{\pi}{2} z}{z^2(z+1)}$ has singular points at

$z_1=0$ and $z_2=-1$.

The function $f(z)$ can be written in the form $f(z) = \frac{\cos \frac{\pi}{2} z}{z^2} = \frac{\varphi(z)}{z^2}$,

where $\varphi(z) = \frac{\cos \frac{\pi}{2} z}{z+1}$ is an analytic function in the neighbourhood of the point $z_1=0$ and $\varphi(0)=1 \neq 0$. By theorem 1.6, the point $z_1=0$ is a double pole. Using formula (1.27), we get:

$$\operatorname{res} f(0) = \frac{1}{1!} \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{z^2 \cos \frac{\pi}{2} z}{z^2(z+1)} \right) = \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{\cos \frac{\pi}{2} z}{z+1} \right) =$$

$$= \lim_{z \rightarrow 0} \frac{-\frac{\pi}{2}(z+1)\sin\left(\frac{\pi}{2}z\right) - \cos\left(\frac{\pi}{2}z\right)}{(z+1)^2} = -1.$$

Let's consider the point $z_2 = -1$. We have:

$$\lim_{z \rightarrow -1} \frac{\cos \frac{\pi}{2}z}{z^2(z+1)} = \left[\frac{0}{0} \right] = \lim_{z \rightarrow -1} \frac{-\frac{\pi}{2}\sin \frac{\pi}{2}z}{3z^2 + 2z} = \frac{\pi}{2}.$$

The point $z_2 = -1$ is a removable singular point of the function $f(z)$, therefore, $\text{res } f(-1) = 0$.

c) The point $z = 0$ is a zero of the function $\varphi(z) = \sin 2z - 2\sin z$ and $\psi(z) = (\sin z - z)\sin z$. Let's determine the order of the zero $z = 0$ for functions $\varphi(z)$ and $\psi(z)$. Using the known expansion

$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$, we have:

$$\begin{aligned} \varphi(z) &= \sin 2z - 2\sin z = 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \dots - 2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \\ &= -\frac{2^3 - 2}{3!}z^3 + \frac{2^5 - 2}{5!}z^5 - \dots = z^3\varphi_1(z), \end{aligned}$$

where $\varphi_1(z) = -\frac{2^3 - 2}{3!} + \frac{2^5 - 2}{5!}z^2 - \dots$, $\varphi_1(0) = -1 \neq 0$.

Analogously, we have:

$$\begin{aligned} \psi(z) &= (\sin z - z)\sin z = \left(-\frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \\ &= z^4\left(-\frac{1}{3!} + \frac{z^2}{5!} - \dots\right)\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right) = z^4\psi_1(z), \end{aligned}$$

where $\psi_1(z) = \left(-\frac{1}{3!} + \frac{z^2}{5!} - \dots\right)\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)$, $\psi_1(0) = -\frac{1}{6} \neq 0$.

$$f(z) = \frac{\sin 2z - 2\sin z}{(\sin z - z)\sin z} = \frac{z^3\varphi_1(x)}{z^4\psi_1(x)} = \frac{\varphi_1(x)}{z\psi_1(x)}.$$

By theorem 1.6, the point $z=0$ is the simple pole. Using formula (1.26), we get:

$$\operatorname{res} f(0) = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} \frac{z \varphi(x)}{\psi(x)} = \lim_{z \rightarrow 0} \frac{z \cdot z^3 \varphi_1(x)}{z^4 \psi_1(x)} = \lim_{z \rightarrow 0} \frac{\varphi_1(x)}{\psi_1(x)} = \frac{\varphi_1(0)}{\psi_1(0)} = 6.$$

d) The function $f(z) = \frac{\sin \frac{1}{z}}{1-z}$ has singular points at $z_1=0$ and $z_2=1$.

Since the limit $\lim_{z \rightarrow 0} \frac{\sin \frac{1}{z}}{1-z}$ does not exist then the point $z_1=0$ is the essential singular point. To find the residue of $f(z)$ at the point $z_1=0$, we

expand $f(z) = \frac{\sin \frac{1}{z}}{1-z}$ into the Laurent series around $z_1=0$:

$$\begin{aligned} \sin \frac{1}{z} &= \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots \quad \text{and} \quad \frac{1}{1-z} = 1 + z + z^2 + \dots \\ f(z) &= \frac{1}{1-z} \cdot \sin \frac{1}{z} = \left(1 + z + z^2 + \dots\right) \left(\frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots\right) = \\ &= \left(1 - \frac{1}{3!} + \frac{1}{5!} - \dots\right) \frac{1}{z} + \left(-\frac{1}{3!} + \frac{1}{5!} - \dots\right) \frac{1}{z^2} + \dots + \\ &\quad + \text{the regular part of the Laurent series.} \\ \operatorname{res} f(0) &= c_{-1} = 1 - \frac{1}{3!} + \frac{1}{5!} - \dots = \sin 1. \end{aligned}$$

The function $f(z)$ can be written in the form $f(z) = \frac{\sin \frac{1}{z}}{1-z} = \frac{\varphi(z)}{1-z}$,

where $\varphi(z) = \sin \frac{1}{z}$ is an analytic function in the neighbourhood of the point $z_2=1$ and $\varphi(1) = \sin 1 \neq 0$. By theorem 1.6, the point $z_2=1$ is a simple pole. By formula (1.26), we have:

$$\operatorname{res} f(1) = \lim_{z \rightarrow 1} \frac{(z-1) \sin \frac{1}{z}}{1-z} = -\lim_{z \rightarrow 1} \sin \frac{1}{z} = -\sin 1. \bullet$$

1.8.2 Application of Residues

Theorem 1.7 (*Basic Residue Theorems*). If the function $f(z)$ is analytic everywhere in a closed domain $\bar{D}$, except at a finite number of isolated singularities z_k ($k = 1, 2, \dots, n$) lying inside this domain, then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{res} f(z_k), \quad (1.29)$$

where C is a contour lying inside the domain D and containing the points $z_1, z_2, \dots, z_n$, the orientation of C is positive.

Theorem 1.8. Let the function $f(x)$ be a rational function, $f(x) = \frac{P_m(x)}{Q_n(x)}$, where $P_m(x)$, $Q_n(x)$ are the polynomials of degree m and n , respectively. If $f(x)$ is a continuous at all points of real axis, ($Q_n(x) \neq 0$) and $n \geq m + 2$, then

$$\int_{-\infty}^{+\infty} f(x) dx = 2\pi i \sigma, \quad (1.30)$$

where σ is the sum of the residues of $f(x) = \frac{P_m(x)}{Q_n(x)}$ at the poles lying in the upper half-plane $\operatorname{Im} z > 0$.

Example 1.28. Evaluate the following integrals:

$$\text{a) } \oint_{|z|=3} \frac{z-3}{(z+1)(z^2+4)^2} dz; \quad \text{b) } \oint_{|z-2i|=1} (z-2i)^2 \sin \frac{1}{z-2i} dz.$$

Solution

a) The function $f(z) = \frac{z-3}{(z+1)(z^2+4)^2} = \frac{z-3}{(z+1)(z+2i)^2(z-2i)^2}$ has singular points at $z_1 = -1$, $z_2 = -2i$ and $z_3 = 2i$. These points lie inside the circle $|z| = 3$ (Fig. 1.19). By formula (1.29), we get:

$$\oint_{|z|=3} \frac{z-3}{(z+1)(z^2+4)^2} dz = 2\pi i (\text{res } f(-1) + \text{res } f(2i) + \text{res } f(-2i)).$$

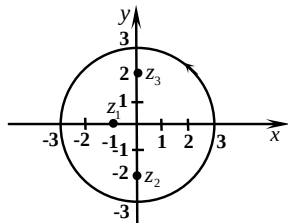


Figure 1.19

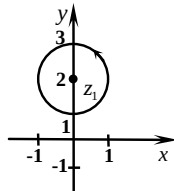


Figure 1.20

The point $z_1 = -1$ is a simple pole. By formula (1.26),

$$\text{res } f(-1) = \lim_{z \rightarrow -1} \left[(z+1) \cdot \frac{z-3}{(z+1)(z^2+4)^2} \right] = -\frac{4}{25}.$$

The points $z_2 = -2i$ and $z_3 = 2i$ are the double poles. By formula (1.27),

$$\begin{aligned} \text{res } f(-2i) &= \frac{1}{1!} \lim_{z \rightarrow -2i} \frac{d}{dz} \left((z+2i)^2 \cdot \frac{z-3}{(z+1)(z-2i)^2(z+2i)^2} \right) = \\ &= \lim_{z \rightarrow -2i} \frac{-2(z^2 - 4z - 3 + 4i)}{(z+1)^2(z-2i)^3} = \frac{2}{25} - \frac{27}{800}i. \end{aligned}$$

$$\begin{aligned} \text{res } f(2i) &= \frac{1}{1!} \lim_{z \rightarrow 2i} \frac{d}{dz} \left((z-2i)^2 \cdot \frac{z-3}{(z+1)(z+2i)^2(z-2i)^2} \right) = \\ &= \lim_{z \rightarrow 2i} \frac{-2(z^2 - 4z - 3 - 4i)}{(z+1)^2(z+2i)^3} = \frac{2}{25} + \frac{27}{800}i. \end{aligned}$$

$$\oint_{|z|=3} \frac{z-3}{(z+1)(z^2+4)^2} dz = 2\pi i \left(-\frac{4}{25} + \frac{2}{25} - \frac{27}{800}i + \frac{2}{25} + \frac{27}{800}i \right) = 0.$$

b) The function $f(z) = (z-2i)^2 \sin \frac{1}{z-2i}$ has a singular point at $z = 2i$. This point lies inside the circle $|z-2i|=1$ (Fig. 1.20). Let's find the kind of this singular point. We have:

$$\begin{aligned} \sin \frac{1}{z-2i} &= \frac{1}{z-2i} - \frac{1}{3!(z-2i)^3} + \frac{1}{5!(z-2i)^5} - \dots \\ f(z) &= (z-2i)^2 \sin \frac{1}{z-2i} = (z-2i)^2 \left(\frac{1}{z-2i} - \frac{1}{3!(z-2i)^3} + \right. \\ &\quad \left. + \frac{1}{5!(z-2i)^5} - \dots \right) = (z-2i) - \frac{1}{3!(z-2i)} + \frac{1}{5!(z-2i)^3} - \dots \end{aligned}$$

The point $z_0 = 2i$ is an essential singularity point because the principal part of the expansion of the integrand contains an infinite number of terms. By (1.30), we get:

$$\oint_{|z-2i|=1} (z-2i)^2 \sin \frac{1}{z-2i} dz = 2\pi i \operatorname{res} f(2i), \quad \operatorname{res} f(2i) = c_{-1} = -\frac{1}{3!} = -\frac{1}{6}.$$

$$\oint_{|z-2i|=1} (z-2i)^2 \sin \frac{1}{z-2i} dz = 2\pi i \cdot \left(-\frac{1}{6} \right) = -\frac{\pi i}{3}.$$

Example 1.29. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+4)^2(x^2+9)^2}$.

Solution

The function $f(z) = \frac{1}{(z^2+4)^2(z^2+9)^2} = \frac{1}{(z+2i)^2(z-2i)^2(z+3i)^2(z-3i)^2}$

has two double poles $z_1 = 2i$ and $z_2 = 3i$ which lie in the upper half-plane $\operatorname{Im} z > 0$. Therefore, applying formula (1.30), we obtain:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dx}{(x^2+4)^2(x^2+9)^2} &= 2\pi i (\operatorname{res} f(2i) + \operatorname{res} f(3i)). \\ \operatorname{res} f(2i) &= \lim_{z \rightarrow 2i} \frac{d}{dz} \left((z-2i)^2 \cdot \frac{1}{(z+2i)^2(z-2i)^2(z^2+9)^2} \right) = \end{aligned}$$

$$\begin{aligned}
&= \lim_{z \rightarrow 2i} \frac{d}{dz} \left(\frac{1}{(z+2i)^2(z^2+9)^2} \right) = \lim_{z \rightarrow 2i} \frac{-2(3z^2+4zi+9)}{(z^2+9)^3(z+2i)^3} = -\frac{11}{4000}i, \\
&\text{res } f(3i) = \lim_{z \rightarrow 3i} \frac{d}{dz} \left((z-3i)^2 \cdot \frac{1}{(z^2+4)^2(z-3i)^2(z+3i)^2} \right) = \\
&= \lim_{z \rightarrow 3i} \frac{d}{dz} \left(\frac{1}{(z^2+4)^2(z+3i)^2} \right) = \lim_{z \rightarrow 3i} \frac{-2(3z^2+6zi+4)}{(z^2+4)^3(z+3i)^3} = -\frac{41}{13500}i, \\
&\int_{-\infty}^{+\infty} \frac{dx}{(x^2+4)^2(x^2+9)^2} = 2\pi i \left(\frac{11}{4000}i - \frac{41}{13500}i \right) = \frac{31\pi}{54000} \bullet
\end{aligned}$$

2 OPERATIONAL CALCULUS

2.1 Laplace Transform

Let $f(t)$ be a function of a real variable t . The function $f(t)$ is called an **original function** if it satisfies the following conditions:

a) $f(t) \equiv 0$ for $t < 0$;

b) $f(t)$ is the piecewise continuous function for $t \geq 0$ ($f(t)$ has a finite number of discontinuities of the first kind on every finite interval of the t -axis);

c) there exist constants $M > 0$ and $s \geq 0$, such that

$$|f(t)| \leq M e^{st} \quad \text{for all } t > 0.$$

The Heaviside unit function is defined as $\eta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$ This

function is the simplest example of the original function.

We will briefly write the original function $f(t)$, implying that

$$f(t)\eta(t) = \begin{cases} 0, & t < 0, \\ f(t), & t \geq 0. \end{cases}$$

The *Laplace transform* of the original function $f(t)$ is called the function $F(p)$ of a complex variable $p = s + i\sigma$, which is defined by

$$F(p) = \int_0^{\infty} f(t) \cdot e^{-pt} dt, \quad (2.1)$$

provided (2.1) converges.

The functions $f(t)$ and $F(p)$ form the *transform pair*, which we'll denote by

$$f(t) \leftrightarrow F(p).$$

$L(f(t))$ is another notation for the Laplace transform of $f(t)$.

Example 2.1. Find the Laplace transform of the Heaviside unit function.

Solution

By (2.1), for $\text{Re } p > 0$ we obtain:

$$F(p) = \int_0^{\infty} 1 \cdot e^{-pt} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-pt} dt = \lim_{b \rightarrow \infty} \left(-\frac{1}{p} e^{-pt} \Big|_0^b \right) = \lim_{b \rightarrow \infty} \frac{1 - e^{-pb}}{p} = \frac{1}{p},$$

$$\eta(t) \leftrightarrow \frac{1}{p}, \quad \operatorname{Re} p > 0. \bullet$$

Example 2.2. Find the Laplace transform of $f(t) = e^{\alpha t}$, where α is a constant.

Solution

By (2.1),

$$F(p) = \int_0^{\infty} e^{\alpha t} \cdot e^{-pt} dt = \int_0^{\infty} e^{-(p-\alpha)t} dt.$$

We know from Example 2.1 that

$$\int_0^{+\infty} e^{-pt} dt = \frac{1}{p}, \quad \operatorname{Re} p > 0.$$

Replacing p by $p - \alpha$ in the last equality, we get

$$F(p) = \int_0^{+\infty} e^{-(p-\alpha)t} dt = \frac{1}{p-\alpha}, \quad \operatorname{Re} p > \operatorname{Re} \alpha.$$

Hence, $e^{\alpha t} \leftrightarrow \frac{1}{p-\alpha}$, $\operatorname{Re} p > \operatorname{Re} \alpha$. $\bullet$

Tables of Laplace transforms. Using definition of the Laplace transform and its properties, the table of Laplace transforms have been compiled. The brief table of Laplace transforms is located in Appendix A.

Properties of Laplace transform

Theorem 2.1 (Linearity property). If $f_k(t) \leftrightarrow F_k(p)$ and λ_k are constants ($k = \overline{1, n}$), then

$$\sum_{k=1}^n \lambda_k f_k(t) \leftrightarrow \sum_{k=1}^n \lambda_k F_k(p).$$

Example 2.3. Use Theorem 2.1 and the known transform pair $e^{\omega t} \leftrightarrow \frac{1}{p - \alpha}$ to find the Laplace transform for the function $\sinh \omega t$ ($\omega \neq 0$).

Solution

By definition,

$$\sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}.$$

$$\text{Therefore, } \sinh \omega t = \frac{1}{2}e^{\omega t} - \frac{1}{2}e^{-\omega t} \leftrightarrow \frac{1}{2} \frac{1}{p - \omega} - \frac{1}{2} \frac{1}{p + \omega} = \frac{\omega}{p^2 - \omega^2}. \bullet$$

Theorem 2.2 (Similarity Theorem). If $f(t) \leftrightarrow F(p)$ and $a > 0$, then $f(at) \leftrightarrow \frac{1}{a} F\left(\frac{p}{a}\right)$.

Proof

$$f(at) \leftrightarrow \int_0^{\infty} f(at) e^{-pt} dt = \left| \begin{array}{l} t_1 = at, \\ \frac{1}{a} dt_1 = dt \end{array} \right| = \frac{1}{a} \int_0^{\infty} f(t_1) e^{-\frac{pt_1}{a}} dt_1 = \frac{1}{a} F\left(\frac{p}{a}\right). \bullet$$

Example 2.4. Use Theorem 2.2 and the known transform pair $\sinh t \leftrightarrow \frac{1}{p^2 - 1}$ to find the Laplace transform for the function $\sinh \omega t$ ($\omega \neq 0$).

Solution

$$\sinh \omega t \leftrightarrow \frac{1}{\omega} \frac{1}{\left(\frac{p}{\omega}\right)^2 - 1} = \frac{\omega}{p^2 - \omega^2}. \bullet$$

Theorem 2.3 (Shifting Theorem). If $f(t) \leftrightarrow F(p)$ and $a \in C$, then $e^{at} f(t) \leftrightarrow F(p - a)$.

Proof

$$e^{at} f(t) \leftrightarrow \int_0^{\infty} e^{at} f(t) e^{-pt} dt = \int_0^{\infty} f(t) e^{-(p-a)t} dt = F(p-a). \bullet$$

Example 2.5. Use Theorem 2.3 and the known Laplace transform of $\sin 5t$ to find the Laplace transform for the function $f(t) = e^{-9t} \cdot \sin 5t$.

Solution

Using the table of Laplace transforms (see Appendix A), we get

$$\sin 5t \leftrightarrow \frac{5}{p^2 + 25}.$$

By Theorem 2.3 for $a = -9$, we have

$$e^{-9t} \cdot \sin 5t \leftrightarrow \frac{5}{(p+9)^2 + 25}. \bullet$$

The *shifted* (or *delayed*) *unit step function* is defined as

$$\eta(t-a) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$

Theorem 2.4 (*Delay Theorem*). If $f(t) \leftrightarrow F(p)$ and $\tau \in R$, $\tau > 0$, then

$$f(t-\tau) \cdot \eta(t-\tau) \leftrightarrow e^{-\tau p} F(p).$$

Example 2.6. Use Theorem 2.4 and the known Laplace transform for $f(t) = t^2 \eta(t)$ to find the Laplace transform for the function $f(t) = (t-5)^2 \eta(t-5)$.

Solution

Using the table of Laplace transforms (see Appendix A), we get

$$t^2 \eta(t) \leftrightarrow \frac{2}{p^3}.$$

By Theorem 2.4, we obtain

$$(t-5)^2 \eta(t-5) \leftrightarrow e^{-5p} \cdot \frac{1}{p^3}.$$

But if we need to find the Laplace transform for the function $f(t) = (t-5)^2 \eta(t)$ we get:

$$\begin{aligned} f(t) &= (t-5)^2 \eta(t) = (t^2 - 10t + 25) \eta(t) = \\ &= t^2 \eta(t) - 10t \eta(t) + 25 \eta(t) \leftrightarrow \frac{2}{p^3} - \frac{10}{p^2} + \frac{25}{p}. \bullet \end{aligned}$$

Theorem 2.5 (*Differentiation of an Original Function*). Let the function $f(t)$ and its derivatives $f^{(k)}(t)$, $k=1,2,\dots,n$, be the original functions, and $f(t) \leftrightarrow F(p)$. Then

$$f'(t) \leftrightarrow pF(p) - f(0), \quad (2.2)$$

$$f''(t) \leftrightarrow p^2 F(p) - pf(0) - f'(0),$$

.....

$$f^{(n)}(t) \leftrightarrow p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \dots - f^{(n-1)}(0). \quad (2.3)$$

Proof

$$\begin{aligned} f'(t) \leftrightarrow \int_0^\infty f'(t) \cdot e^{-pt} dt &= \left| \begin{array}{l} u = e^{-pt}, \quad du = -p e^{-pt} dt \\ dv = f'(t) dt, \quad v = f(t) \end{array} \right| = \\ &= f(t) e^{-pt} \Big|_0^\infty + p \int_0^\infty f(t) \cdot e^{-pt} dt = -f(0) + pF(p). \end{aligned}$$

We get $f'(t) \leftrightarrow pF(p) - f(0)$. Using this formula, we find $f''(t)$:

$$f''(t) = (f'(t))' \leftrightarrow p(pF(p) - f(0)) - f'(0) = p^2 F(p) - pf(0) - f'(0).$$

Analogously, we find $f'''(t)$:

$$\begin{aligned} f'''(t) &= (f''(t))' \leftrightarrow p(p^2 F(p) - pf(0) - f'(0)) - f''(0) = \\ &= p^3 F(p) - p^2 f(0) - p f'(0) - f''(0). \end{aligned}$$

Using formula (2.2) $(n-1)$ times, we get formula (2.3). •

Formula (2.3) is used for solving the initial value problems for the linear differential equations with constant coefficients:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = f(x),$$

$$y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}.$$

Theorem 2.6 (Differentiation of a Transform). If $f(t) \leftrightarrow F(p)$, then

$$\begin{aligned} F'(p) &\leftrightarrow -t f(t), \\ F''(p) &\leftrightarrow t^2 f(t), \\ &\dots\dots\dots \\ F^{(n)}(p) &\leftrightarrow (-1)^n t^n f(t). \end{aligned} \quad (2.4)$$

Example 2.7. Find transforms of the following functions:

a) $f(t) = t \cdot \cosh 8t$; **b)** $f(t) = t^3 \cdot \sin 2t$.

Solution

a) Using the table of Laplace transforms (see Appendix A), we get

$$\cosh 8t \leftrightarrow \frac{p}{p^2 - 64}.$$

By Theorem 2.6, we obtain:

$$\begin{aligned} -t \cosh 8t &\leftrightarrow \left(\frac{p}{p^2 - 64} \right)', \\ t \cosh 8t &\leftrightarrow - \left(\frac{p}{p^2 - 64} \right)' = - \frac{p'(p^2 - 64) - p(p^2 - 64)'}{(p^2 - 64)^2} = \frac{p^2 + 64}{(p^2 - 64)^2}. \end{aligned}$$

$$\text{Thus, } F(p) = \frac{p^2 + 64}{(p^2 - 64)^2}.$$

b) Using the table of Laplace transforms (see Appendix A), we get

$$\sin 2t \leftrightarrow \frac{2}{p^2 + 4}.$$

Applying theorem 2.6, we have:

$$(-1)^3 \cdot t^3 \cdot \sin 2t \leftrightarrow \left(\frac{2}{p^2 + 4} \right)''', \quad t^3 \cdot \sin 2t \leftrightarrow - \left(\frac{2}{p^2 + 4} \right)''',$$

$$\left(\frac{2}{p^2+4}\right)' = -\frac{4p}{(p^2+4)^2}, \quad \left(\frac{2}{p^2+4}\right)'' = \left(-\frac{4p}{(p^2+4)^2}\right)' = \frac{12p^2-16}{(p^2+4)^3},$$

$$\left(\frac{2}{p^2+4}\right)''' = \left(\frac{12p^2-16}{(p^2+4)^3}\right)' = \frac{192p-48p^3}{(p^2+4)^4}.$$

$$\text{Hence, } F(p) = -\frac{192p-48p^3}{(p^2+4)^4} = \frac{48(p^3-3p)}{(p^2+4)^4}.$$

Theorem 2.7 (Integration of an Original Function). If $f(t) \leftrightarrow F(p)$, then $\int_0^t f(\tau) d\tau \leftrightarrow \frac{F(p)}{p}$ (integration of the original function corresponds to division of the transform by p).

Example 2.8. Find the Laplace transform of the function $\int_0^t \sin \frac{\tau}{5} d\tau$.

Solution

Using the table of Laplace transforms (see Appendix A), we get:

$$\sin \frac{t}{5} \leftrightarrow \frac{1/5}{p^2 + \left(\frac{1}{5}\right)^2} = \frac{5}{25p^2 + 1}. \text{ Applying Theorem 2.7, we obtain:}$$

$$\int_0^t \sin \frac{\tau}{5} d\tau \leftrightarrow \frac{5}{p(25p^2 + 1)}.$$

Theorem 2.8 (Integration of a Transform). If $\frac{f(t)}{t}$ is the original function and $f(t) \leftrightarrow F(p)$, then

$$\frac{f(t)}{t} \leftrightarrow \int_p^\infty F(p) dp$$

(Integration of the transform corresponds to division of the original function by t).

Example 2.9. Find the transform of the function $f(t) = \frac{1-e^{-3t}}{t}$.

Solution

Using the table of Laplace transforms (see Appendix A), we find

$1-e^{-3t} \leftrightarrow \frac{1}{p} - \frac{1}{p+3}$. Applying theorem 2.8, we obtain:

$$\begin{aligned} \frac{1-e^{-3t}}{t} &\leftrightarrow \int_p^{+\infty} \left(\frac{1}{\rho} - \frac{1}{\rho+3} \right) dp = \lim_{b \rightarrow +\infty} \int_p^b \left(\frac{1}{\rho} - \frac{1}{\rho+3} \right) dp = \\ &= \lim_{b \rightarrow +\infty} \left(\ln|\rho| - \ln|\rho+3| \right) \Big|_p^b = \lim_{b \rightarrow +\infty} \ln \left| \frac{\rho}{\rho+3} \right| \Big|_p^b = \lim_{b \rightarrow +\infty} \left(\ln \left| \frac{b}{b+3} \right| - \ln \left| \frac{p}{p+3} \right| \right) = \\ &= \lim_{b \rightarrow +\infty} \left(\ln \left| 1 - \frac{3}{b+3} \right| - \ln \left| \frac{p}{p+3} \right| \right) = -\ln \left| \frac{p}{p+3} \right|. \end{aligned}$$

Hence, $F(p) = -\ln \left| \frac{p}{p+3} \right|$. •

The convolution $f(t)*g(t)$ of two functions $f(t)$ and $g(t)$ is defined by the following formula

$$f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

It can be shown that $f*g = g*f$. This means that

$$\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Theorem 2.9 (The Convolution Theorem). If $f_1(t) \leftrightarrow F_1(p)$ and $f_2(t) \leftrightarrow F_2(p)$, then

$$f_1(t)*f_2(t) \leftrightarrow F_1(p) \cdot F_2(p).$$

Example 2.10. Evaluate the Laplace transform of the convolution integral $f(t) = \int_0^t (t - \tau)^3 \cos 5\tau d\tau$.

Solution

The given function is the convolution of two functions $f_1(t) = t^3$ and $f_2(t) = \cos 5t$. Using the table of Laplace transforms (see Appendix A), we find:

$$F_1(p) = \frac{3!}{p^4} = \frac{6}{p^4}, \quad F_2(p) = \frac{p}{p^2 + 25}.$$

Applying theorem 2.9, we obtain:

$$f(t) = f_1(t) * f_2(t) \leftrightarrow F_1(p) \cdot F_2(p) = \frac{6}{p^4} \cdot \frac{p}{p^2 + 25} = \frac{6}{p^3(p^2 + 25)}.$$

$$\text{Thus, } F(p) = \frac{6}{p^3(p^2 + 25)}.$$

2.2 Inverse Laplace Transform

Previously, we defined the Laplace transform of $f(t)$ by

$$F(p) = \int_0^{+\infty} e^{-pt} f(t) dt.$$

We'll also say that $f(t)$ is an inverse Laplace transform of $F(p)$.

Another notation is $L^{-1}(F(p)) = f(t)$.

The following methods are used to find the original function $f(t)$ if the Laplace transform $F(p)$ is known:

1) using the table of Laplace transforms (see Appendix A) and properties of the Laplace transform;

2) if $F(p)$ is a proper rational fraction, then it is decomposed into partial fractions, then using the properties of the Laplace transform the originals for each partial fraction are found.

Example 2.11. Use the table of Laplace transforms (Appendix A) to find the original functions for the given Laplace transforms:

$$a) F(p) = \frac{p}{p^2 + 9}; \quad b) F(p) = \frac{1}{p^2 - 36}.$$

Solution

a) Using formula 8 (see Appendix A), we get:

$$F(p) = \frac{p}{p^2 + 9} = \frac{p}{p^2 + 3^2} \leftrightarrow \cos 3t.$$

b) Using formula 9 (see Appendix A), we obtain:

$$F(p) = \frac{1}{p^2 - 36} = \frac{1}{6} \cdot \frac{6}{p^2 - 6^2} \leftrightarrow \frac{1}{6} \sinh 6t. \bullet$$

Example 2.12. Find the original functions for the given Laplace transforms:

$$\begin{aligned} a) F(p) &= \frac{7p + 11}{p^2 + 12p + 61}; & b) F(p) &= \frac{2p - 7}{p^2 + 10p - 56}; \\ c) F(p) &= \frac{3p^2 - 2p + 38}{(p^2 + 16)(p + 5)}; & d) F(p) &= \frac{9p^2 - 11p + 23}{(p - 4)(p^2 + 4p + 9)}; \\ e) F(p) &= \frac{5 \cdot e^{-3 \cdot p}}{(p + 8)^3}; & f) F(p) &= \frac{1}{(p^2 + 4)^2}. \end{aligned}$$

Solution

$$a) F(p) = \frac{7p + 11}{p^2 + 12p + 61}.$$

Let's complete the square in the denominator:

$$F(p) = \frac{7p + 11}{p^2 + 12p + 61} = \frac{7p + 11}{(p^2 + 2 \cdot 6p + 36) + 25} = \frac{7p + 11}{(p + 6)^2 + 5^2}.$$

Because of the form of the denominator, we represent $F(p)$ in the following form:

$$F(p) = \frac{7((p + 6) - 6) + 11}{(p + 6)^2 + 5^2} = \frac{7(p + 6) - 31}{(p + 6)^2 + 5^2} = \frac{7(p + 6)}{(p + 6)^2 + 5^2} - \frac{31}{(p + 6)^2 + 5^2} =$$

$$= 7 \cdot \frac{p+6}{(p+6)^2 + 5^2} - \frac{31}{5} \cdot \frac{5}{(p+6)^2 + 5^2}.$$

Using formulas 14 and 13 (see Appendix A), we find the original function $f(t)$ for the given transform $F(p)$:

$$f(t) = 7e^{-6t} \cos 5t - \frac{31}{5} e^{-6t} \sin 5t.$$

$$b) F(p) = \frac{2p-7}{p^2+10p-56}.$$

We factor the denominator of $F(p)$:

$$F(p) = \frac{2p-7}{(p+14)(p-4)}.$$

In order to find the original function, we decompose the obtained rational fraction into partial fractions.

$$\begin{aligned} \frac{2p-7}{(p+14)(p-4)} &= \frac{A}{p+14} + \frac{B}{p-4}, \\ \frac{2p-7}{(p+14)(p-4)} &= \frac{A(p-4)+B(p+14)}{(p+14)(p-4)}, \\ 2p-7 &= A(p-4)+B(p+14). \end{aligned}$$

Using the collocation method, we find the unknown coefficients:

$$\left. \begin{array}{l} p = -14 \\ p = 4 \end{array} \right| \begin{array}{l} -35 = -18A \\ 1 = 18B \end{array} \Rightarrow \begin{cases} A = \frac{35}{18}, \\ B = \frac{1}{18}. \end{cases}$$

$$F(p) = \frac{2p-7}{(p+14)(p-4)} = \frac{35/18}{p+14} + \frac{1/18}{p-4} = \frac{35}{18} \cdot \frac{1}{p+14} + \frac{1}{18} \cdot \frac{1}{p-4}.$$

Using formula 4 (see Appendix A), we find the original function $f(t)$ for the given transform $F(p)$:

$$f(t) = \frac{35}{18} e^{-14t} + \frac{1}{18} e^{4t}.$$

$$c) F(p) = \frac{3p^2 - 2p + 38}{(p^2 + 16)(p + 5)}.$$

In order to find the original function, we decompose the rational fraction $\frac{3p^2 - 2p + 38}{(p^2 + 16)(p + 5)}$ into partial fractions.

$$\begin{aligned}\frac{3p^2 - 2p + 38}{(p^2 + 16)(p + 5)} &= \frac{Ap + B}{p^2 + 16} + \frac{C}{p + 5}, \\ \frac{3p^2 - 2p + 38}{(p^2 + 16)(p + 5)} &= \frac{(Ap + B)(p + 5) + C(p^2 + 16)}{(p^2 + 16)(p + 5)}, \\ 3p^2 - 2p + 38 &= (Ap + B)(p + 5) + C(p^2 + 16), \\ 3p^2 - 2p + 38 &= Ap^2 + 5Ap + Bp + 5B + Cp^2 + 16C, \\ 3p^2 - 2p + 38 &= (A + C)p^2 + (5A + B)p + 5B + 16C.\end{aligned}$$

To find the unknown coefficients, we use the method of undetermined coefficients:

$$\left. \begin{array}{l} p^2 \\ p \\ p^0 \end{array} \right| \begin{array}{l} 3 = A + C, \\ -2 = 5A + B, \\ 38 = 5B + 16C \end{array} \Rightarrow \begin{cases} A = 0, \\ B = -2, \\ C = 3. \end{cases}$$

$$F(p) = \frac{-2}{p^2 + 16} + \frac{3}{p + 5} = -\frac{1}{2} \cdot \frac{4}{p^2 + 4^2} + 3 \cdot \frac{1}{p + 5}.$$

Using formulas 7 and 4 (see Appendix A), we find the original function $f(t)$ for the given transform $F(p)$:

$$f(t) = -\frac{1}{2} \sin 4t + 3e^{-5t}.$$

$$d) F(p) = \frac{9p^2 - 11p + 23}{(p - 4)(p^2 + 4p + 9)}.$$

In order to find the original function, we decompose the rational fraction $\frac{9p^2 - 11p + 23}{(p - 4)(p^2 + 4p + 9)}$ into partial fractions:

$$\frac{9p^2 - 11p + 23}{(p - 4)(p^2 + 4p + 9)} = \frac{A}{p - 4} + \frac{Bp + C}{p^2 + 4p + 9},$$

$$\frac{9p^2 - 11p + 23}{(p-4)(p^2 + 4p + 9)} = \frac{A(p^2 + 4p + 9) + (Bp + C)(p-4)}{(p-4)(p^2 + 4p + 9)},$$

$$9p^2 - 11p + 23 = A(p^2 + 4p + 9) + (Bp + C)(p-4),$$

$$9p^2 - 11p + 23 = (A+B)p^2 + (4A-4B+C)p + 9A-4C.$$

Combining the collocation method and the method of undetermined coefficients, we find the unknown coefficients A , B and C :

$$\begin{array}{l|l} p=4 & 123 = 4A, \\ p^2 & 9 = A+B, \\ p & -11 = 4A-4B+C \end{array} \Rightarrow \begin{cases} A=3, \\ B=6, \\ C=1. \end{cases}$$

$$F(p) = \frac{3}{p-4} + \frac{6p+1}{p^2+4p+9}.$$

$$\begin{aligned} F(p) &= \frac{3}{p-4} + \frac{6(p+2)-11}{(p+2)^2 + (\sqrt{5})^2} = \\ &= 3 \cdot \frac{1}{p-4} + 6 \cdot \frac{p+2}{(p+2)^2 + (\sqrt{5})^2} - \frac{11}{\sqrt{5}} \cdot \frac{\sqrt{5}}{(p+2)^2 + (\sqrt{5})^2}. \end{aligned}$$

Using formulas 4, 14 and 13 (see Appendix A), we find the original function $f(t)$ for the given transform $F(p)$:

$$f(t) = 3e^{4t} + 6e^{-2p} \cos \sqrt{5}t - \frac{11}{\sqrt{5}} e^{-2p} \sin \sqrt{5}t.$$

$$e) F(p) = \frac{5 \cdot e^{-3 \cdot p}}{(p+8)^3}.$$

Using formula 5 (see Appendix A) and delay theorem with $\tau=3$, we obtain:

$$\frac{5}{(p+8)^3} = \frac{5}{2!} \cdot \frac{2!}{(p+8)^3} \leftrightarrow \frac{5}{2!} t^2 e^{-8t}, \quad \frac{5e^{-3p}}{(p+8)^2} \leftrightarrow \frac{5(t-3)^2 e^{-8(t-3)}}{2!} \eta(t-3).$$

$$\text{Hence, } f(t) = \frac{5(t-3)^2 e^{-8(t-3)}}{2!} \eta(t-3).$$

$$f) F(p) = \frac{1}{(p^2+4)^2}.$$

By theorem 2.9, we obtain:

$$\begin{aligned}
 F(p) &= \frac{1}{(p^2 + 4)^2} = \frac{1}{p^2 + 4} \cdot \frac{1}{p^2 + 4} \leftrightarrow \int_0^t \frac{1}{2} \sin 2(t - \tau) \cdot \frac{1}{2} \sin 2\tau d\tau = \\
 &= \frac{1}{4} \int_0^t \sin 2(t - \tau) \cdot \sin 2\tau d\tau = \frac{1}{4} \cdot \frac{1}{2} \int_0^t (\cos 2(t - 2\tau) - \cos 2t) d\tau = \\
 &= \frac{1}{8} \cdot \left(-\frac{1}{4} \sin 2(t - 2\tau) - \cos 2t \cdot \tau \right) \Bigg|_0^t = \frac{1}{16} \sin 2t - \frac{1}{8} t \cos 2t.
 \end{aligned}$$

Thus, $f(t) = \frac{1}{16} \sin 2t - \frac{1}{8} t \cos 2t$. •

2.3 Application of Operational Calculus to Solving Linear Differential Equations with Constant Coefficients

Statement of the problem. Using Laplace transformation, solve the initial value problem:

$$y'' + a_1 y' + a_2 y = f(t), \quad y(0) = y_0, \quad y'(0) = y_1. \quad (2.5)$$

Let $y(t) \leftrightarrow Y(p)$, $f(t) = F(p)$, then by the differentiation theorem for the original function we have:

$$y'(t) \leftrightarrow pY(p) - y_0, \quad y''(t) \leftrightarrow p^2 Y(p) - py_0 - y_1.$$

Taking Laplace transforms of both sides of the differential equation in (2.5), we get the operator equation

$$p^2 Y(p) - py_0 - y_1 + a_1(pY(p) - y_0) + a_2 Y(p) = F(p). \quad (2.6)$$

We solve the equation and find the unknown function $Y(p)$:

$$Y(p) = \frac{F(p) + py_0 + a_1 y_0 + y_1}{p^2 + a_1 p + a_2}. \quad (2.7)$$

Applying the inverse Laplace transform to (2.7), we get the solution of (2.5).

Analogously, the initial value problem for linear differential equation (NLDE) of the n^{th} order with constant coefficients can be solved.

Example 2.13. Use the Laplace transform to solve the initial value problem:

$$y'' - 4y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution

Let $y(t) \leftrightarrow Y(p)$, then by the differentiation theorem for the original we have:

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 1,$$

$$y''(t) \leftrightarrow p^2Y(p) - py(0) - y'(0) = p^2Y(p) - p - 2.$$

We get the operator equation:

$$(p^2Y(p) - p - 2) - 4(pY(p) - 1) + 3Y(p) = 0.$$

Solving the obtained equation for $Y(p)$, we obtain:

$$(p^2 - 4p + 3)Y(p) = p - 2,$$

$$Y(p) = \frac{p - 2}{p^2 - 4p + 3},$$

In order to find the original function $y(t)$, we factor the denominator and decompose $Y(p)$ into partial fractions:

$$Y(p) = \frac{p - 2}{(p - 1)(p - 3)} = \frac{A}{p - 1} + \frac{B}{p - 3},$$

$$p - 2 = A(p - 3) + B(p - 1),$$

$$\left. \begin{array}{l} p = 1 \\ p = 3 \end{array} \right| \begin{array}{l} -1 = -2A \\ 1 = 2B \end{array} \Rightarrow \begin{cases} A = \frac{1}{2}, \\ B = \frac{1}{2}. \end{cases}$$

$$Y(p) = \frac{1}{2} \frac{1}{p - 1} + \frac{1}{2} \frac{1}{p - 3}$$

Using the table of Laplace transforms (see Appendix A) we find:

$$\frac{1}{p - 1} \leftrightarrow e^t, \quad \frac{1}{p - 3} \leftrightarrow e^{3t}.$$

Therefore, $y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{3t}$. •

Example 2.14. Use the Laplace transform to solve the initial value problem:

$$y'' - 2y' - 8y = 6e^{-2t}, \quad y(0) = 2, \quad y'(0) = -5.$$

Solution

Let $y(t) \leftrightarrow Y(p)$ then by the differentiation theorem for the original function we have:

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 2,$$

$$y''(t) \leftrightarrow p^2 Y(p) - p y(0) - y'(0) = p^2 Y(p) - 2p + 5.$$

In addition, using the table of Laplace transforms (see Appendix A) we get $6 \cdot e^{-2t} \leftrightarrow \frac{6}{p+2}$. The operator equation has the following form:

$$p^2 Y(p) - 2p + 5 - 2(pY(p) - 2) - 8Y(p) = \frac{6}{p+2}.$$

Solving the obtained equation for $Y(p)$, we get:

$$(p^2 - 2p - 8)Y(p) = 2p - 9 + \frac{6}{p+2},$$

$$(p-4)(p+2)Y(p) = \frac{2p^2 - 5p - 12}{p+2},$$

$$Y(p) = \frac{2p^2 - 5p - 12}{(p-4)(p+2)^2}.$$

This is the operator solution. In order to find the original function $y(t)$, we

decompose the rational fraction $\frac{2p^2 - 5p - 12}{(p-4)(p+2)^2}$ into partial fractions:

$$\frac{2p^2 - 5p - 12}{(p-4)(p+2)^2} = \frac{A}{p-4} + \frac{B}{(p+2)^2} + \frac{C}{p+2},$$

$$\frac{2p^2 - 5p - 12}{(p-4)(p+2)^2} = \frac{A(p+2)^2 + B(p-4) + C(p-4)(p+2)}{(p-4)(p+2)^2},$$

$$2p^2 - 5p - 12 = A(p+2)^2 + B(p-4) + C(p-4)(p+2),$$

$$\begin{array}{l|l} p=4 & 0=36A, \\ p=-2 & 6=-6B, \\ p^2 & 2=A+C. \end{array} \Rightarrow \begin{cases} A=0, \\ B=-1, \\ C=2. \end{cases}$$

$$Y(p) = \frac{-1}{(p+2)^2} + \frac{2}{p+2} = -\frac{1}{(p+2)^2} + 2 \cdot \frac{1}{p+2}.$$

Using the table of Laplace transforms (see Appendix A), we find:

$$\frac{1}{(p+2)^2} \leftrightarrow t e^{-2t}, \quad \frac{1}{p+2} \leftrightarrow e^{-2t}.$$

So, $y(t) = -t e^{-2t} + 2e^{-2t}$. •

Example 2.15. Use the Laplace transform to solve the initial value problem:

$$y'' + 3y' - 10y = 47 \cos 3t - \sin 3t, \quad y(0) = 3, \quad y'(0) = -1.$$

Solution

Let $y(t) \leftrightarrow Y(p)$ then by the differentiation theorem for the original function we have:

$$y' \leftrightarrow pY(p) - y(0) = pY(p) - 3,$$

$$y'' \leftrightarrow p^2 Y(p) - py(0) - y'(0) = p^2 Y(p) - 3p + 1.$$

Using the table of transforms (Appendix A), we obtain:

$$47 \cos 3t - \sin 3t \leftrightarrow 47 \cdot \frac{p}{p^2 + 9} - \frac{3}{p^2 + 9} = \frac{47p - 3}{p^2 + 9}.$$

The operator equation has the following form:

$$(p^2 Y(p) - 3p + 1) + 3(pY(p) - 3) - 10Y(p) = \frac{47p - 3}{p^2 + 9}.$$

Solving the obtained equation for $Y(p)$, we get:

$$(p^2 + 3p - 10)Y(p) = \frac{47p - 3}{p^2 + 9} + 3p + 8,$$

$$(p - 2)(p + 5)Y(p) = \frac{47p - 3 + (3p + 8)(p^2 + 9)}{p^2 + 9},$$

$$Y(p) = \frac{3p^3 + 8p^2 + 74p + 69}{(p-2)(p+5)(p^2+9)}.$$

This is the operator solution. In order to find the original function $y(t)$, we decompose $Y(p)$ into partial fractions:

$$\begin{aligned} \frac{3p^3 + 8p^2 + 74p + 69}{(p-2)(p+5)(p^2+9)} &= \frac{A}{p-2} + \frac{B}{p+5} + \frac{Cp+D}{p^2+9}, \\ \frac{3p^3 + 8p^2 + 74p + 69}{(p-2)(p+5)(p^2+9)} &= \\ &= \frac{A(p+5)(p^2+9) + B(p-2)(p^2+9) + (Cp+D)(p-2)(p+5)}{(p-2)(p+5)(p^2+9)}, \\ 3p^3 + 8p^2 + 74p + 69 &= A(p+5)(p^2+9) + B(p-2)(p^2+9) + \\ &\quad + Cp(p+5)(p-2) + D(p+5)(p-2), \\ \left. \begin{array}{l} p=2 \\ p=-5 \\ p^3 \\ p^0 \end{array} \right| \begin{array}{l} 273 = 91 \cdot A, \\ -476 = -238 \cdot B, \\ 3 = A + B + C, \\ 69 = 45A - 18B - 10D. \end{array} &\Rightarrow \left\{ \begin{array}{l} A=3, \\ B=2, \\ C=-2, \\ D=3. \end{array} \right. \end{aligned}$$

Therefore,

$$Y(p) = \frac{3}{p-2} + \frac{2}{p+5} - \frac{2p}{p^2+9} + \frac{3}{p^2+9}.$$

Using the table of Laplace transforms (see Appendix A), we find:

$$\frac{1}{p-2} \leftrightarrow e^{2t}, \quad \frac{1}{p+5} \leftrightarrow e^{-5t}, \quad \frac{p}{p^2+9} \leftrightarrow \cos 3t, \quad \frac{3}{p^2+9} \leftrightarrow \sin 3t.$$

$$\text{So, } y(t) = 3e^{2t} + 2e^{-5t} - 2\cos 3t + \sin 3t. \bullet$$

Example 2.16. Use the Laplace transform to solve the initial value problem

$$y'' + 2y' + y = t^2, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution

Let $y(t) \leftrightarrow Y(p)$ then by the differentiation theorem for the original function we have:

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 1,$$

$$y''(t) \leftrightarrow p^2Y(p) - p y(0) - y'(0) = p^2Y(p) - p.$$

Using the table of Laplace transforms (see Appendix A), we get

$$t^2 \leftrightarrow \frac{2}{p^3}.$$

The operator equation has the following form:

$$p^2Y(p) - p + 2(pY(p) - 1) + Y(p) = \frac{2}{p^3}.$$

Solving the obtained equation for $Y(p)$, we get:

$$(p^2 + 2p + 1)Y(p) = \frac{2}{p^3} + p + 2,$$

$$Y(p) = \frac{p^4 + 2p^3 + 2}{p^3(p+1)^2}.$$

This is an operator solution. In order to find the original function, we

decompose $\frac{p^4 + 2p^3 + 2}{p^3(p+1)^2}$ into partial fractions:

$$\frac{p^4 + 2p^3 + 2}{p^3(p+1)^2} = \frac{A}{p^3} + \frac{B}{p^2} + \frac{C}{p} + \frac{D}{(p+1)^2} + \frac{E}{p+1},$$

$$p^4 + 2p^3 + 2 = A(p+1)^2 + Bp(p+1)^2 + Cp^2(p+1)^2 + Dp^3 + Ep^3(p+1),$$

$$p^4 + 2p^3 + 2 = A(p^2 + 2p + 1) + B(p^3 + 2p^2 + p) + C(p^4 + 2p^3 + p^2) + Dp^3 + E(p^4 + p^3),$$

$$\begin{array}{l|l} p=0 & 2 = A \\ p=-1 & 1 = -D \\ p^4 & 1 = C + E \\ p^2 & 0 = A + 2B + C, \\ p & 0 = 2A + B, \end{array} \Rightarrow \begin{cases} A = 2, \\ B = -4, \\ C = 6, \\ D = -1, \\ E = -5. \end{cases}$$

$$Y(p) = \frac{2}{p^3} - \frac{4}{p^2} + \frac{6}{p} - \frac{1}{(p+1)^2} - \frac{5}{p+1}.$$

Using the table of Laplace transforms (see Appendix A), we find:

$$\frac{2}{p^3} \leftrightarrow t^2, \frac{1}{p^2} \leftrightarrow t, \frac{1}{p} \leftrightarrow 1, \frac{1}{(p+1)^2} \leftrightarrow te^{-t}, \frac{1}{p+1} \leftrightarrow e^{-t}.$$

Therefore, $y(t) = t^2 - 4t + 6 - te^{-t} - 5e^{-t}$. •

2.4 Application of Operational Calculus to Solving Systems of Differential Equations with Constant Coefficients

The solution of the initial value problem for the systems of linear differential equations with constant coefficients is performed according to the same scheme as for the linear differential equations. Applying the Laplace transformation to the equations of the system reduces it to a system of linear algebraic equations in the transforms of the sought functions. Using inverse Laplace transform to the found transforms we obtain the solutions of the initial value problem for the given system.

Example 2.17. Use Laplace transform to solve the initial value problem for the system

$$\begin{cases} x' = 4x - y, & x(0) = -3, \\ y' = -x + 4y, & y(0) = 6. \end{cases}$$

Solution

Let $x(t) \leftrightarrow X(p)$, $y(t) \leftrightarrow Y(p)$ then by the differentiation theorem for the original function we have:

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) + 3,$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 6.$$

We write the considered system in the operator form:

$$\begin{cases} pX(p) + 3 = 4X(p) - Y(p), \\ pY(p) - 6 = -X(p) + 4Y(p). \end{cases} \Rightarrow \begin{cases} (p-4)X(p) + Y(p) = -3, \\ X(p) + (p-4)Y(p) = 6. \end{cases}$$

Using Cramer's rule, we solve the obtained system for $X(p)$ and $Y(p)$:

$$\Delta = \begin{vmatrix} p-4 & 1 \\ 1 & p-4 \end{vmatrix} = p^2 - 8p + 15, \quad \Delta_1 = \begin{vmatrix} -3 & 1 \\ 6 & p-4 \end{vmatrix} = -3p + 6,$$

$$\Delta_2 = \begin{vmatrix} p-4 & -3 \\ 1 & 6 \end{vmatrix} = 6p - 21,$$

$$X(p) = \frac{-3p+6}{p^2-8p+15}, Y(p) = \frac{6p-21}{p^2-8p+15} \text{ is the operator solution.}$$

In order to find the original function $x(t)$, we decompose $X(p)$ into partial fractions:

$$X(p) = \frac{-3p+6}{p^2-8p+15} = \frac{-3p+6}{(p-5)(p-3)} = \frac{A}{p-5} + \frac{B}{p-3},$$

$$-3p+6 = A(p-3) + B(p-5),$$

$$\left. \begin{array}{l} p=5 \\ p=3 \end{array} \right| \begin{array}{l} -9=2A \\ -3=-2B \end{array} \Rightarrow \begin{cases} A = -\frac{9}{2}, \\ B = \frac{3}{2}. \end{cases}$$

$$X(p) = \frac{-9/2}{p-5} + \frac{3/2}{p-3}.$$

We decompose $Y(p)$ into partial fractions to find the original function $y(t)$:

$$Y(p) = \frac{6p-21}{p^2-8p+15} = \frac{6p-21}{(p-5)(p-3)} = \frac{A}{p-5} + \frac{B}{p-3},$$

$$6p-21 = A(p-3) + B(p-5),$$

$$\left. \begin{array}{l} p=5 \\ p=3 \end{array} \right| \begin{array}{l} 9=2A \\ -3=-2B \end{array} \Rightarrow \begin{cases} A = \frac{9}{2}, \\ B = \frac{3}{2}. \end{cases}$$

$$Y(p) = \frac{9/2}{p-5} + \frac{3/2}{p-3}.$$

Using the table of Laplace transforms (see Appendix A), we obtain:

$$x(t) = -\frac{9}{2}e^{5t} + \frac{3}{2}e^{3t}, \quad y(t) = \frac{9}{2}e^{5t} + \frac{3}{2}e^{3t}.$$

So, the solution of the given system has the following form:

$$\begin{cases} x(t) = -\frac{9}{2}e^{5t} + \frac{3}{2}e^{3t}, \\ y(t) = \frac{9}{2}e^{5t} + \frac{3}{2}e^{3t}. \end{cases}$$

Example 2.18. Use Laplace transform to solve the initial value problem for the system

$$\begin{cases} x' = 2x + 4y + 5\cos 2t, & x(0) = 0, \\ y' = -x - 2y + 3\sin 2t, & y(0) = 0. \end{cases}$$

Solution

Let $x(t) \leftrightarrow X(p)$, $y(t) \leftrightarrow Y(p)$ then by the differentiation theorem for the original function we have:

$$\begin{aligned} x'(t) &\leftrightarrow pX(p) - x(0) = pX(p), \\ y'(t) &\leftrightarrow pY(p) - y(0) = pY(p). \end{aligned}$$

Using the table of transforms (Appendix A), we obtain:

$$5\cos 2t \leftrightarrow \frac{5p}{p^2 + 4}, \quad 3\sin 2t \leftrightarrow \frac{6}{p^2 + 4}.$$

We write the system in the operator form:

$$\begin{cases} pX(p) = 2X(p) + 4Y(p) + \frac{5p}{p^2 + 4}, \\ pY(p) = -X(p) - 2Y(p) + \frac{6}{p^2 + 4}, \end{cases} \Rightarrow \begin{cases} (p-2)X(p) - 4Y(p) = \frac{5p}{p^2 + 4}, \\ X(p) + (p+2)Y(p) = \frac{6}{p^2 + 4}. \end{cases}$$

Using Cramer's rule, we solve the obtained system for $X(p)$ and $Y(p)$:

$$\Delta = \begin{vmatrix} p-2 & -4 \\ 1 & p+2 \end{vmatrix} = p^2, \quad \Delta_1 = \begin{vmatrix} \frac{5p}{p^2 + 4} & -4 \\ \frac{6}{p^2 + 4} & p+2 \end{vmatrix} = \frac{5p^2 + 10p + 24}{p^2 + 4},$$

$$\Delta_2 = \begin{vmatrix} p-2 & \frac{5p}{p^2 + 4} \\ 1 & \frac{6}{p^2 + 4} \end{vmatrix} = \frac{p-12}{p^2 + 4}.$$

$$X(p) = \frac{5p^2 + 10p + 24}{p^2(p^2 + 4)}, \quad Y(p) = \frac{p - 12}{p^2(p^2 + 4)} \text{ is the operator solution.}$$

In order to find the original function $x(t)$, we decompose $X(p)$ into partial fractions:

$$\begin{aligned} X(p) &= \frac{5p^2 + 10p + 24}{p^2(p^2 + 4)} = \frac{A}{p^2} + \frac{B}{p} + \frac{Cp + D}{p^2 + 4}, \\ 5p^2 + 10p + 24 &= A(p^2 + 4) + Bp(p^2 + 4) + p^2(Cp + D), \\ 5p^2 + 10p + 24 &= (B + C)p^3 + (A + D)p^2 + 4Bp + 4A. \end{aligned}$$

$$\begin{array}{l|l} p^3 & 0 = B + C, \\ p^2 & 5 = A + D, \\ p^1 & 10 = 4B, \\ p^0 & 24 = 4A. \end{array} \Rightarrow \begin{cases} A = 6, \\ B = \frac{5}{2}, \\ C = -\frac{5}{2}, \\ D = -1. \end{cases}$$

$$X(p) = \frac{6}{p^2} + \frac{5/2}{p} + \frac{-5/2 p - 1}{p^2 + 4} = 6 \cdot \frac{1}{p^2} + \frac{5}{2} \cdot \frac{1}{p} - \frac{5}{2} \cdot \frac{p}{p^2 + 4} - \frac{1}{p^2 + 4}.$$

We decompose $Y(p)$ into partial fractions to find the original function $y(t)$:

$$\begin{aligned} Y(p) &= \frac{p - 12}{p^2(p^2 + 4)} = \frac{A}{p^2} + \frac{B}{p} + \frac{Cp + D}{p^2 + 4}, \\ p - 12 &= A(p^2 + 4) + Bp(p^2 + 4) + p^2(Cp + D), \\ p - 12 &= (B + C)p^3 + (A + D)p^2 + 4Bp + 4A, \end{aligned}$$

$$\begin{array}{l|l} p^3 & 0 = B + C, \\ p^2 & 0 = A + D, \\ p^1 & 1 = 4B, \\ p^0 & -12 = 4A. \end{array} \Rightarrow \begin{cases} A = -3, \\ B = \frac{1}{4}, \\ C = -\frac{1}{4}, \\ D = 3. \end{cases}$$

$$Y(p) = \frac{-3}{p^2} + \frac{1/4}{p} + \frac{-1/4 p + 3}{p^2 + 4} = -3 \cdot \frac{1}{p^2} + \frac{1}{4} \cdot \frac{1}{p} - \frac{1}{4} \cdot \frac{p}{p^2 + 4} + 3 \cdot \frac{1}{p^2 + 4}.$$

Using the table of Laplace transforms (see Appendix A), we find the originals for $X(p)$ and $Y(p)$:

$$x(t) = 6t + \frac{5}{2} - \frac{5}{2} \cos 2t - \frac{1}{2} \sin 2t, \quad y(t) = -3t + \frac{1}{4} - \frac{1}{4} \cos 2t + \frac{3}{2} \sin 2t.$$

Consequently, the solution of the given system is

$$\begin{cases} x(t) = 6t + \frac{5}{2} - \frac{5}{2} \cos 2t - \frac{1}{2} \sin 2t, \\ y(t) = -3t + \frac{1}{4} - \frac{1}{4} \cos 2t + \frac{3}{2} \sin 2t. \end{cases} \bullet$$

REFERENCES

1. Валяшек В. Б. Теорія функцій комплексної змінної. Конспект лекцій для студентів технічних спеціальностей усіх форм навчання / В. А. Кривень, В. Б. Валяшек, А. В. Каплун, О. П. Ясній. – Тернопіль : ТНТУ, 2015. – 87 с.
2. Вища математика : підручник: у 2 кн. / Г. Л. Кулініч, Є. Ю. Таран, В. М. Бурим та ін.; за ред. Г. Л. Кулініча. – 2-ге вид., перероб. і доп. – К. : Либідь, 2003. – Кн. 2. Спеціальні розділи. – 2003. – 368 с.
3. Войцеховський О. А. Операційне числення : навчальний посібник з вищої математики для студентів усіх спеціальностей / О. А. Войцеховський. – Вінниця : ВДТУ, 2000. – 148 с.
4. Воронка Р. О. Англо-український математичний словник / Укл. : Р. О. Воронка, М. І. Кратко, Є. В. Мейнарович, В. О. Павленко; ред. І. А. Черненко. – К. : НВП «Дидактик», 1993. – 224 с.
5. Дубовик В. П. Вища математика : навч. посіб. для студ. вищ. навч. закл. / В. П. Дубовик, І. І. Юрик. – К. : Ігнатекс-Україна, 2013. – 648 с.
6. Мартиненко М. А. Теорія функції комплексної змінної. Операційне числення: навч. посібник / М. А. Мартиненко, І. І. Юрик. – Київ : «Слово», 2013. – 296 с.
7. Овчинников П. П. Вища математика : підруч. для студ. : у 2 ч. Ч. 2: Диференціальні рівняння. Операційне числення. Ряди та їх застосування. Стійкість за Ляпуновим. Рівняння математичної фізики. Оптимізація та керування. Теорія ймовірностей. Числові методи; за заг. ред. П. П. Овчинникова. – К. : Техніка, 2004. – 792 с.
8. Приклади розв'язання типових завдань розрахункової роботи з вищої математики з теми «Елементи операційного числення» для студентів технічних спеціальностей денної форми навчання / Укл. : Н. М. Антоненко. – Запоріжжя : НУ «Запорізька політехніка», 2024. – 40 с.
9. Тевяшев А. Д. Вища математика у прикладах та задачах. Ч. 3. Диференціальні рівняння Ряди Функції комплексної змінної Операційне числення / А. Д. Тевяшев, О. Г. Литвин, Г. М. Кривошеєва, та ін. – Харків : ХНУР, 2002. – 596 с.
10. Ahlfors L. V. Complex Analysis / L. V. Ahlfors. – McGrah Hill, 1966. – 317 p.

11. Beerends R. J. Fourier and Laplace Transforms / R. J. Beerends, H. G. ter Morsche, J. C. van den Berg and E. M. van de Vrie Frontmatter. – Cambridge University Press, 2003. – 458 p.
12. Bird J. Higher Engineering Mathematics / J. Bird. – Amsterdam-Tokyo : Newnes, 2010. – 705 p.
13. Kodaira K. Introduction to Complex Analysis / K. Kodaira. – Cambridge University Press, 2007. – 418 p.

Appendix A

Table of Laplace Transforms

№	Original function $f(t)$	Laplace transform $F(p)$	№	Original function $f(t)$	Laplace transform $F(p)$
1	C	$\frac{C}{p}$	12	$\cos(t - \tau)$	$\frac{p \cdot e^{-\tau \cdot p}}{p^2 + 1}$
2	$t^n,$ $n = 1, 2, 3, \dots$	$\frac{n!}{p^{n+1}}$	13	$e^{\lambda t} \cdot \sin \omega t$	$\frac{\omega}{(p - \lambda)^2 + \omega^2}$
3	$t^\alpha,$ $\alpha > -1$	$\frac{\Gamma(\alpha + 1)}{p^{\alpha + 1}}$	14	$e^{\lambda t} \cdot \cos \omega t$	$\frac{p - \lambda}{(p - \lambda)^2 + \omega^2}$
4	$e^{\lambda t},$ $\lambda = a + i \cdot b$	$\frac{1}{p - \lambda}$	15	$e^{\lambda t} \cdot \sinh \omega t$	$\frac{\omega}{(p - \lambda)^2 - \omega^2}$
5	$t^n e^{\lambda t}$	$\frac{n!}{(p - \lambda)^{n+1}}$	16	$e^{\lambda t} \cdot \cosh \omega t$	$\frac{p - \lambda}{(p - \lambda)^2 - \omega^2}$
6	$t^\alpha e^{\lambda t}$	$\frac{\Gamma(\alpha + 1)}{(p - \lambda)^{\alpha + 1}}$	17	$t \cdot \sin \omega t$	$\frac{2p\omega}{(p^2 + \omega^2)^2}$
7	$\sin \omega t$	$\frac{\omega}{p^2 + \omega^2}$	18	$t \cdot \cos \omega t$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$
8	$\cos \omega t$	$\frac{p}{p^2 + \omega^2}$	19	$t \cdot \sinh \omega t$	$\frac{2p\omega}{(p^2 - \omega^2)^2}$
9	$\sinh \omega t$	$\frac{\omega}{p^2 - \omega^2}$	20	$t \cdot \cosh \omega t$	$\frac{p^2 + \omega^2}{(p^2 - \omega^2)^2}$
10	$\cosh \omega t$	$\frac{p}{p^2 - \omega^2}$	21	$t^n \cdot \sin \omega t$	$\frac{\text{Im}(p + i\omega)^{n+1}}{(p^2 + \omega^2)^{n+1}} \cdot n!$
11	$\sin(t - \tau),$ $\tau > 0$	$\frac{e^{-\tau \cdot p}}{p^2 + 1}$	22	$t^n \cdot \cos \omega t$	$\frac{\text{Re}(p + i\omega)^{n+1}}{(p^2 + \omega^2)^{n+1}} \cdot n!$

Appendix B
Table of Derivatives
($u = u(x)$ is a differentiable function)

1. $(C)' = 0, C - \text{const};$	11. $(\tan u)' = \frac{1}{\cos^2 u} \cdot u';$
2. $(x)' = 1;$	12. $(\cot u)' = -\frac{1}{\sin^2 u} \cdot u';$
3. $(u^n)' = n \cdot u^{n-1} \cdot u';$	13. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u';$
4. $(\sqrt{u})' = \frac{1}{2\sqrt{u}} \cdot u';$	14. $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u';$
5. $(a^u)' = a^u \cdot \ln a \cdot u', a - \text{const};$	15. $(\arctan u)' = \frac{1}{1+u^2} \cdot u';$
6. $(e^u)' = e^u \cdot u';$	16. $(\cot^{-1} u)' = -\frac{1}{1+u^2} \cdot u';$
7. $(\log_a u)' = \frac{1}{u \cdot \ln a} \cdot u';$	17. $(\sinh u)' = \cosh u \cdot u';$
8. $(\ln u)' = \frac{1}{u} \cdot u';$	18. $(\cosh u)' = \sinh u \cdot u';$
9. $(\sin u)' = \cos u \cdot u';$	19. $(\tanh u)' = \frac{1}{\cosh^2 u} \cdot u';$
10. $(\cos u)' = -\sin u \cdot u';$	20. $(\coth u)' = -\frac{1}{\sinh^2 u} \cdot u'.$

Appendix C
Table of Basic Integrals
($u = u(x)$ is a differentiable function)

1. $\int du = u + C ;$	12. $\int \frac{du}{\sin^2 u} = -\cot u + C ;$
2. $\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1 ;$	13. $\int \frac{du}{\sin u} = \ln \left \tan \frac{u}{2} \right + C ;$
3. $\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C ;$	14. $\int \frac{du}{\cos u} = \ln \left \tan \left(\frac{u}{2} + \frac{\pi}{4} \right) \right + C ;$
4. $\int \frac{du}{u} = \ln u + C ;$	15. $\int \sinh u du = \cosh u + C ;$
5. $\int a^u du = \frac{a^u}{\ln a} + C ;$	16. $\int \cosh u du = \sinh u + C ;$
6. $\int e^u du = e^u + C ;$	17. $\int \frac{du}{\cosh^2 u} = \tanh u + C ;$
7. $\int \sin u du = -\cos u + C ;$	18. $\int \frac{du}{\sinh^2 u} = -\coth u + C ;$
8. $\int \cos u du = \sin u + C ;$	19. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a} + C ;$
9. $\int \operatorname{tgu} du = -\ln \cos u + C ;$	20. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left \frac{u-a}{u+a} \right + C ;$
10. $\int \cot u du = \ln \sin u + C ;$	21. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln \left u + \sqrt{u^2 \pm a^2} \right + C ;$
11. $\int \frac{du}{\cos^2 u} = \tan u + C ;$	22. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C ;$

$23. \int \sqrt{a^2 - u^2} du = \frac{1}{2} u \sqrt{a^2 - u^2} +$ $+ \frac{1}{2} a^2 \arcsin \frac{u}{a} + C ;$	$25. \int \frac{f'(x)}{f(x)} dx = \ln f(x) + C ;$
$24. \int \sqrt{u^2 \pm a^2} du = \frac{1}{2} u \sqrt{u^2 \pm a^2} \pm$ $\pm \frac{1}{2} a^2 \ln \left u + \sqrt{u^2 \pm a^2} \right + C ;$	$26. \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C .$