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THEORETICAL ELECTRICAL ENGINEERING

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This book is devoted to my Teacher and Friend,
Professor Rassalskiy Alexander Nikolaevich.

PREFACE

The purpose of this book is to provide a firm foundation in the study of electrical engineering course for the students engaging in different specialties at electro-technical department.

The studying of electrical engineering involves the analysis, design, and applications of devices and systems for conversion, processing, and transmission of electrical energy and information. Owing to the enormous progress made in field of electrical engineering during the last few decades, the scope of its applications has become virtually unlimited. In view of current progress and future prospects, it is necessary for an engineer to acquire a basic knowledge of electrical engineering. I have a hope that the textbook will help to achieve of this goal. The book treats the main principal areas of electrical engineering in sufficient depth to impart real understanding. It is not intended to be a survey. I have lectured electrical engineering at the University for almost thirty years. My experience prompts me how it is better to inform the students about this subject, and help them to understand in the peripetias of our rather complicated but such an interesting subject. The additional complicacy in electrical engineering studying by my students consists in teaching of material in the English language.

Nowadays young specialists in any field have to be able to associate easily with his foreign colleagues in the performance of his official duty. As it is known English is taken in the capacity of international language of intercourse among the specialists of different fields all over the world. Thanks to thorough knowledge both electrical engineering and technical English, the graduates of the groups with English teaching are well needed at the present. I took upon myself the liberty of writing down this manual. I hope that this textbook will help both the lectures and the students of other Universities. The prompt rhythm of a today's life, integration of manufacture, expansion of partner communications (connections) with foreign firms demand from the expert not only a profound knowledge of the chosen trade, but also skill to communicate with foreign partners in English, accepted today as language of business dialogue all over the world.

The Zaporozhye national technical university (ZNTU) is one of the first in Ukraine which began preparation of students from groups with English language of training after a speciality "Electric machines and devices". I have been teaching Electrical Engineering for thirty five years and I have been doing it in English for already eighty years. Now similar groups start to be created at many universities of the country.

I hope, that this book which as a matter of fact is the brief abstract of my lectures, will be interesting both to my colleagues, and students, interested persons not only it is good to know theoretical to the electrical engineer, but also thoroughly to understand subtleties of terminology of the given subject used in all advanced countries of the world.

The ten major chapters of the book are: Physical Bases of Electrical Engineering, Electrical Circuits and its Elements, Linear Direct Current Circuit, Single-Phase Sinusoidal Alternating Current Circuit, Three-Phase Circuits, Transients, Non-Linear Direct Current Circuits, Periodical Non-Sinusoidal Currents in Linear Electrical Circuits, Magnetic Theory and Circuits, and Four-Terminal Networks. I believe that this ordering of the material is easier for the students to assimilate. At first they study electric circuit analysis. It is a prerequisite for the study of other major areas of electrical engineering. The basic circuit elements, the concepts of voltage, current, ideal sources and resistors form the basis for further understanding of direct current circuit calculation. The next chapter discusses the energy storage elements – the inductive coil and the capacitor, different *RLC* circuits, their particularities and their calculation methods. Simple circuits are used to illustrate the concepts of transient response. It is easier for students to understand more general circuits handled using the differential operators. In this text I try to excite student's interest in the subject when I consider different aspects of electrical engineering passing from simple notions to more complicated.

To my opinion, such statement of a material will help them to study the course more profoundly, and employ their knowledge in subsequent studies. By preparation of this lecture course I used known textbooks, collections and grants (1 … 20), and methodical workings out developed of Electrical Engineering chair of Zaporozhye National Technical University.

Finally, I want to express my gratitude for the useful comments and suggestions provided by the colleagues who reviewed this textbook during the course of its development, especially to Professors of Zaporozhye National Technical University Vladimir Petrovich Metelskiy and to Professor of Kharkiv National Technical University "KPI" Vladimir Milyh. I am confident that their many suggestions have helped to improve the precision and clarity of the treatment.

INTRODUCTION

All various electrical cars, devices and systems cannot be designed, developed and maintained truly without knowledge of organic laws of electromagnetism, without ability qualitatively and quantitatively to analyze various modes, to carry out necessary calculations, without a habit of correct application of electrical and magnetic sizes and their measurement by means of measuring devices and systems. Electrical Engineering is a base common-technical course for electrotechnical and specialities. The given grant is made according to the typical program and is intended for helping to students which study in a direction "Electro-mecanics" by independent preparation.

The course "Electrical Engineering" requires the student to have completed the basic knowledge in algebra, trigonometry, differential equations and elementary calculus, as well as basic physics. This course includes the studying of parities between electrical and magnetic values, the methods of calculation and the analysis of electromagnetic processes, the principles of devices which use electromagnetic energy. It has exclusively great meaning for formation of a scientific outlook of the students who study special electro-technical disciplines.

The two basic models are used for the analysis of electromagnetic phenomena and processes: electrical circuits and an electromagnetic field. Accordingly and Electrical Engineering course is conditionally divided into two parts: the theory of electrical and magnetic circuits and the theory of an electromagnetic field.

The theory of circuits follows from the approached replacement of the real electro-technical device by its idealized equivalent circuit. This scheme contains the parts of networks on which required voltages and currents are defined. Electromagnetic processes are considered to be concentrated in separate circuits and the elements of these circuits, and are quantitatively described by the means of concepts of an electromotive force, a current and a voltage.

The theory of an electromagnetic field studies the electromagnetic processes occurring in considered space and described by means of concepts: the intensity of electrical field, a magnetic induction, relative dielectrical and magnetic permeabilities. It investigates the intensity of electrical and magnetic fields and with their help - such phenomena, as radiation of electromagnetic energy, the distribution of volume charges, the density of currents, etc.

The given textbook is devoted to the theory of linear and nonlinear electrical and magnetic circuits. The maintenance of a course and the sequence of a statement of a material in it are corresponded to the program of Electrical Engineering course for electro-technical and specialities of universities. The purpose of this book is to provide a firm foundation in the study of electrical engineering. This subject is basically concerned with providing efficient, convenient, and reliable means for transforming, processing, and transmitting energy and information. Hence, the study of electrical engineering involves the analysis, design, and applications of devices and systems for conversion, processing, and transmission of electrical energy. Basic knowledge of this subject is necessary to every specialist because the scope of its applications has become virtually unlimited. The importance of electrical engineering to those in other fields of engineering stems from the fact that in almost every walk of life we are concerned with one or more aspects of electrical engineering - the conversion, transmission, processing, and storing of energy and intelligence. Our aim is to present a unified overview of major areas of electrical engineering.

As a result of studying of this course the student should know the basic methods of the analysis and calculation of constant processes in linear and nonlinear circuits with the concentrated parameters, in linear circuits of non - sinusoidal current, in linear circuits with the distributed parameters. He must learn the basic methods of transient calculation in the specified circuits and to be able to use them into practice.

The textbook comprises following basic sections: the theory of linear circuits of a direct current; the theory of linear circuits of a sinusoidal current; three-phase electrical circuits; linear circuits at periodic not sinusoidal currents; bases of the theory of passive two-port networks and filters; transients in linear electrical circuits; nonlinear electrical and magnetic circuits at constants and alternating currents and magnetic streams in stationary modes; electrical circuits with the distributed parameters in a constant mode.

For the purpose of acquisition and fastening of skills of methods of the calculations, which necessary for successful studying of following applied courses, in the book after each theme questions and problems for self-checking are offered. Answers are resulted in all problems.

LIST OF SYMBOLS

B	magnetic induction; susceptance;
C	capacitance;
C_{km}	self-capacitance;
D	demagnetizing factor, determinant;
E	electric field strength, a voltage generator; electromotive force (e.m.f.).
e_L	e.m.f. of self-induction;
e_M	e.m.f. of mutual induction;
F	force, a magnetomotive force (m.m.f.);
Q	electrical charge;
f	frequency, a function;
G	conductance;
H	magnetic intensity;
H_c	coercive force;
I	current (an effective value);
i	current (an instantaneous value);
J	current source;
K_I	current transmission ratio;
K_U	voltage transmission ratio;
L	inductance or a self-inductance;
l	length;
M	mutual inductance;
N	number of turns of a coil;
P	active power;
Q	reactive power;
q	charge;
R	resistance;
NR	non-linear resistance;
r	distance between two points;
S	apparent or total power; cross-section area;
T	period of a function;
t	time;
U	voltage (an effective value);
U_M	magnetic potential difference;
F	magnetomotive force;
u	voltage (an instantaneous value);
W_E	electric energy;

W_M	magnetic energy;
X	reactance;
X_L	inductive reactance;
X_C	capacitive reactance;
Y	admittance or total reactive conductance;
Z	complex impedance.

Greek letter symbols

α	phase shift constant;
Δ	determinant;
δ	air gar, decay factor;
ε	permittivity;
ε_o	permittivity of free space (vacuum);
η	efficiency;
μ_o	permeability of a vacuum;
μ_r	relative permeability;
μ_a	complex permeability;
τ	time constant;
Φ	magnetic flux;
φ	potential; angle shift between a current and a voltage;
ψ	epoch angle; flux linkage;
ρ	resistivity; wave resistance;
ω	angular velocity or angular frequency;
ω_r	resonant frequency;
∇	nabla operator.

Subscripts

eq	equivalent;
L	load;
l	line;
in	input, internal;
oc	open circuit;
ph	phase;
sc	short circuit;
ss	steady-state;
t	transient.

Chapter 1

PHYSICAL BASES OF ELECTRICAL ENGINEERING

The science of electrical engineering is based on just a few experimentally established fundamental laws. The principles and concepts of many engineering devices are very often the same in spite of differences in appearance and arrangements. In the interest of stressing the importance of these basic laws, attention is focused on the historical frame of reference as well as the final experimentation which culminated in their strikingly simple formulations. Once the fundamental laws are studied and understood, a considerable amount of respective will have been gained. In turn, this will facilitate the understanding of those branches of engineering where the appropriate laws provide the corresponding foundation.

For example, it will be seen that the field of *electric circuit theory* emanates from the fundamental results achieved by Coulomb (discovered in 1785), Ohm (1827), Faraday (1831), and Kirchhoff (1857). The dates denote the years in which the laws, which today bear their names, were first published. Likewise, further in this book, it will be seen that the whole subject matter of electromagnetic devices and electromechanical energy conversion can be treated and analyzed by applying just two of the fundamental laws - Ampere's law (1825) and Faraday's law of induction (1831).

In this chapter we will study electrical circuits. The electrical circuit is the set of devices intended for transfer, distribution and mutual transformation electrical and other kinds of energy if the processes proceeding in devices, can be described by means of concepts about electromotive force, a current and a voltage.

The electrical circuit generally consists of sources and receivers of electrical energy and the wires connecting them forming closed ways for passage of a current.

We will study how each of circuit elements behaves individually and how, when they are interconnected, their interaction is governed by circuit laws.

And there are only three basic factors regarding the operation of all electrical *dc* circuits: voltage, current and resistance. It is not enough to know only what they do; you must also understand the reason why they behave as they do. If you can understand why, everything in electricity will become much easier for you, and you'll be much better in whatever area of electricity you work.

Once we have covered the three primary factors related to electrical currents in circuits, we will move on to different associated subjects. So, it is important that you understand them fully.

1.1. The Three Primary Factors

There are three primary factors regarding the operation of all electrical circuits. They are voltage, current flow and resistance. These are the fundamental things that control every electrical circuit everywhere. Introducing these three factors, one can use a comparison to the flow of water to provide a comparative illustration of how electricity operates.

Voltage is the force that pushes the current through electrical circuits. It is represented in formulas with the letter U. It's measured in *volts*. The scientific definition of volts is the work necessary to force 1 ampere of current to flow through a resistance of 1 ohm.

In comparing electrical systems to water systems, voltage is comparable to water pressure. The more pressure there is, the faster the water will flow through the system. Likewise, with electricity, the higher the voltage (electrical pressure), the more current will flow through any electrical system.

Current is the rate of flow of electrical charges. The scientific description for current is the intensity of flow of the particles carrying electric charges. It is represented in formulas with the letter I. The unit of current is *ampere* (A). Current compares with the rate of flow in a water system, which is typically measured in gallons per unit. In simple terms, electricity is thought to be the flow of electrons through a conductor.

Resistance is oppositional to the flow of electricity. Some materials, such as glass, offer a very high resistance to the flow of electricity. Other materials, such as metals, offer little resistances to the flow of electricity. To continue with the water illustration, a drinking straw would have a high resistance to water flow. In formulas it is represented by the symbol R. Resistance is measured in *ohms*, and ohms are represented by the Greek capital letter omega (Ω).

A more modern term for the opposition to the flow of electrical current is *impedance*. Impedance is the total opposition to the flow of electricity. It is important to differentiate between impedance and resistance. Resistance is the more traditional but it is less accurate term. Resistance is a fine term for direct-current circuit only. Impedance can be used for either direct- and alternating-current circuits. We will consider this difference in detail further.

1.2. Origins Inside of Atom

Electricity begins in atoms. Atoms are regarded as the smallest particles that retain the properties of an element. Figure 1.2 shows the simplest atom, hydrogen.

Elements are substances that cannot be changed, decomposed by ordinary types of chemical change, or made by chemical union. A molecule is the smallest unit quantity of matter that can exist by itself and retain all the properties of the original substances. It consists of one or more atoms.

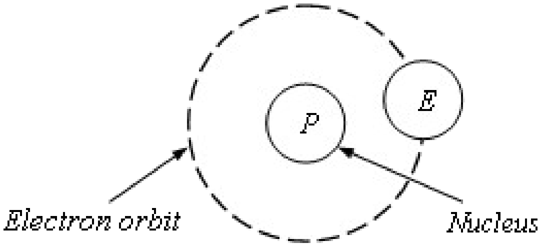


Fig. 1.1. Hydrogen atom with one proton and one electron

Atoms are composed of two main parts: a hard central core known as a nucleus, and electrons that move around the nucleus.

There are three types of particles that make up atoms: *protons* are positively charged particles; *electrons* are negatively charged particles; *neutrons* are particles that have no charge at all.

The nucleus contains two types of particles: one is known as proton and carries a positive charge, the other is neutron which is electrically neutral, that is it carries no charge.

So, protons and neutrons always form the nucleus of an atom, and electrons always move around the nucleus. They carry the smallest negative charge and have a negligible mass because of their submicroscopic sizes.

As an example we can see a hydrogen atom in Fig. 1.1.

The positive charge of a proton is numerically equal to the negative charge of an electron. The positive charges and the negative charges balance themselves. Normally, the atom is electrically neutral, because it consists of as many protons as electrons.

1.3. Charge and Electric Forces

Electrical charge and its movement are the most basic items of interest in electrical engineering.

The basic component of charge is an electron, which carries a negative charge 1.6×10^{-19} *C*, where *C* is the unit of charge given in coulombs. One coulomb of charge therefore represents a tremendous number of electrons.

The nuclei of atoms contain an equal amount of positive charge in the heavier protons.

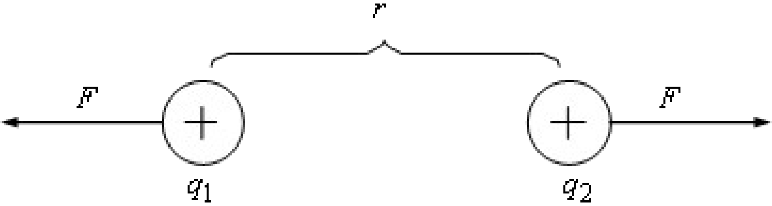


Fig. 1.2. Illustration of Coulomb's Law

Charges of the same sign tend to repel each other, and charges of opposite sign tend to be attracted together. Thus, charges exert forces on each other. It is the electric force that we are interested in utilizing and controlling.

For example, consider two charges q_1 and q_2, separated by some distance r. The force F exerted on one charge by the other varies inversely with the square of the separation between them and directly proportional to the strengths of the charges according to *Coulomb's Law* (see Fig.1.2):

$$F = k \cdot \frac{q_1 \cdot q_2}{r^2} = \frac{q_1 \cdot q_2}{4\pi \cdot \varepsilon \cdot \varepsilon_0 \cdot r^2}. \tag{1.1}$$

where q_1 and q_2 are the charges expressed in coulombs, r is the distance between the charges in meters, $\varepsilon_o = 8.854 \times 10^{-12}$ *F/m* is the *permittivity* of free space (vacuum) and ε is the relative permittivity of medium. The relative permittivity is a dimensionless quantity. The 4π is a proportionality constant which appears whenever Coulomb's law is expressed in the International System of Units (SI).

1 4. Voltage

Coulomb's Law serves as the starting point for the study of a considerable part of electrical engineering. It is our purpose next to establish additional background that follows naturally from Coulomb's Law and that will be useful later in the book.

Since charges exert forces on other charges, energy must be expended in moving a charge in the vicinity of other charges. The unit of energy is the joule (*J*), where one joule is the energy expended in the application of one newton of force in moving an object through a distance of one meter ($J = N \cdot m$).

For example, consider Fig. 1.3. Moving a charge *q* from point "*a*" to point "*b*" in the presence of some other charge *Q* requires a net expenditure of energy. The force of *Q* may oppose the movement of *q* over certain portions of the route, while over other portions this force may be in a direction so as to aid the movement of *q*. Thus, it reasonably follows that if we move the charge from point "*a*" to point "*b*" and return it to point "*a*", the net expenditure of energy will be zero.

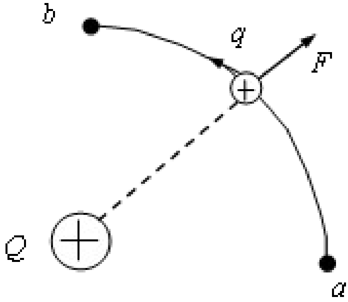


Fig. 1.3. - Voltage (potential difference) between two points in terms of the work required for moving a charge between those points.

This result is explained by the fact that charge *Q* has a type of force field around it which tends to repel charges of like sign and attract charges of unlike sign. If we move a charge *q* from point "*a*" to point "*b*" and energy W_{ab} is expended, and it is worthwhile at this point to use a description which stresses not so much the total work imparted but rather the total work per unit charge, which is called *voltage*.

So, we can say that the voltage *U* between these two points is the work required per unit charge:

$$U_{ab} = \frac{W_{ab}}{q}\,. \tag{1.2}$$

Hence, the voltage is the potential difference between two points. The unit of voltage is *volt* (*V*), where 1*V* = 1*J*/1*C*. Knowing the direction (polarity) of this voltage, or electrical potential, is as important as knowing its magnitude, since this will determine whether energy is to be expended by us or by the force field when a charge is moved between these two points.

Thus, the voltage across a branch of a circuit is the potential difference existing at the terminals of that branch. In a branch containing no e.m.f. the current moves from a point of higher potential to one of lower potential. In this case the charge gives up energy. Conversely, when a unit charge moves from a point of lower potential to one of higher potential it receives energy. The potential difference (p.d.) at the terminals of a resistance is also called the voltage across a resistance. It is also referred with such terms as *voltage rise* or *voltage drop*. In this text the latter term will be used. The positive direction of a voltage drop is the positive direction of the current in the resistance.

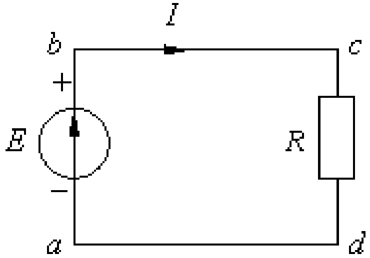


Fig. 1.4. - Example of the electric circuit.

Look at Fig. 1.4 for an illustration of the application of these terms. Current is assumed flowing in a clockwise direction, which means that either positive charges are moving clockwise or negative charges are moving counterclockwise.

As the positive charges move through the voltage rise from *a* to *b* they receive energy so that the potential of *b* is higher than *a*. However, as they move from *c* to *d* they undergo a voltage drop, thereby losing energy to the device appearing between these two points. It is interesting to note that the stream of charge gains energy in one part of the circuit (at the active element) and gives it up at the other part (the passive element). Of course, the total energy remains unchanged in accordance with the law of conversation of energy.

The positive direction of voltage in diagrams is shown by an arrow. The arrowhead should point from the terminal designated by the first letter of the subscript to the terminal designated by the second. Thus, the positive direction for U_{12} is shown by the arrow pointing from 1 to 2. It is important to understand that we need not be concerned with designating the correct point oh higher potential, since this can easily be reversed by using a negative sign with the numeral value of the voltage.

1.5. Current and Magnetic Forces

Current is the rate of movement of electrical charge. Consider Fig. 1.5, *a* in which we have shown a charge moving along a cylinder.

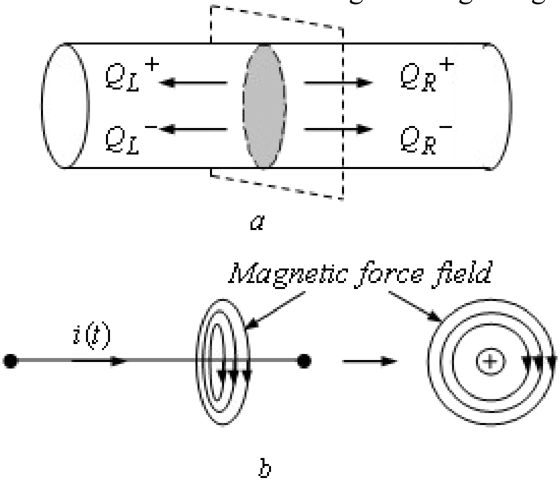


Fig. 1.5. - Electrical current *a*) net movement of positive charge; *b*) magnetic force field around a current-carrying wire

A certain amount of positive charge Q_R^+ and negative charge Q_R^- is moving to the right across some cross section of the cylinder, and similar quantities Q_L^+ and Q_L^- are moving to the left. The net positive charge moving to the right is

$$Q = Q_R^+ - Q_R^- - Q_L^+ + Q_L^- \qquad (1.3)$$

since negative charge moving to the right is equivalent to positive charge moving to the left. If we observe the charge crossing the area at certain time interval Δt, the current i directed to the right is the rate of movement of net positive charge to the right per unit of time:

$$i = \frac{Q}{\Delta t}. \tag{1.4}$$

The unit of current is ampere (A), where one ampere of current is the movement of one coulomb of net positive charge past the cross-sectional area of a wire per one second (s): $1A = 1C/1s$.

The rate of charge movement may not be constant but may vary with time. To determine the current at a specific time, we reduce the time of observation to a very small interval about that time and obtain which gives us the current as the function of time t.

$$i(t) = \frac{dq}{dt}. \tag{1.5}$$

In most metallic conductors, such as wires, current is exclusively the movement of free electrons in the wire. Since electrons are negative, the charges are thus moving in a direction opposite to the direction of the current designation. The net positive charge movement is nevertheless in the direction we designate for i.

Electrical current produces forces, just as electrical charges do. The forces produced by electrical currents are referred to as magnetic forces, and they possess many of the properties of ordinary bar magnets.

A current-carrying wire has a magnetic force field around it just as an electrical charge has an electrical force field around it. This magnetic force field appears as concentric circles around the wire; at a constant radius r from the wire, the magnetic force is the same (Fig. 1.5, *b*).

The electrical force field of a positive charge always points away from the charge, and vice versa for the negative charge. If we place an ordinary bar magnet of a compass in the vicinity of the wire, the compass needle will align with the direction of the magnetic force field.

Electric current is defined as the time rate of change of charge passing through a specified area. The moving charges may be positive or negative; the area may be the cross-sectional area of a wire or some other suitable spatial area where charges are in motion.

Expressed mathematically we can write

$$i = \frac{dq}{dt}\,. \tag{1.6}$$

In this equation i denotes the instantaneous electric current and q represents the net charge, which may be of the positive as well as negative kind. The amount of current is measured in amperes.

This term commemorates one of the great discoverers of electricity Andre-Marie Ampere -a French physicist of the 19th century who did groundbreaking work on the relationships between electricity and magnetism.

1.6. Gauss's Law

This law is an important consequence of Coulomb's Law and provides additional useful knowledge, which is needed in the work that follows in Chapter 2. In thus connection consider that a sphere of radius r is put around a point charge q as depicted in Fig. 1.6.

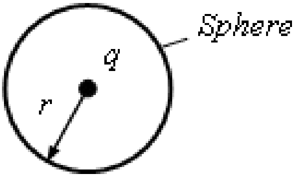


Fig. 1.6. Configuration for deriving Gauss's electric flux law

The electrical engineers have to take into account important vectors: $\overrightarrow{D}$. and $\overrightarrow{E}$.

$\overrightarrow{D}$ is called the *electric displacement* or *electric induction*. It is defined as the property of the field which is evidenced by its ability to induce charges on conductors placed in it.

The *electric field strength* (or *electric intensity*) $\overrightarrow{E}$ is a vector quantity defined at any point in a field by magnitude and direction.

If we consider a stationary positive charge so small that it will not bring about any changes in the distribution of charges on the bodies producing the field, the ratio of the force acting on the charge to the magnitude of the charge gives the electric field strength at that point:

$$\overrightarrow{E} = \frac{\overrightarrow{F}}{q}\,.$$

So, the *electric field strength is numerically equal to the force acting on a unit charge*.

Taking into account all above-stated, the Gauss theorem may be stated in following ways.

1. The surface integral of the normal component of the electric displacement, i.e. the total normal displacement flux, over a closed surface enclosing a certain volume is equal to the algebraic sum of the free charges within this surface:

$$\oint \vec{D}\, \vec{ds} = \sum q_f . \tag{1.7}$$

Rewriting this equation for our sphere yields

$$q = D\,A = 4\pi r^2 D . \tag{1.8}$$

The quantity *D* is identified as an *electric flux density* because it is combined with an area term $A = 4\pi r^2$ =area of sphere, to yield charge, which is also called *electric flux*.

2. Since $\vec{D} = \varepsilon_o \varepsilon \vec{E}$, then Gauss' theorem for a homogeneous and isotropic medium can be written as

$$\oint \vec{E}\, \vec{dS} = \frac{\sum q_f}{\varepsilon_o \varepsilon} ,$$

that is, the surface integral of the normal component of the electric field strength, or the total normal field strength flux, over a closed surface is equal to the total free electric charge within the surface divided by the product $\varepsilon_o \varepsilon$.

Either of the two expressions has a field of application of its own. It is important to stress that the normal vector flux is dependent solely on the total charge within the closed surface and is independent of the contribution of the charge outside it.

1.7. Ohm's Law

This law is perhaps one of the first things learned about electricity in any elementary course on the subject. It is not at all unlikely that the reader was introduced to Ohm's law in its simplest form in his physics course. This is mentioned by the way of further stressing its importance in the study of electrical engineering.

George Simon Ohm (1787 - 1854) was a professor of mathematics and physics who devoted considerable time and effort to experiments dealing with voltaic cells and conductors. These experiments included the effect of temperature on the resistivity of various metals.

In spite of this extensive laboratory work, however, the law that bears his name was the result of a mathematical analysis of the galvanic circuit based on an analogy between the flow of electricity and the flow of heat.

By following the formulation of Fourier's heat conduction equation and using electric field intensity as analogous to temperature gradient, Ohm was able to show that the current flow in a circuit composed of a battery and conductors can be expressed as

$$I = \frac{S}{\rho}\frac{du}{dl} \tag{1.9}$$

where the derivative term denotes the electric field gradient, and $S = 4\pi r^2$ =area of sphere. In the language used by Ohm u was called the electroscopic force by way of representing the volume density of electricity at a point in the conductor - a terminology consistent with the analogous situation in heat flow through a solid described in terms of the quantity of heat per unit volume. In the case where a conductor of uniform cross-sectional area is used, Eq. (1.9) may be written as

$$I = \frac{S}{\rho}\frac{U}{l} = \frac{U}{R} \tag{1.10}$$

where U is the potential difference in volts appearing across the conductor of length l, S is the cross-section area; ρ is the property of the material called the *resistivity*; R is the resistance of the conductor in ohms.

In the case of a rectilinear conductor which has constant cross-section R is found as

$$R = \rho \cdot \frac{l}{S} \tag{1.11}$$

where S - the conductor cross-section, (m^2).

Equation (1.10) is a mathematical description of Ohm's law. It states that for any given potential difference the strength of the current in a wire is proportional to the potential difference between its ends and inversely proportional to the resistance, which in turn is dependent upon the composition of the wire.

Ohm's law may be alternatively expressed as

$$U = I\,R\,. \tag{1.12}$$

In this form it states that for any given potential difference, the amount of current produced is inversely proportional to the resistance, which in turn is dependent upon the composition of the wire.

1.8. Instantaneous power and energy of an electric current. The Joule-Lenz's law

When current flows through a section of a circuit an electric field fulfils work. There are following power transformations. At first the energy of a source changes into the energy of the field. Then the field gives the energy to the charged particles, moving in the conductors of a circuit.

The energy of movement (the energy of a current) will change into internal (thermal) energy of conductors, and all allocated energy dissipates in environment after some increase in their temperature. For an interval of time Δt a charge $\Delta q = I \cdot \Delta t$ proceeds through a section of a circuit. The work fulfilled by electric field in the part of a circuit:

$$\Delta A = (\varphi_a - \varphi_b)\Delta q = U \cdot I \cdot \Delta t \tag{1.13}$$

where $U = U_{ab} = \varphi_a - \varphi_b$ is the voltage across a considered section. Taking into account Ohm's law it is possible to present (1.13) expression into the following state:

$$\Delta A = R \cdot I^2 \cdot \Delta t = \frac{U^2}{R} \cdot \Delta t\ . \tag{1.14}$$

Joule-Lenz's law: the work ΔA of electric current *I* which flows through a motionless conductor with resistance *R*, is equal to the quantity of thermal energy which is allocated in this conductor for the interval of time Δt :

$$W_T = \Delta A = R \cdot I^2 \cdot \Delta t\,. \tag{1.15}$$

The power of an electric current is equal to the relation of current work ΔA to an interval of time Δt for which this work has been made:

$$P = \frac{\Delta A}{\Delta t} = U \cdot I = R \cdot I^2 = \frac{U^2}{R}. \qquad (1.16)$$

The work of electric current is measured in *joules* (*J*), and power is measured in *watts* (*Wt*).

The energy derivative in time, i.e. the speed of energy receipt in time, represents instantaneous power of section of a circuit:

$$p = \frac{dW}{dt} = u\frac{dq}{dt} = ui. \qquad (1.17)$$

Instantaneous power is an algebraic value. It is positive when current and voltage coincide in sign (when a section of the circuit has got the energy from other part of a circuit). It is negative if current is opposite to voltage (a section gives out the energy to the circuit).

1.9. Electromotive force

For making constantly existing electric field in a conductor, it is necessary a current (voltage) source be in current-carrying circuit. If a moving of charges in a source is excited by the external forces which have not been caused by an electrostatic field, the energy of external fields, exciting these forces, will be converted into electric energy. In direct current generators external fields are excited by an electromagnetic induction, in galvanic cells - by chemical reactions, in thermogenerators - by heating.

Thus, the surplus of positive charges is made up at one terminal marked out by a plus sign, and the surplus of negative charges is made up at the other terminal marked out by a minus sign. An electric field is created inside a source and in external electric circuit as a result of separation of charges.

The sources of electric energy are characterized *by an electromotive force (e.m.f.)*. The e.m.f. is equal to the operation of an external forces expended on displacement of an individual positive charge inside a source from a terminal with a lower potential to a terminal with a higher potential, and is marked out as *E*.

As the external forces act only in the current source between its electrodes, the e.m.f. is the source description, and it does not depend on an external network. Irrespective of the nature of external forces, the e.m.f. of a source is numerically equal to a voltage between terminals of the energy source in the absence of a current. The instantaneous value of e.m.f. is marked out $e(t)$. The electromotive force is measured in volts (V).

1.10. Basic Characteristics of a Magnetic Field

Just as in the space enclosing electric charges, there is an electric field, and in the space enclosing moving charges and constant magnets, there is a force field which is termed as *magnetic* one.

The major singularity of a magnetic field is that it is created only by moving charges, and it acts only on moving charged particles.

The parameters of a magnetic field are *magnetic induction* (or flux density) B, *magnetic intensity* (or magnetic field strength) H, and *magnetization* J.

The magnetic induction B at any point in a magnetic field is a vector quantity that determines the e.m.f. induced in an elementary conductor that is moving through the field at that point.

It numerically equals to the meaning of force F with which the magnetic field acts in the given point on individual charge q which moves with individual velocity v and has a direction, perpendicular to force and velocity vectors, coinciding with a translation of the right screw at its twirl from a force direction to a direction of velocity of a particle with a plus charge.

$$B = \frac{F}{qv}\,. \tag{1.18}$$

If everywhere the vector B is the same numerically and in the direction, the field is considered to be homogeneous. The unit of magnetic induction is the *tesla* (T).

The lines of the magnetic field force are the lines of a magnetic flux. In each point of these lines, the vector of the magnetic induction is directed on a tangent to this line. Each line of a magnetic field is the line of the equal induction.

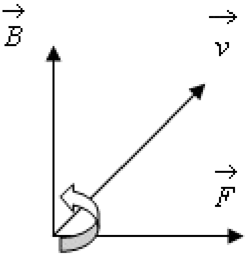


Fig. 1.7. The displacement of the vector of magnetic induction

The vector of magnetic induction $\vec{B}$ is directed perpendicularly to the plane which it is organized by mutually perpendicular vectors of velocity $\vec{v}$ and force $\vec{F}$ (Fig. 1.7).

The magnetic intensity *H* at any point in a magnetic field is defined as a force which produces, or is associated with, the magnetic induction at that point.

The magnetization *J* at any point in a magnetic field is defined as the magnetic moment per unit volume.

The three quantities are related thus

$$B = \mu_o (H + J) \tag{1.19}$$

where μ_o is a permeability of a vacuum (free space). The magnetic intensity and the magnetization are measured in ampere per meter.

The magnetization is in the same direction as magnetic intensity at that point and is directly proportional to the magnetic intensity within the material. Thus

$$J = \chi H \tag{1.20}$$

where χ is the *magnetic susceptibility* of the material and is in turn a function of *H*. Substituting Eq. (1.20) in Eq. (1.19) and putting $1 + \chi = \mu_r$, we get

$$B = \mu_o \mu_r H \tag{1.21}$$

where $\mu_o = 4\pi \times 10^{-7} = 1.256 \times 10^{-6}\, H/m$.

The magnetic flux *Φ* through the area *S* is the surface integral of the normal component of the magnetic induction vector over that area, or

$$\Phi = \int_S B dS \tag{1.22}$$

where dS is an element of area S. Magnetic flux is measured in webers.

1.11. The Biot-Savart-Laplace's Law

According to the Biot-Savart-Laplace's law the magnetic induction in a free space at a point A distant from the element of length dl carrying current I is given by

$$\vec{dB} = \frac{\mu_r \mu_o}{4\pi} \cdot \frac{[I\,\vec{dl} \times \vec{r}\,]}{r^3} \tag{1.23}$$

where $\vec{dl}$ is the vector which in module is equal to the length of an element of a conductor dl and coincides in direction with a current; $\vec{r}$ is a radius vector which is drawn from element dl to a point of the field A; r - the module of radius vector $\vec{r}$

The direction dB which is determined by a vector product (see Eq. 1.23), is perpendicularly to each of vector factors dl and r , that is, it is perpendicularly to the plane in which they are placed.

The direction of a magnetic induction vector is found by the right-handed screw rule: the direction coincides with a rotation of a head of the right screw if its translational movement coincides with the direction of current in the element dl .

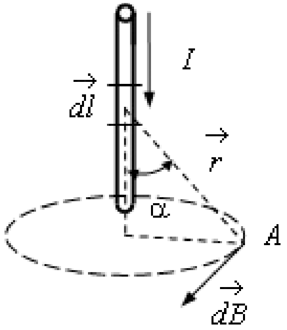


Fig. 1.8. A straight conductor carrying current

The vector module is found by the expression

$$dB = \frac{\mu_r \mu_o}{4\pi} \cdot \frac{I \cdot dl}{r^2} \cdot \sin\alpha \tag{1.24}$$

where α is the angle between vector $\overrightarrow{dl}$ and $\vec{r}$ (Fig. 1.8).

1.12. Ampere's Circuital Law

Electric currents passing through a conductor produces a magnetic field in its neighbourhood.

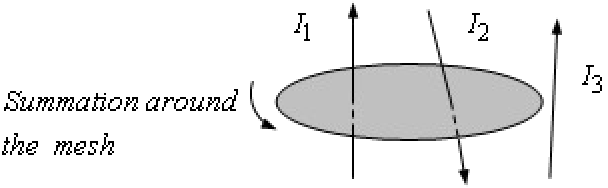


Fig. 1.9. Closed path and the currents bounded by that path

The quantitative relationship between the line integral of the magnetic intensity vector *H* around any closed path and the total current bounded by that path is given by

$$\oint H dl = \sum I \, . \tag{1.25}$$

This is most often known as *Ampere's circuital law*. It states that the line integral of the magnetic field intensity over a closed path is equal to the total current threading through that path. It can be applied to the circulation of magnetic intensities due to current-carrying conductors. The current is considered to be positive if it organises the right screw with the direction of summation around the mesh, and it is considered to be negative - otherwise.

For example, for the contour which is shown in Fig. 1.9, the circulation of a vector of magnetic intensity $\overrightarrow{H}$ is proportional ($I_1 - I_2$) and does not depend on I_3 .

1.13. The law of electromagnetic induction

In 1831, Michael Faraday (1791 - 1867) showed that electricity could be produced from magnetism. He demonstrated that induced currents could be made to flow in a circuit whenever (1) current in a neighbouring circuit is established of interrupted, (2) a magnet is brought near a closed circuit, and (3) a closed circuit is moved about in the presence of a magnet or other closed current-carrying circuits.

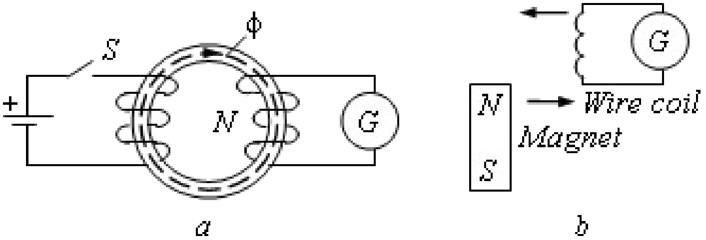


Fig. 1.10. Illustration Faraday's law of induction

Figure 1.10, *a*, shows one of the experiments made by Faraday. By closing the switch he was able to observe a deflection in the galvanometer *G* connected to the second circuit. (A galvanometer is a sensitive instrument used to measure small currents and voltages).

Moreover, upon opening the switch, he again observed a deflection but this time in a reverse direction. Figure 1.10, *b*, shows schematically still another arrangement used by Faraday to show how electricity is produced from magnetism. If either the magnet was moved toward the circuit or the circuit toward the magnet, the galvanometer registered a deflection.

Faraday was a disciplined and meticulous experimenter who kept careful and complete records of his laboratory work. From these descriptions it has been possible to formulate Faraday's law of induction mathematically as follows:

$$e = -\frac{d\Psi}{dt} = -w\,\frac{d\Phi}{dt}\,. \tag{1.27}$$

The quantity e denotes the instantaneous value of electromotive force induced in a closed circuit having a flux linkage of Ψ weber-turns. In those instances where the magnetic flux Φ penetrates all the turns of the coil as shown in Fig. 1.10, *a*, Faraday's law may be expressed by the following form of Eq. (1.27).

The negative sign is due to Emil Lenz, who subsequent to Faraday's experiments pointed out that the direction of the induced current is always such as to oppose the action that produced it. This reaction is commonly called *Lenz's law*.

Faraday's law as embodied in Eq. 1.27 is one of two basic relationships upon which the entire theory of electromagnetic and electromechanical energy conversion devices is based.

In fact, soon after Faraday's work of genius was published in 1831 explanations were at last possible for the phenomena observed by Oersted, Ampere, and other experimenters. The foundation was now established to facilitate the ensuing rapid development of electric motors and generators.

Faraday was also the first experimenter to identify the *electromotive force of self-induction* which manifested itself whenever circuits carrying current in long wires or circuits wound with many turns were disconnected.

The American inventor Joseph Henry also independently discovered the current of self-induction but not before Faraday. Both experimenters were able to demonstrate that a changing current produced an e.m.f. of self-induction in a coil of wire which varied directly with the time rate of change of current. Expressed mathematically,

$$e = -L\frac{di}{dt} \tag{1.28}$$

where L is a proportionality factor called the coefficient of self-inductance which is dependent upon the medium and some physical dimensions. This result is equivalent to Faraday's law of induction as expressed in Eq. (1.27).

Moreover, Eq. (1.28) is extremely important in the development of electric circuit theory. It serves to identify one of the three basic circuit parameters - *inductance*. The other two parameters, resistance and capacitance, have already been identified with Ohm's and Coulomb's laws, respectively.

Summary review questions

1. What is termed as the electric circuit and what devices can it consist of? Give a general and inclusive description of electric current.
2. What electrical parameters describe the state of electric circuits and the processes which are flowing past in them?
3. Give a general and inclusive description of electric current and illustrate your statement with appropriate symbols.

4. State Coulomb's law and identify the kind of information that can be deduced from it.

5. What is permittivity? Describe how this quantity can be found for various materials (media).

6. Assume that a circuit consists of a battery connected in series with a resistor. Describe the action that take place in the circuit as an electron is made to circulate around the circuit.

7. Distinguish among voltage, voltage drop, voltage rise, and potential difference.

8. How are the parameters of devices of equivalent circuit of a receiver and a generator of electric energy found in direct-current circuits?

9. Does capacitance exist between two thin current-carrying conductors? Explain.

10. Describe how the resistance property of a conducting material can be determined using Ohm's law. Can this procedure be applied to nonlinear materials? Explain. Can this procedure be applied to devices that exhibit discontinuities? Explain.

11. What does Lenz's law state? When is instantaneous power positive and is it negative?

12. Draw a series circuit comprising a battery, a switch, a resistor, and a coil. Describe the effect of the law of electromagnetic induction when the switch is suddenly opened after being allowed to remain closed for a period of time. How does Lenz's law manifest itself in this action?

13. What does Ampere's circuital law state? Where can it be applied?

14. State Ampere's law and illustrate its application to show that the net force is zero in the case where two long, current-carrying conductors are placed in quadrature with each other.

15. A positive point charge of value 10^{-6} C is fixed at a point in a vacuum. An identical point charge is then brought to a distance 10 cm away. Find the potential energy of the second charge.

Chapter 2

ELECTRIC CIRCUIT AND ITS ELEMENTS

2.1. The basic notions of an electric circuit

As electric circuit is termed a totality of the devices intended for generation, transmission, conversion and the use of electric current. The electromagnetic processes in the electric circuit can be presented by means of such integral concepts, as a current, a voltage and an electromotive force.

Any electric circuit is an interconnection of electric devices such as energy sources, energy converters or loads, and conductors that connect the source and the loads.

Energy sources are devices converting chemical, mechanical or other forms of energy into electric one. Supplied electric energy is further transformed by energy converters (load) to other forms of energy (mechanical work, heat, light and so on).This chapter deals with direct-current (d.c.) circuits.

The energy converters of electric energy are devices in which an electromagnetic energy is transformed into other kinds of energy: mechanical, thermal, chemical, the energy of light radiation etc. Besides, some part of energy is reserved in electric and magnetic fields of a circuit. The converters are also called receivers or loads.

For description and theoretical examination of circuit properties the notions "two-terminal network" and "four-terminal network" are introduced. A part of an electric circuit of the arbitrary configuration, observed concerning any two terminals (poles) is termed a *two-terminal network*. There are passive and active two-terminal networks. The networks which do not contain energy sources are called passive ones. The networks containing one or several energy sources are termed active ones.

The properties of transfer system are described by means of notion "four-terminal network". A *four-terminal network* is an electric circuit with two input and two output terminals. A four-terminal network usually is an intermediate link between a source of supply and a load.

A *direct current* is defined as a unidirectional current unvarying with time. It is flow of particles carrying electric charges (free electrons in metals, and ions in liquids). Current flow is caused by electric field established by the energy source of a given circuit.

A source of electric energy is specified by the magnitude and the direction of electromotive force (e.m.f.) it generates, and its internal resistance. In the international system the unit of current I is *ampere* (*A*), that of the e.m.f. E, the *volt* (*V*), and that of resistance R, the *ohm*.

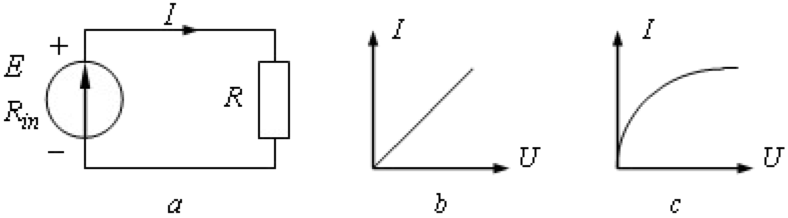


Fig. 2.1. A graphic representation of an electric circuit and volt-ampere characteristics

A graphic representation of an electric circuit is called a circuit diagram (Fig. 2.1, a). We represent resistances by rectangles, an energy source, by a circle with an arrow inside it to indicate the positive direction of the e.m.f.

In any electric circuit the energy converter and the conductors connecting it to the source make up an *external circuit* in which current flows from "+" side to the "-" side of the source. Inside the source, current flow is in the opposite direction, i.e. from the "-" side to the "+" side. The relation between the current through a resistance and the voltage across the same resistance is called its *volt-ampere (volt-current) characteristic.* When presented graphically, voltages are laid off the abscissa, and the currents, as ordinates.

There are two different types of volt-ampere (*V/A*) characteristics. One is a straight line; the other is a curve. Both are shown in Fig. 2.1, *b* and *c*, respectively. Resistive elements for which the volt-ampere characteristic is a straight line are called *linear*, and the electric circuits containing only linear resistances are called *linear circuits*. Resistive elements for which the volt-ampere characteristic is other than a straight line are called *non-linear*, and so the electric circuits containing them are called *non-linear circuits*.

All electric circuits actually are nonlinear. Linear they can be considered only in the limited ranges of values of currents and voltages.

The volt-ampere characteristics may be of different forms. Such volt-ampere characteristics for various resistive elements (2 - for a linear resistive element, 1 and 3 - for nonlinear resistive elements) are presented in Fig. 2.1. In an appearance from a linear resistive element each point of volt-ampere characteristic of a nonlinear element is defined by two parameters: static resistance and differential resistance.

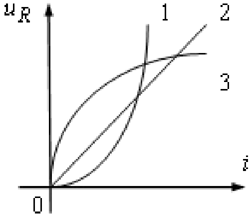


Fig. 2.2. The volt-ampere characteristics

Static or *integrated resistance* of a resistive element is numerically equal to the ratio of voltage on element terminals to the force of a current flowing through it. This resistance is proportional to a tangent of an inclination angle α of a straight line which is led from the beginning of coordinates to a working point 1 of the characteristic, in relation to an axis of currents (Fig. 2.3, *a*):

$$R_{in} = \frac{u_{1R}}{i_1} = \frac{m_u}{m_i} \cdot \operatorname{tg} \alpha = m_R \cdot \operatorname{tg} \alpha \qquad (2.1)$$

where m_u, m_i, m_R - corresponding scales for a voltage, a current and a resistance.

Dynamic or *differential resistance* of a resistive element is numerically equal to the ratio of an infinitesimal increment of the voltage on element clips to an infinitesimal increment of a current flowing through it.

This resistance is proportional to a tangent of an inclination angle *β* to a tangent which is led to a working point 1 of the characteristic, in relation to an axis of currents (Fig. 2.3, *b*):

$$R_d = \left.\frac{du_R}{di}\right|_{i_1} = \frac{m_u}{m_i} \cdot \operatorname{tg}\beta = m_R \cdot \operatorname{tg}\beta \qquad (2.2)$$

In the case if a volt-ampere characteristic is linear, the values of R_{in} and R_d don't depend on a choice of a working point and are equal to each other.

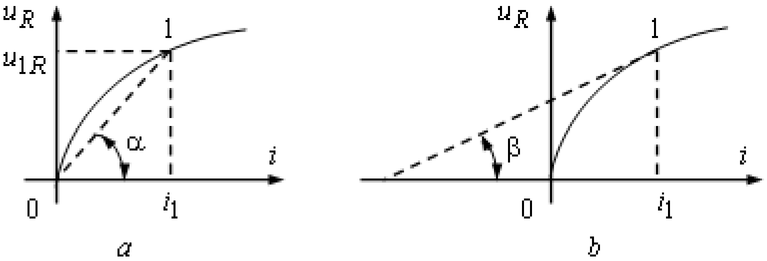


Fig. 2.3. The volt-ampere characteristics for determining the static and dynamic resistances

Similarly it is possible to represent coulomb-volt characteristics for capacitor elements and weber-ampere characteristics for inductive elements. In the limited range of voltage or current changes it is possible to neglect their nonlinearity and consider them to be linear. In this case the values of resistances, capacitances and inductances are constant.

2.2. Ideal Voltage and Current Sources

All electric circuits are driven by sources. These sources may be independent of their network variables, as in the case of synchronous generator that are used to supply energy to homes and many industrial loads; or they may be of a dependent type, such as is frequently encountered in electronic circuits. Attention here is confined to the independent type. However, independence of a circuit variable such as current or voltage does not imply independence of time. Accordingly, many independent sources are often time varying, as with the synchronous generator whose phase voltage is described by $e(t) = Em \sin \omega t$, where ω is determined by the speed of rotation of the generator.

The distinguishing feature of an independent voltage source is that the value of the voltage is not dependent on either the magnitude or direction of the current flowing through the source. Figure 2.5, *a* depicts the model representation of such a two-terminal source.

A voltage source, or voltage generator, connected in series with a resistance R_{in} ("source resistance") equal to the internal resistance of the real energy source is presented in Fig. 2.4, *a*. The equivalent voltage generator is "idealized" in that the voltage it generates is thought of as being constant, independent of the current flowing through it, and equal to the e.m.f. of the real energy source. The internal resistance of this idealized generator is zero. In practice a voltage generator is shown in diagrams by a circle with an arrow inside and the letter E outside the circle. The arrow points in the positive direction of e.m.f (this is the direction in which the potential inside the generator increases).

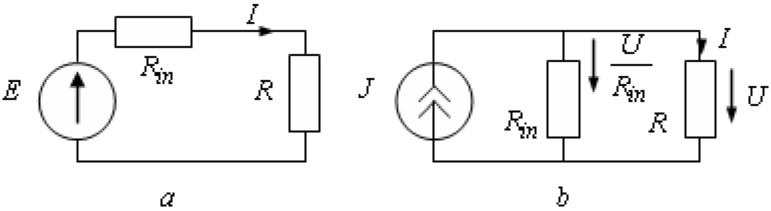


Fig. 2.4. Electric circuits

An *ideal current source* is a two-terminal element which supplies its specified current to the circuit in which it is placed independently of the value and direction of the voltage appearing across its terminals. An ideal *current source,* or *current generator*, connected in parallel with a resistance R_{in} equal to the internal resistance of the real energy source is depicted in Fig. 2.4, *b*.

The ideal independent current source is a two-terminal element which supplies its specified current to the circuit in which it is placed independently of the value and direction of the voltage appearing across its terminals.

The equivalent current generator is "idealized" if the current J is independent of the load resistance R and equal to the quotient of the e.m.f. of the real source and its internal resistance, or $J = E/R_{in}$. In an ideal current source, its internal resistance is infinitely large.

In diagrams a current generator is symbolized by a circle with an arrow inside and letter J outside. The letter may have a subscript (say, I_k). The arrow points in the positive direction of J .

The current through the load (the resistance R) is the same in both equivalent circuits (a and b in Fig. 2.4), and equal to the current

$$I = \frac{E}{R + R_{in}} . \tag{2.3}$$

The behaviour of voltage and current generators can be presented in chart form, as in Fig. 2.5, a and b, where the current through an idealized source is related to the voltage across its terminals.

The plot of Fig. 2.5, a applies to a voltage generator and shows the voltage-current characteristic of an ideal voltage source. Observe that the characteristic is horizontal to the current axis, which is entirely consistent with the fact that any value of current magnitude and direction can be associated with the voltage source. Of course, it is assumed that the voltage source is not being operated beyond its energy capability, otherwise, the ideal representation is no longer valid.

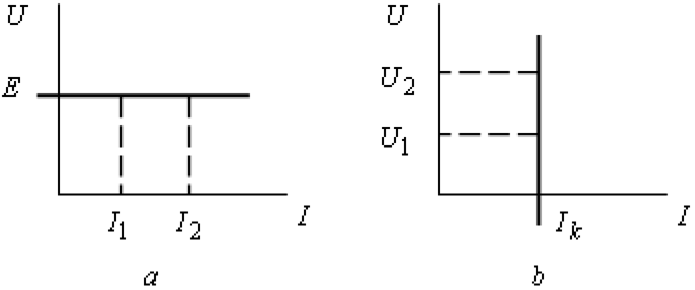


Fig. 2.5. The voltage-current characteristics

A notable point about an ideal voltage source is its zero internal resistance. This is readily demonstrated by invoking Ohm's law in incremental form as it applies to the volt-ampere characteristic. Thus we can write

$$R = \frac{\Delta U}{\Delta I} = \frac{\Delta U}{I_2 - I_1} = \frac{0}{I_2 - I_1} = 0 \qquad (2.4)$$

where ΔU denotes the change in source voltage associated with a given change in current, $\Delta I = I_2 - I_1$. From Fig. 2.6 it is clear that the change in voltage associated with the change in current is zero. This result applies to all ideal voltage sources.

The plot of Fig. 2.4, b applies to a current generator and shows the voltage-current characteristic of an ideal current source.

By applying the similar analysis to the voltage source, it is easy to demonstrate that the internal resistance of an ideal current source is equal to infinite:

$$R = \frac{\Delta U}{\Delta I} = \frac{U_2 - U_1}{\Delta I} = \frac{U_2 - U_1}{0} = \infty . \qquad (2.5)$$

Accordingly when one "looks" into the terminals of a current source, one "sees" an open circuit. This is a useful bit of information to keep at hand when doing circuit analysis that involves current sources.

2.3. Real Energy Sources

As there are no technical devices in which in this or that form there would be no irreversible transformations of energy, in real energy sources there are losses of energy and they are replaced with two ideal elements at their calculation:

- ideal electromotive force source (e.m.f. source) to which the resistive element is connected in series;
- ideal current source to which the resistive element is connected. in parallel.

The Resistance of an active resistive element is called an internal resistance of an energy source.

Real sources possess final capacity and their external characteristics are not parallel to axes, and cross them in the characteristic points. They cross both axes of coordinates and these points of crossing correspond to a zero current through a source and to a zero voltage. The volt-ampere characteristic of a real source is presented in Fig. 2.6.

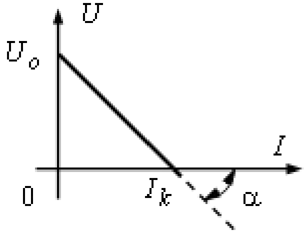


Fig. 2.6. The volt-ampere characteristic of a real source

The rate with a zero current and a nonzero voltage drop (when a load resistance is equal infinity), is called an idling mode or an open circuit, and the rate with a zero voltage drop and a nonzero current on an exit (when a load resistance is equal zero) is called a short circuit rate.

The inclination of the external characteristic is defined by internal resistance of a source which it is equal:

$$R_i = \frac{U_o}{I_к} \qquad (2.6)$$

where U_o is an idling voltage, I_k is a short circuit current.

The value of e.m.f. of an ideal source is numerically equal to the idling voltage of a real source. Accordingly the value a current of an ideal source is numerically equal to a short circuit current of a real source. The equivalent circuit of real e.m.f. to which the load is connected, is presented in Fig. 2.7, a , an equivalent circuit of a real current source to which the load is connected - in Fig. 2.7, b .

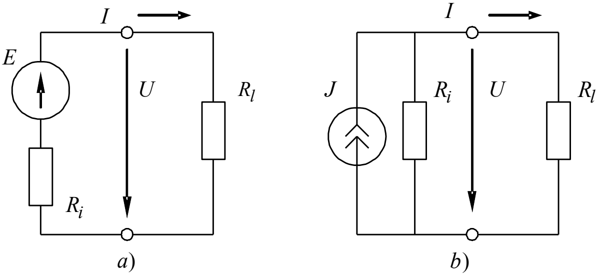


Fig. 2.7. The equivalent circuits of a real e.m.f. and a real current source

For a real e.m.f. source the external characteristic is described by the equation:

$$U = U_o - R_i I = E - R_i I \ . \qquad (2.7)$$

And for a real current source the external characteristic is described by the equation:

$$I = I_к - \frac{U}{R_i} = J - \frac{U}{R_i} \ . \qquad (2.8)$$

Both equivalent circuits of these real sources of electric energy (Fig. 2.7, a , and Fig. 2.7, b) are equivalent (they have the same volt-ampere characteristic) from the point of view of currents, voltages and powers in external parts of any electric circuit, and can be applied in the dependence on the purposes and convenience of concrete representation. Therefore, in the course of calculations it is possible to replace any real e.m.f. source with a real current source, and on the contrary, using a parity:

$$R_i = \frac{U_o}{I_k} = \frac{E}{J}. \tag{2.9}$$

Energy sources can give out the energy to other part of a circuit (to work in a generator mode) and to take it away from a circuit (to work in a consumer mode).

2-4. The resistance parameter

Resistance is one of three basic parameters of electric circuit theory. Resistance is introduced as the proportionality factor in Ohm's law relating current to potential difference. Ordinarily, the free electrons associated with the atoms of a conducting material undergo random thermal motion and so do not constitute a net current flow. However, the application of a voltage source to such a conductor produces an electric field within the body of the conductor which then imparts a directed velocity component to the random motion of the electrons. As the electrons respond to the electric field by moving in a direction opposite to it, they encounter frequent collisions with the atoms in the lattice structure of the conducting material, which in turn produces an irreversible heat loss. In a general way resistance can be described as that property of a circuit element which offers opposition to the flow of current and in so doing converts electric energy into heat energy.

A circuit element is described from the circuit viewpoint when it is expressed in terms of an associated current and potential difference. Since by Ohm's law we have

$$R = \frac{U}{I}. \tag{2.10}$$

It follows that this equation embodies the circuit-viewpoint description of resistance. Eq. 2.10 is a linear algebraic expression when the proportionality factor R is independent of the current. Moreover, Eq 2.10 is valid for negative as well as positive values of current.

As the quantitative characteristic of ability of this element to transform electric energy to other kinds of energy is used a parameter named *active resistance* (or simply *resistance*).

The instantaneous power arriving in a resistive element:

$$p_R = u_R i = Ri^2 = \frac{u_R^2}{R}. \tag{2.11}$$

Hence, parameter R is numerically equal:

$$R = \frac{p_R}{i^2}. \tag{2.12}$$

The instantaneous power in a resistive element is square-law function of a current or a voltage. It cannot accept a negative value, hence, energy always arrives from a source to the element.

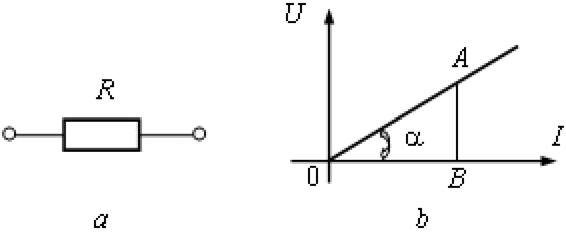


Fig. 2.8. A resistive circuit element

The quantity of thermal energy W_T allocated in a resistive element with resistance R at flowing current i since some moment of time $t = 0$ till considered moment t is equal:

$$W_T = \int_0^t Ri^2 dt. \tag{2.13}$$

This formula mathematically reflects the main property of a resistive element - irreversible transformation of electromagnetic energy to thermal energy. The quantity of thermal energy at direct current flowing is defined by equation (Joule-Lenz's law):

$$W_T = RI^2 t. \tag{2.14}$$

The value opposite to resistance is named *an active conductance or conductivity*:

$$G = \frac{1}{R}. \tag{2.15}$$

A resistor is a real object, the most approached on its properties to a resistive element. Every resistor has its own *resistance*.

Active resistance is measured in *Ohm (Ω),* active conductivity is measured in *Siemens* (*Sm*) or in *Mho*.

For the elementary resistor is a rectilinear conductor with constant cross-section section. Its resistance is defined by expression

$$R = \rho \cdot \frac{l}{S} = \frac{1}{\gamma} \cdot \frac{l}{S} \qquad (2.16)$$

where ρ is a constant depending on the nature of the material; γ - specific electric conductivity, l - a length of a conductor, S - the of cross-section area.

A resistive circuit element is depicted in Fig. 2.8, *a*.

The dependence of voltage from current is called volt-ampere characteristic (Fig. 2.8, *b*). If a resistance does not depend on the current we have direct proportionality between voltage and current. Such a value is called a linear resistance. In common case all resistors are non-linear. We can see a volt-ampere characteristic where linear resistance is proportional to tg α

$$R = \frac{U}{I} = \frac{m_U \cdot AB}{m_I \cdot OB} = \frac{m_U}{m_I} \cdot tg\,\alpha \qquad (2.17)$$

where m_U and m_I are scales for a voltage and a current correspondently.

2-5. The inductance parameter

Inductance was first discovered by Faraday. In a general way inductance can be characterized as that property of a circuit element by which energy is capable of being stored in a magnetic flux field. A significant and distinguishing feature of inductance, however, is that it makes itself felt in a circuit only when there is a changing current.

Thus, although a circuit element may have inductance by virtue of its geometrical and magnetic properties, its presence in the circuit is not exhibited unless there is a time rate of change of current.

This aspect of inductance is particularly stressed when we consider it from the circuit viewpoint.

Suppose we form a length of wire into a loop and pass a current through this wire as shown in Fig. 2.9, *a*.

A magnetic force field is generated around the wire in the form of magnetic flux ψ. This magnetic flux $\psi = L \cdot i$ and the current producing it are linearly related by a quantity called the *inductance* of a loop.

The inductance depends on the shape of the loop (or coil), but it is independent of the magnitude of the current flowing in it.

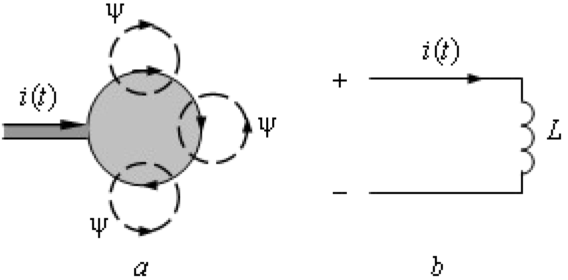


Fig. 2.9. A single loop

Faraday's law states that the magnetic flux induces the voltage in the loop which tends to produce a current in the loop. This induced current and associated magnetic flux tend to oppose the change in the original current and its associated magnetic flux. This induced voltage is related to the time rate of change of the magnetic flux:

$$u_L = \frac{d\psi}{dt} = L \cdot \frac{di}{dt}. \qquad (2.18)$$

Substituting $\psi = L \cdot i$, we obtain, for an inductance which does not change with time (i.e., the physical shape of the loop is constant),

$$u_L = L \cdot \frac{di}{dt}. \qquad (2.19)$$

Hence, the current-voltage relationship involving the inductance parameter is expressed by Eq. (2.19) and is repeated here for convenience.

This is illustrated in Fig. 2.9, *b*, where the circuit symbol for the inductor is shown. In general both i and u_L are functions of time. The potential difference u_L, appearing across the terminals of the inductance parameter when a changing current flows into terminal.

Any circuit element that exhibits the property of inductance is called an *inductor* and is denoted by the symbolism shown in Fig. 2.9, *b*. In the ideal sense the inductor is considered to be resistanceless, although practically it must contain the wire resistance out of which the inductor coil is formed.

Note that the current through an inductor at a particular time depends on the values of the inductor voltage at all prior instants of time. It follows from Eq. (2.19) that an appropriate defining equation for inductance is

$$L = \frac{u_L}{di/dt}\,. \tag{2.20}$$

Thus by recording the potential difference at a given time instant across the terminals of an inductor and dividing by the corresponding derivative of the current time function we determine the inductance parameter. Note that the units on inductance are volt-second / ampere. For simplicity this is more commonly called the Henry (H).

A linear inductor is one for which the inductance parameter is independent of current. As current flows through an inductor it creates a space flux. When this flux permeates air, strict proportionality between current and flux prevails so that the inductance parameter stays constant for all values of current.

However, to express the current in terms of the potential difference across the inductor, Eq. (2.20) must be transposed to read as follows:

$$di = \frac{1}{L} u_L \; dt\,.$$

In integral form this becomes

$$i(t) = \frac{1}{L} \int_0^t u_L \, dt + i(0)\,. \tag{2.21}$$

Eq. (2.21) thus reveals that the current in an inductor is dependent upon the integral of the voltage across its terminals as well as the initial current in the coil at the start of integration.

An examination of Eqs. (2.18) and (2.21) reveals an important property of inductance: the current in an inductor cannot change abruptly in zero time. This is made apparent from Eq. (2.18) by noting that a finite change in current in zero time calls for an infinite voltage to appear across the inductor, which is physically impossible.

On the other hand, Eq. (2.21) shows that in zero time the contribution to the inductor current from the integral term is zero so that the current immediately before and after application of voltage to the inductor is the same. In this sense, then, we may look upon inductance as exhibiting the property of inertia.

When the flux is made to penetrate iron, however, it is possible for large currents to upset the proportional relationship between the current and the flux it produces. In such a case the inductor is called nonlinear. But we will only consider linear inductors in this text.

If current does not change, the value of voltage across an inductive element is equal to zero.

Therefore when there is a steady-state rate in direct current circuit the inductive element is not considered, and the part of the circuit which contains such an element is considered as a usual conductor.

The current in inductance can be defined through the voltage u_L :

$$i = \frac{1}{L}\int_{-\infty}^{t} u_L dt \,. \tag{2.22}$$

The instantaneous power of the inductive element:

$$p_L = u_L i = Li\frac{di}{dt} \,. \tag{2.23}$$

It is positive when the current and a voltage have the same signs (the inductive element consumes energy from other parts of electric circuit and reserves it, transforming into the energy of magnetic field), and negative, when the current and voltage have different signs (the inductive element gives out the stored energy to other parts of a circuit).

The energy of magnetic field which is reserved in the inductive element by any moment of time:

$$W_L = W_M = \int_{-\infty}^{t} p_L dt = L\int_{0}^{i} idi = \frac{Li^2}{2} = \frac{\psi^2}{2L} \,. \tag{2.24}$$

2-6. The capacitance parameter

A charge and a voltage are related through some proportionality factor which can be reasonably expected to depend upon the medium in which the charges find themselves as well as upon the geometrical configuration prevailing. This proportionality is called *capacitance*.

It is the third basic parameter of electric circuit theory. In a general way capacitance can be characterized as that property of a circuit element in which energy is capable of being stored in an electric field. A significant and distinguishing feature of capacitance is that its influence in an electric circuit is manifested only when there exists a *changing potential difference* across the terminals of the circuit element. This aspect of capacitance is easily apparent when treated from a circuit viewpoint.

Capacitance is introduced as the proportionality factor relating the charge between two metal surfaces (or conductors) to the corresponding potential difference existing between them. Thus

$$q = C \cdot u_C . \tag{2.25}$$

where q represents the charge and u_C denotes the potential difference.

Lowercase letters are used here to stress the instantaneous nature of the quantities.

To obtain a definition of capacitance from a circuit viewpoint it is necessary to introduce current into the formulation of Eq. 2.20. This is readily accomplished by substituting Eq. 2.25 into the general expression for current as given by Eq. 1.5.

Thus

$$i(t) = \frac{dq}{dt} = C \cdot \frac{du_C}{dt} . \tag{2.26}$$

This expression shows the manner in which the current flowing through a capacitance parameter is related to the potential difference appearing across it.

Any circuit element showing the property of yielding a current which is directly proportional to the rate of change of the voltage across its terminals is called a capacitor. A capacitor usually consists of large metal surfaces separated by small distances.

A circuit element that exhibits the property of capacitance is called a capacitor and is denoted by the scheme shown in Fig. 2.10, *a*. And Fig. 2.10, *b* shows the vector diagram for this case.

With the establishment of current-voltage relationship of Eq. 2.26 the definition of capacitance from a circuit viewpoint readily follows:

$$C = \frac{i}{du_C / dt} . \tag{2.27}$$

Moreover, from the terms appearing on the right side of Eq. 2.27 it is seen that the unit of capacitance is ampere-second/volt or coulomb/volt. However, for convenience this quantity is defined as the *farad*. Hence the unit of capacitance is the farad.

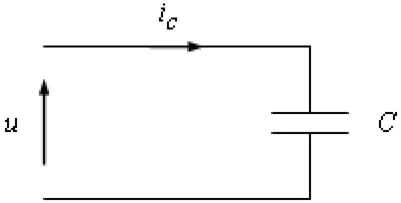


Fig. 2.10. The capacitor circuit

Equation 2.26 expresses the capacitor current in terms of the capacitor voltage. One can transpose it and integrate to express the capacitor voltage in terms of the capacitor current.

$$u_C = \frac{1}{C}\int_0^t i \cdot dt\,. \tag{2.28}$$

Electric circuit devices capable to reserve the energy of electric or magnetic field are called reactive.

The symbol of a resistor is presented in Fig. 2.11, *a*, the symbol of an inductor - in Fig. 2.11, *b*, *and* the symbol of a capacitor - in Fig. 2.11, *c*.

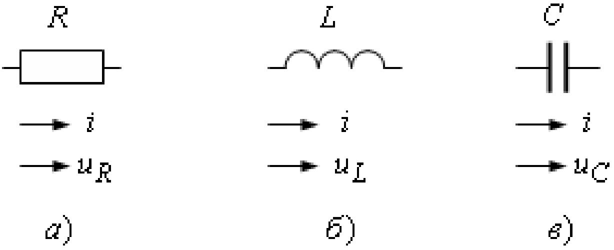


Fig.2.11. The symbols of elements

All basic formulas which describe the properties of reactive elements of an electric circuit are represented in Table 2.1.

Table 2.1

	Current	Voltage	Power	Energy
R	$i = \frac{u_R}{r}$	$u_R = ri$	$p_r = i^2 R$	$W_r = \int_0^t r i^2 d$
L	$i = \frac{1}{L} \int_{-\infty}^{t} u_L d$	$u_L = L \frac{di}{dt}$	$p_L = Li \frac{di}{dt}$	$W_L = \frac{Li^2}{2}$
C	$i = C \frac{du_C}{dt}$	$u_C = \frac{1}{C} \int_{-\infty}^{t} i dt$	$p_C = Cu_C \frac{du_C}{dt}$	$W_C = \frac{C u_C^2}{2}$

As we can see from this table, only in a resistive element the current and voltage are connected among themselves by an algebraic parity. But there are integral-differential parities between current and voltage in inductive and capacitive elements.

Summary review questions

1. What is the electric circuit?
2. What is the source of electric energy?
3. What is a two-terminal network?
4. Explain the difference between the passive two-terminal network and the active one?
5. What is the value of the internal resistance of an ideal voltage source? Explain.
6. What is the value of the internal resistance of an ideal current source? Explain.
7. Give the definitions of resistive, inductive and capacitive elements.
8. Which parameters are the main for resistive, inductive and capacitive elements?
9. What is called the external characteristic of a source of electric energy?
10. Explain the difference among external characteristics of an ideal e.m.f. source, an ideal current source and a real source of electric energy?

11 Explain why the voltage across the terminals of a capacitor is not allowed to change abruptly.

12. What is the definition of power? How is it related to energy?
13. Distinguish between the terms *resistor* and *resistance*.
14. How does temperature affect the value of resistance? Why it is important to know this?

15. Define the terms *resistivity*, *circular mils*, and *conductivity*.

16. Distinguish between the terms *inductor* and *inductance*.

17. Explain why it is that the current flowing through an inductor is not allowed to change instantaneously.

18. By what factor is the inductance of a coil increased when the number of turns is doubled?

19. Distinguish between the terms *capacitor* and *capacitance*.

20. What does it state Ampere's circuital law. Where can it be applied?

21. What dependence is called a volt-ampere characteristic?

22. Give the definitions of a node, a junction, a branch and a contour? What is called an independent contour?

23. When an ideal current source is placed as part of closed circuit, comment on what factors determine the voltage that appears across the current source terminals.

24. When an ideal voltage source is placed as part of a closed circuit, comment on the factors that determine the magnitude of the current associated with the voltage source.

25. Identify the various resistor types and comment on their corresponding tolerance ratings.

26. When the power of the inductive element is positive or negative? Explain.

27. What elements are called reactive? Why are they called so?

28. What are the units for resistance, inductance and capacitance?

29. Describe the inductance of a coil from a circuit viewpoint, energy viewpoint and geometrical viewpoint. Identify the important information about inductance that flows from a geometrical description and which cannot readily be obtained from the other two descriptions.

30. What is the principle cause of the deviation of inductance from linearity?

31. Describe capacitance from a circuit, an energy, and a geometric viewpoint. Identify the important information about capacitance that flows from a geometric description and which cannot readily be deduced from the other two descriptions.

32. Does a capacitor make its presence known in a circuit at all times? Explain.

Chapter 3

LINEAR DIRECT CURRENT CIRCUITS

3.1. Ohm's Law for a Branch

This law is perhaps one of the first things learnt about electricity in any elementary course on the subject. It is not at all unlikely that the reader was introduced to Ohm's law in its simplest form in his physics course. This is mentioned by way of further stressing its importance in the study of electric engineering.

George Simon Ohm (1787-1854) was a professor of mathematics and physics who devoted considerable time and effort to experiments dealing with voltaic cells and conductors.

These experiments included the effects of temperature on the resistivity of various metals. In spite of this extensive laboratory work, however, the law that bears his name was the result of a mathematical analysis of the galvanic circuit based on an analogy between the flow of electricity and the flow of heat. This work was described in a pamphlet published in 1827.

A mathematical description of Ohm's law:

$$I = \frac{U}{R} \tag{3.1}$$

where I is the current in amperes flowing in the circuit,

U is the potential difference in volts appearing across the conductor,

R is the resistance of the conductor in ohms.

So, the current in any electric circuit is equal to the voltage or electromotive force (e.m.f.) imposed upon this circuit, divided by the entire resistance of the circuit.

Ohm's law states that the strength of the current in a wire is directly proportional to the voltage between its ends, and inversely proportional to the resistance.

Accordingly, the amount of voltage is equal to the amount of current multiplied by the amount of resistance.

Ohm's law may be alternatively expressed as

$$U = I \cdot R \tag{3.2}$$

In this form it states that for any given potential difference, the amount of current produced is inversely proportional to the resistance, which in turn is dependent upon the composition of the wire.

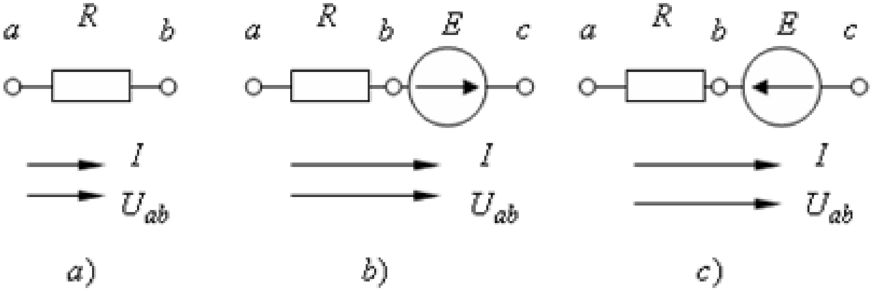


Fig. 3.1. Illustration of Ohm's law

Ohm's Law for a circuit branch with no e.m.f. in it is related its current and voltage. Thus, for Fig. 3.1, *a*

$$U_{ab} = I \cdot R \tag{3.3}$$

or

$$I = \frac{U_{ab}}{R} = \frac{\varphi_a - \varphi_b}{R} \tag{3.4}$$

Ohm's Law for a branch with a resistance and the e.m.f. gives the current in the branch in terms of the potential difference existing at its terminals and the e.m.f. it contains. Thus, for Fig. 3.1, *b* we have

$$I = \frac{\varphi_a - \varphi_b + E}{R} = \frac{U_{ac} + E}{R} \tag{3.5}$$

Similarly, when we have a circuit branch with a resistance and e.m.f. opposite to the current in it (See Fig. 3.1, *c*):

$$I = \frac{\varphi_a - \varphi_b - E}{R} = \frac{U_{ab} - E}{R} \tag{3.6}$$

In the general case

$$I = \frac{\varphi_a - \varphi_b \pm E}{R} = \frac{U_{ab} \pm E}{R} \tag{3.7}$$

Equation 3.7 is a mathematical expression of Ohm's Law for a branch containing an e.m.f. The "plus" sign before *E* applies to the case of Fig. 3.1, *b*, and the "minus" sign - to the case in Fig. 3.1, *c*. In a special case, when $E = 0$, Eq. 3.7 is reduced to Eq. 3.5.

3.2. Single- and Multi-Mesh Network

An electric circuit is a closed path for current flow. Electricity needs a complete loop to flow. With a broken path it will not move.

The word *network* is used synonymously with the term circuit and refers to any arrangement of passive and/or active circuit elements which form closed paths. Illustrated in Fig. 3.2 is a typical network.

Electric circuits may be divided into networks providing a single closed path known as a *mesh*, and networks providing several closed paths. An elementary *single-mesh* network in which all the elements carry the same current is shown in Fig. 2.5, *a* . An elementary *multi-mesh* network is shown in Fig. 3.2.

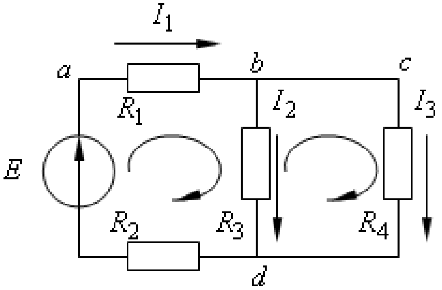


Fig. 3.2. An elementary multi-mesh network

The electric scheme shows the connection of elements in a considered electric circuit. Active and passive elements are "electric" components of the scheme. Branches and nodes are "geometrical" components of the circuit

A *node* of a network is an equipotential surface at which two or more circuit elements are joined. Thus in Fig. 3.2 terminals *a*, *b*, *c* and *d* are nodes. It is also called a s*imple node* (the node which may be eliminated).

Junction or difficult node is that point in a network where three or more circuit elements are joined. In the network of Fig. 3.2 there are two junction points - *b* and *d*.

A section of the circuit between two junctions is called an *electric branch*. It is formed by one or several elements which are connected by simple nodes. The same electric current flows in all elements of a branch. The elements which are in one branch are connected *in series*. Branches which have two common nodes are connected *in parallel*. We can see three branches in this circuit (see Fig 3.2).

A loop is any closed path of the network. Examples of loops in the figure are *abda*, *dbcd* and *abcda*.

A mesh is the most elementary form of a loop. It is property of a planar circuit diagram and must be so identified that it cannot be further divided into other loops. In the circuit both loops *abda* and *dbcd* qualify as meshes, but *abcda* cannot because it enclose the first two loops. *The closed mesh* is a set of branches forming a way, moving along which it is possible to return to a starting point, not passing more than once through each branch. The contour containing at least one branch, not entering into other contours is called *as independent*.

Depending on the way of connection of elements of an electric and circuit one must distinguish *the ramified* and *non-ramified* circuits. In *non-ramified* circuits all elements are connected in series. In *ramified circuits* elements incorporate in series, in parallel, in "delta" or in "wye" connections.

3.3. Kirchhoff's Laws

By the middle of the nineteenth century Gustav Robert Kirchhoff (1824-1887) had published the first systematic formulation of the principles governing the behavior of electric circuits. He advanced no new experimental facts or concepts but verily restated familiar principles. His work was embodied in two laws - a current and voltage law – which together are known as Kirchhoff's laws. It is upon these laws that electric circuit theory is based.

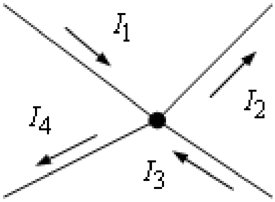


Fig. 3.3. Illustration of the Kirchhoff's current law

Kirchhoff's current law (abbreviated KCL) states that *the sum of the currents entering or leaving a junction point at any instant is equal* *to zero*. A junction point is that place in a circuit where two or more circuit elements are joined together. It is often called an independent node.

KCL or the first Kirchhoff's law may be also formulated in two ways: *(1) The algebraic sum of the currents flowing into a junction is zero.*

(2) The total current entering any junction of a circuit is equal to the total current leaving that junction.

The current law may be expressed mathematically as

$$\sum_{k=1}^{n} I_k = 0 \tag{3.8}$$

where *n* denotes the number of circuit elements connected to the node in question and Σ is the Greek symbol used to indicate summation. For the circuit of Fig. 3.3 there are four circuit branches joined at a node.

Referring to Fig. 3.3 and assuming that the currents entering a junction are positive and those leaving it are negative, from the first statement it follows

$$I_1 - I_2 + I_3 - I_4 = 0. \tag{3.9}$$

The minus signs are used because these currents are defined as leaving rather than entering a node. Equation (3.9) may be rewritten as:

$$I_1 + I_3 = I_2 + I_4. \tag{3.10}$$

And it states in the mathematical terms that the current entering the node is equal to the sum of the currents leaving it.

In countless experiments performed over the past century Kirchhoff's current law has never been found to be invalid. This is understandable when it is realized that the current law is nothing more than a restatement of the principle of conservation of charge.

This principle states that the number of electrons passing per second must be the same for all points in a circuit. Accordingly, if the summation of all the currents at a node were not to add up to zero, there would have to occur an accumulation of charge at the node. For the sake of argument suppose the sum of currents at some node where not zero but rather one ampere. Then charge would accumulate at the rate of 1 C for each second. We have already learned, however, that by Coulomb's law a charge accumulation of this kind would produce explosive forces. But experiment shows no evidence of this whatsoever. Any accumulation of charge at the node means an accumulation of mass.

So, physically, Kirchhoff's current law implies that there can be no accumulation of electric charge at any junction of a circuit.

Kirchhoff's voltage law (abbreviated KVL) or the second Kirchhoff's law may likewise be stated in two ways:

(1) at any time instant *the algebraic sum of voltages in any closed circuit is zero*. Essentially this law is a restatement of the law of conversation of energy.

A generalized formulation of KVL is written as

$$\sum_{k=1}^{n} U_k = 0 . \tag{3.11}$$

(2) The net voltage drop round a closed circuit equals the net e.m.f. acting in the same direction round the circuit:

$$\sum_{k=1}^{m} R_k \cdot I_k = \sum_{k=1}^{n} E_k . \tag{3.12}$$

The terms enter the respective sum with the "plus" sign if they are in the direction of summation round the circuit, and with the "minus" sign, if they are in the opposite direction.

Kirchhoff's laws are used in circuit problems to find the branch currents. Since each branch carries a current of its own, there are as many unknown currents as there are branches. Before writing down the Kirchhoff's equations for solution, one should:

(a) draw the meshes, that is assume a positive direction for each current by placing an arrow along the respective branch;

(b) assume a positive direction for summation round each mesh (loop) so that the Kirchhoff's voltage law can be written by inspection of the meshes.

For uniformity the same direction of summation round all the meshes should be chosen, for example, clockwise.

Let there be m branches and n nodes in a network. Then for the equations to be linearly independent, as many of them should be written by the Kirchhoff's current law as there are nodes minus one, or (n - 1). By the Kirchhoff's voltage law one should write as many equations as there are branches minus the number of equations written by the Kirchhoff's current law, or $m - (n - 1)$.

In writing equations by the Kirchhoff's voltage law it is important to choose the meshes successively so that each new mesh includes at least one branch not already included in a mesh. Such meshes are called *independent.*

Example 2. Find the branch currents in the network of Fig. 3.2, for $E = 100$ V, $R_1 = 5\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 10\ \Omega$ and $R_4 = 20\ \Omega$.

Solution: Assume a positive direction for current flow in each branch. The network has two nodes. So, only one equation can be written by Kirchhoff's current law:

$$I_1 - I_2 - I_3 = 0\,.$$

From Kirchhoff's voltage law there must be $n - (m - 1) = 3 - (2 - 1) = 2$ equations. Assume clockwise summation round the meshes. Thus for the left-hand mesh (loop) in Fig. 3.2 one can write

$$(R_1 + R_2)\cdot I_1 + R_3 \cdot I_2 = E\,.$$

For the right-hand loop we can write

$$-R_3 \cdot I_2 + R_4 \cdot I_3 = 0\,.$$

To the right of this equation we have zero because there are no e.m.f. in this loop. And the voltage drop $R_3 I_2$ has the "-" sign because I_2 flows against the direction of summation.

Solving these three equations simultaneously gives $I_1 = 3.75$ A; $I_2 = 2.5$ A; $I_3 = 1.25$ A. As stated earlier, it is necessary to put arrows on the graph to mark the assumed positive direction of current flow before writing equations. Frequently, the sense of current flow is a sheer guess. Then a minus sign in an answer indicates an incorrect guess. In this problem we can see that all the currents are positive. It means that we have chosen the positive directions of currents correctly.

3.4. Series and Parallel Combinations of Resistances

Very often in circuit analysis it is necessary to deal with several elements in a closed-loop circuit which exhibits the property of dissipating heat. In the circuits which supply electric energy to homes and commercial establishments, for example, several resistive elements are frequently combined to carry the same current. Thus in a circuit which supplies electric power to a lamp, three resistances are found: the internal resistance of the distribution transformer located beneath the street, the lamp resistance, and the resistance of the wires used to conduct the electric power to the lamp. Similarly, radio and television circuits as well as industrial electronic circuitry employ series combinations of various resistors to achieve specific desirable objectives. To analyze such circuits we must know how to treat resistances in series. By definition, circuit elements that carry the same current are said to be in series. Thus, *series connection* is such a connection of elements in electric circuit in which each pair of elements is joined by one simple node. Any series connection can include the arbitrary number of resistors and voltage sources. The same current flows through all the elements when they are connected in series. The circuit parameters appearing in Fig. 3.4, *a* are in series, for it is obvious here that the same current flows through each circuit element.

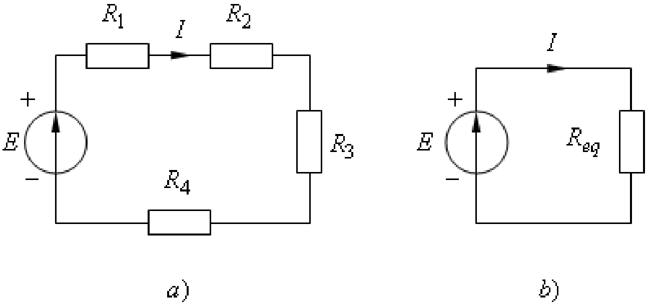


Fig. 3.4. Resistances in series: *a* original configuration: *b* equivalent circuit.

Applying Kirchhoff's voltage law, to the circuit of Fig. 3.4, a reveals a simple rule for handling resistances in series. Calling all voltage drops positive and voltage rises negative, as the circuit is traversed in the assumed current-flow direction, we can write

$$R_1 I + R_2 I + R_3 I + R_4 I = E\ . \tag{3.13}$$

Rearranging yields

$$I \cdot (R_1 + R_2 + R_3 + R_4) = E\ . \tag{3.14}$$

The current I is factored out because it is common to each resistance. Consequently, the quantity in parentheses may be replaced by an equivalent resistance which is given by

$$R_{eq} = R_1 + R_2 + R_3 + R_4\ . \tag{3.15}$$

The Eq. (3.14) may be written simply as

$$I \cdot R_{eq} = E \tag{3.16}$$

where R_{eq} denotes the *equivalent series resistance* of the circuit. It follows, too, from this analysis that the original circuit configuration of Fig 3.4, *a* may be replaced by the equivalent circuit shown in Fig. 3.4, *b*, which is merely a circuit interpretation of Eq. (3.16).

In general, if there are n series-connected resistances in a circuit, the *equivalent series resistance is obtained by taking the sum of the individual resistances.* Expressed mathematically, we have

$$R_{eq} = R1 + R2 + R3 + ... + R_n . \tag{3.17}$$

So, in series combination the equivalent resistance is equal to the sum of values of all resistances. The current can be defined by the next equation $I = U/R_{eq}$. A voltage across any resistance in this connection is defined according to Ohm's law.

Circuit elements are also very frequently found in parallel combination. In the home all electric light bulbs appear in parallel parts with respect to the source voltage. Other circuit elements such as the electric ironer and the electric broiler when used simultaneously are in parallel. By definition circuit elements are said to be in parallel when the same potential difference appears across their terminals. In accordance with this definition, the resistances R_1, R_2 and R_3 in Fig. 3.5, *a* are connected in parallel.

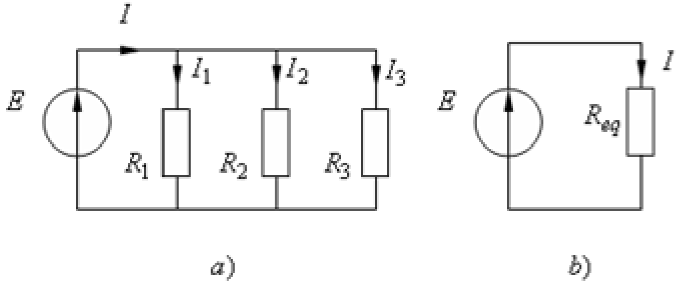


Fig. 3.5. Resistances in parallel: *a* original configuration: *b* equivalent circuit.

It is possible, again by means of circuit analysis, to treat this parallel combination of resistances in terms of an equivalent quantity.

KCL states that the current entering some node is equal to the sum of currents leaving this node. Expressed in equation form we have

$$I = I_1 + I_2 + I_3 . \tag{3.18}$$

However, from Ohm's law as it relates to each resistance, Eq (3.18) may be rewritten

$$I = \frac{E}{R_1} + \frac{E}{R_2} + \frac{E}{R_3}. \tag{3.19}$$

Again by factoring out the common variable, which in this instance is the voltage *E*, there results

$$I = E \cdot \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right). \tag{3.20}$$

The expression in parentheses may be replaced by an equivalent quantity defined as

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{3.21}$$

where R_{eq} denotes the equivalent resistance of the parallel combinations of resistances. Upon substituting Eq. (3.21) into Eq. (3.20) we obtain a simplified equation for the circuit. Thus

$$I = \frac{E}{R_{eq}}. \tag{3.22}$$

Figure 3.5. b, which is the circuit representation of Eq. 3.22, may accordingly be considered as the equivalent circuit of the configuration of Fig. 3.5, *a*.

A general formulation of the foregoing procedure states that the equivalent resistance of *n* parallel-connected resistances is the reciprocal of the sum of the reciprocals of the individual resistances. Expressed in equation form this becomes

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_n}. \tag{3.23}$$

Equation (3.23) deals with the reciprocals of resistance. The unit for this quantity is the siemens, *Sm*. On the basis of the definition for conductance, which was introduce in Chapter 2 (see Eq. 2.15), Eq. 3.23 may also be expressed as

$$G_{eq} = G_1 + G_2 + G_3 + \ldots + G_n. \tag{3.24}$$

Examples illustrating the use of Eqs. (3.15) and (3.24) to simplify circuit analysis are given in the next section.

The expressions for common resistance R_{eq} when two or three resistances are connected in parallel may be expressed as

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2}, \tag{3.25}$$

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_2}} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R} \tag{3.26}$$

3.5. Series-Parallel Circuit

In many practical circuits in electric engineering there occur situations where a circuit element is in series with a parallel combination of other circuit elements. Although these configurations may involve all three of the circuit parameters, we shall, in the remainder of this chapter, confine our attention exclusively to circuits involving only the resistance parameter. We follow this procedure because it is simpler. The theoretical considerations which are required to solve series-parallel combinations of the resistance parameter have already been studied. They are embodied in Kirchhoff's current and voltage laws, Ohm's law, and Eqs. (3.15) and (3.24) which are a consequence of these laws. The procedure for handling series-parallel circuits is best illustrated by examples.

Example 3.1. Let consider the circuit (see Fig. 3.6) with the mixed parallel- series connection.

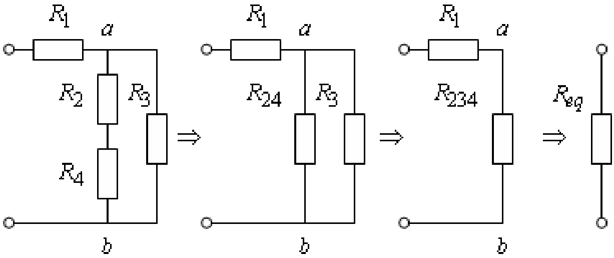


Fig. 3.6. The mixed connection

The values of resistive elements: $R_1 = 10\,\Omega$, $R_2 = 15\,\Omega$, $R_3 = 40\,\Omega$; $R_4 = 25\,\Omega$. The value of the voltage source $U = 120\,V$. One should find the value of current flowing from the voltage source.

Solution: As the first step in the procedure we find the equivalent resistance of the series combination of R_2 and R_4 in the second branch.

Using expression (3.17), we will get:

$$R_{24} = R_2 + R_4 = 15 + 25 = 40\,\Omega.$$

Now each of parallel branches consists of one element, and they form parallel connection with each other. Let's find their equivalent resistance with the help of Eq. (3.23):

$$\frac{1}{R_{234}} = \frac{1}{R_{24}} + \frac{1}{R_3} = \frac{1}{40} + \frac{1}{40} = 0{,}05\ \text{Sm}$$

As a result we have got a series connection of R_1 and R_{234}. One can define the equivalent resistance of the circuit:

$$R_{eq} = R_1 + R_{234} = 10 + 20 = 30\,\Omega.$$

So, now we have the circuit which consists of the voltage source and equivalent resistance. Then we can calculate the current flowing from the voltage source by Ohm's law

$$I = \frac{U}{R_{eq}} = \frac{120}{30} = 4\ A.$$

3.6. Star and Delta Connection of resistive elements

There are connections in complex circuits which cannot be referred to the series or parallel ones. The arrangement of three branches in a network as shown in Fig. 3.8 a is called a *delta connection*, and the one in Fig. 3.8 b as a *star* (or *wye*) *connection*.

Both the star and the delta are connected to the remainder of the network at three junctions. In network analysis it is often convenient to reduce a given network to a simpler arrangement by converting a star into a delta or back. In practice, delta-to-star conversion is used more often than star-to-delta transformation. Replacement of a delta of resistive elements with an equivalent star should be made so that the currents I_a, I_b, I_c and the voltages U_{ab}, U_{bc}, U_{ca} in the not affected part of the electric circuit remained the same after this conversion.

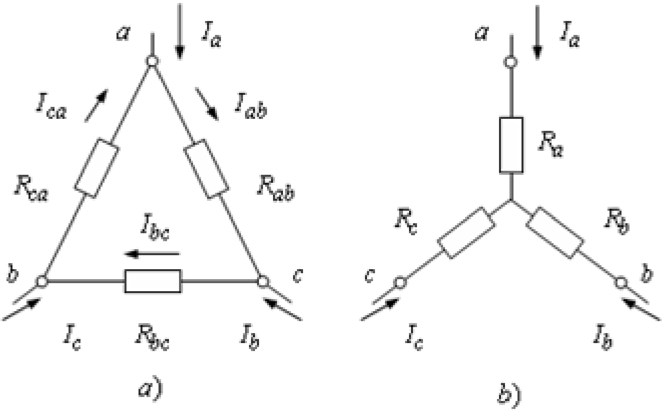


Fig.3.8. Three – terminal delta and wye networks.

Let's make for a delta contour (Fig. 3.8) the equation by Kirchhoff's voltage law, and for nodes *a* and *b* – the equations by the Kirchhoff's current law and we will receive system

$$\begin{cases} R_{ab}I_{ab} + R_{bc}I_{bc} + R_{ca}I_{ca} = 0 \\ I_a + I_{ca} - I_{ab} = 0 \\ I_b + I_{ab} - I_{bc} = 0. \end{cases}$$

Let's solve system concerning current I_{ab} :

$$I_{ab} = \frac{R_{ca}I_a - R_{bc}I_b}{R_{ab} + R_{bc} + R_{ca}} .$$

Then a voltage drop between junctions *a* and *b* in delta connection is equal

$$U_{ab} = R_{ab}I_{ab} = \frac{R_{ab}R_{ca}I_a - R_{ab}R_{bc}I_b}{R_{ab} + R_{bc} + R_{ca}} = \\ = \frac{R_{ab}R_{ca}}{R_{ab} + R_{bc} + R_{ca}} I_a - \frac{R_{bc}R_{ab}}{R_{ab} + R_{bc} + R_{ca}} I. \quad (3.27)$$

Then a voltage drop between junctions *a* and *b* in wye connection is equal

$$U_{ab} = R_a I_a - R_b I_b . \quad (3.28)$$

Equating multipliers at currents in expressions (3.27) and (3.28), we will get the next equations:

$$
\begin{aligned}
R_a &= \frac{R_{ab} \cdot R_{ca}}{R_{ab} + R_{bc} + R_{ca}}; \\
R_b &= \frac{R_{bc} \cdot R_{ab}}{R_{ab} + R_{bc} + R_{ca}}, \\
R_c &= \frac{R_{ca} \cdot R_{bc}}{R_{ab} + R_{bc} + R_{ca}}.
\end{aligned}
\tag{3.29}
$$

From the equations (3.27) it is possible to deduce the resistances of resistive elements of the equivalent delta connection:

$$
\begin{aligned}
R_{ab} &= R_a + R_b + \frac{R_a \cdot R_b}{R_c}, \\
R_{bc} &= R_b + R_c + \frac{R_b \cdot R_c}{R_a}, \\
R_{ca} &= R_c + R_a + \frac{R_c \cdot R_a}{R_b}.
\end{aligned}
\tag{3.30}
$$

3.7. The Method of Direct Application of Ohm's law (the method of curtailment)

The essence of a method of direct application of Ohm's law (it is also named the method of equivalent transformations or the method of curtailment consists in replacement of circuit sections with equivalent resistances. It allows to reduce a complex circuit to the elementary one, i.e. consisting of an energy source and the equivalent resistance. After transformation of a circuit the calculation is reduced to a number of arithmetic operations by Ohm's law. On defining the current through this resistive element, one must carry out return transformation of the equivalent electric scheme to the initial one, calculating voltages across the parts of the circuit and distribution of currents in parallel branches. The method of equivalent transformations is expedient to apply for the solving of problems in which the electric circuit has no more than three contours.

Example 3.2. Let consider the electric circuit with e.m.f. source (see Fig. 3.9). The circuit parameters: $E = 60\,V$, $R_1 = 8\,\Omega$, $R_2 = 56\,\Omega$, $R_3 = 28\,\Omega$, $R_4 = 14\,\Omega$, $R_5 = 7\,\Omega$, $R_6 = 6\,\Omega$. Find currents in all circuit branches.

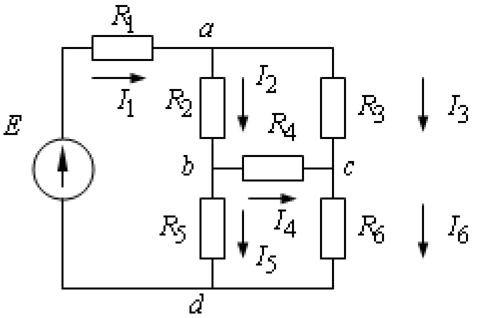
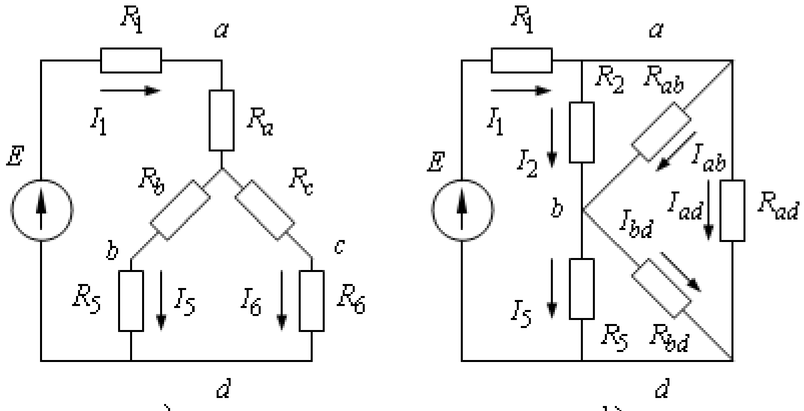


Fig. 3.9

Solution: There are no series or parallel connections of resistive elements in the given electric scheme. But there are three star connections R_1, R_2, R_3; R_2, R_4, R_5; R_3, R_4, R_6, and two delta connections $R_2,\ R_3,\ R_4$ and $R_{4,}\ R_{5,}\ R_6$ here.

As a result of transformation of any of these four last connections the scheme becomes simpler and led to the mixed connection. The result of converting a delta R_2, R_3, R_4 into equivalent star is shown in Fig. 3.10, *a*, and the result of converting of wye R_3, R_4, R_6 into delta connection is shown in Fig. 3.10, *b*.



a) b)

Fig.3.10

According to Eq. (3.28) the resistances of an equivalent star:

$$R_a = \frac{R_2 \cdot R_3}{R_2 + R_3 + R_4} = \frac{56 \cdot 28}{56 + 28 + 14} = 16\ \Omega;$$

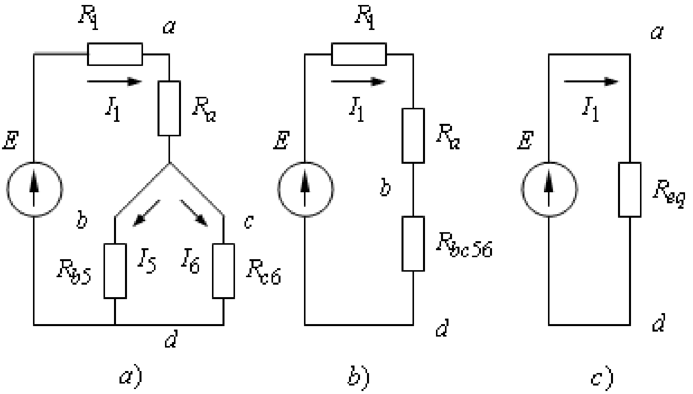
$$R_b = \frac{R_2 \cdot R_4}{R_2 + R_3 + R_4} = \frac{56 \cdot 14}{56 + 28 + 14} = 8\ \Omega;\ \ ;$$

$$R_c = \frac{R_3 \cdot R_4}{R_2 + R_3 + R_4} = \frac{28 \cdot 14}{56 + 28 + 14} = 4\ \Omega.$$

The resistive elements R_b, R_5 and R_c, R_6 are connected consistently. Their equivalent resistance $R_{b5} = R_b + R_5 = 8 + 7 = 15\ \Omega$;

$$R_{\tilde{n}6} = R_{\tilde{n}} + R_6 = 4 + 6 = 10\ \Omega.$$

The resistors R_{b5} и R_{c6} are joined in parallel (Fig. 3.11, a).



a) b) c)

Fig. 3.11

Their equivalent resistance (by Eq. 3.25) is

$$R_{bc56} = \frac{R_{b5} \cdot R_{c6}}{R_{b5} + R_{c6}} = \frac{15 \cdot 10}{15 + 10} = 6\ \Omega.$$

Now the resistors R_1, R_a, R_{bc56} are connected in series (Fig. 3.11, *b*). The equivalent resistance of the circuit:

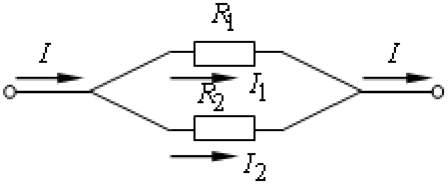
$$R_{eq} = R_1 + R_a + R_{234} = 8 + 16 + 20 = 30\ \Omega.$$

The total current of the circuit (Fig. 3.11, *c*) by Ohm's law:

$$I_1 = \frac{E}{R_{eq}} = \frac{60}{30} = 2\ A.$$

Other currents might be found in several ways. Is this case the more convenient is the following method.

Let we have the circuit which involves the combination of two parallel resistors R_1 and R_2.



The voltage on a ramification in a case of two resistors

$$R \cdot I = R_1 \cdot I_1 = R_2 \cdot I_2 \tag{3.31}$$

where $R = \frac{R_1 \cdot R_2}{R_1 + R_2}$ is the equivalent resistance of two parallel ones.

From Eq. (3.31) we can express the current in the first parallel branch

$$I_1 = I\frac{R}{R_1} = I\frac{R_1 \cdot R_2}{R_1 + R_2} \cdot \frac{1}{R_1} = I\frac{R_2}{R_1 + R_2}. \tag{3.32}$$

Analogically, the current in the second parallel branch

$$I_2 = I\frac{R}{R_2} = I\frac{R_1}{R_1 + R_2} \tag{3.33}$$

Equations (3.32) and (3.33) allow getting so-called "the resolving of the total current" into two parallel branches.

As a voltage across parallel branches is the same, at a parallel connection of energy receivers and a given voltage, the mode of operation of each of them does not influence on the mode of operation of others. The consumers of electrical energy - electric motors, furnaces, heating devices, bulbs calculated for the operation at invariable nominal voltage, are usually connected in parallel to each other.

Then we can calculate the currents in the parallel branches using Eq. (3.32) and (3.33), (see Fig. 3.11, *a*):

$$I_5 = I_1 \frac{R_{c6}}{R_{b5} + R_{c6}} = 2 \cdot \frac{10}{15+10} = 0.8\ A;$$

$$I_6 = I_1 \frac{R_{b5}}{R_{b5} + R_{c6}} = 2 \cdot \frac{15}{15+10} = 1.2\ A$$

For finding currents I_2 and I_3 we have to return to the initial circuit (Fig. 3.9) and use Kirchhoff's laws.

Let's compose the equation by the second Kirchhoff's law for a mesh of the initial scheme which consists of resistors R_4, R_5 and R_6:

$$I_4 R_4 + I_6 R_6 - I_5 R_5 = 0 .$$

From the previous equation:

$$I_4 = \frac{I_5 R_5 - I_6 R_6}{R_4} = \frac{0{,}8 \cdot 7 - 1{,}2 \cdot 6}{14} = -\frac{80}{7} \approx -0{,}114\ A .$$

The "minus" sign says that the true direction of this current does not coincide with the chosen one.

Now we can write equations by the first Kirchhoff's law for the junctions *b* and *c*:

$$I_2 = I_4 + I_5 = -0{,}114 + 0{,}6 = 0{,}486\ A,$$

$$I_3 = I_6 - I_4 = 1{,}2 - (-0{,}114) = 1{,}314\ A$$

which are the desired results.

3.8. The Method of Direct Application of Kirchhoff's Laws

Kirchhoff's laws are the universal method of electric circuits analysis. The essence of the method of direct application of Kirchhoff's laws consists in drawing up and the following calculation of the system of independent equations by Kirchhoff's laws. The quantity of such equations must coincide with the quantity of unknown values. Usually the unknown values are branch currents.

We define the quantity of junctions *K* for an initial circuit, the number of all branches *N*, and the number of the branches containing current sources N_J. The currents in the branches with current sources are known, therefore the quantity of unknown currents is equal to $N - N_J$. The total quantity of equations is equal to the number of unknown currents.

For any junction of the circuit one can compose the equations by the first Kirchhoff's law, but only $(K-1)$ equations will be independent (as the equation for the last junction is a consequence of all previous equations). Therefore we can make $(K-1)$ the equations by the KCL.

Missing $(N-N_J)-(K-1)]$ equations are worked out by the second Kirchhoff's law.

Doing it, we choose the independent meshes embracing all the branches in which there are no current sources. The loop is considered to be independent if one branch which does not enter into remaining loops enters into it at least. The direction of summation round the mesh is chosen arbitrarily.

Beforehand it is necessary to set arbitrarily the directions of branch currents in the network, except branches with current sources (these currents are known and the directions of currents in such branches are defined by current sources).

Example 3.3. Let's consider the direct current circuit (Fig. 3.12). The parameters of energy sources $E_1 = 50\ V$, $E_2 = 30\ V$, $J = 1\ A$, the values of resistances $R_1 = 10\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 20\ \Omega$, $R_4 = 25\ \Omega$, $R_5 = 30\ \Omega$, $R_6 = 35\ \Omega$. Define branch currents in this electric circuit.

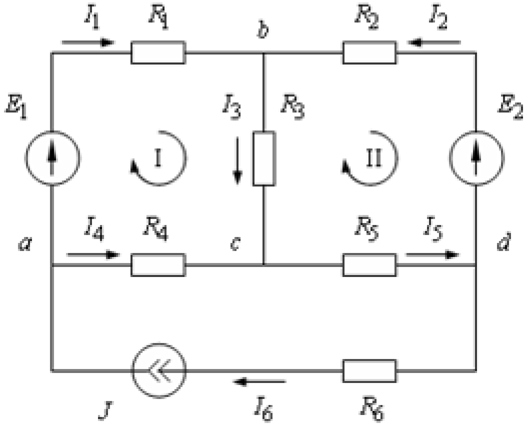


Fig. 3.12

The solution. The number of junctions $K = 3$, the number of branches $N = 6$, the quantity of branches with current sources $N_J = 1$, and the number of unknown currents $N - N_J = 6 - 1 = 5$.

By the first Kirchhoff's law it is possible to make $K-1=4-1=3$ independent equations.

Missing $[(N-N_J)-(K-1)]=[(6-1)-(4-1)]=2$ equations can be composed by the second Kirchhoff's law.

We arbitrarily set the directions of branch currents. Then we can write the equations by the first Kirchhoff's law for junctions *a*, *b* and *c*. We choose the directions of summation round both meshes which contain accordingly elements E_1, R_1, R_3, R_4 and elements E_2, R_5, R_3, R_2 as clockwise. Writing the equations, we consider, that $I_6=J$.

$$\begin{cases} J-I_1-I_4=0; \\ I_1+I_2-I_3=0; \\ I_3+I_4-I_5=0; \\ I_1R_1+I_3R_3-I_4R_4=E_1; \\ -I_2R_2-I_3R_3-I_5R_5=-E_2. \end{cases}$$

For calculation of this system with the help of a computer it is necessary to present it in the matrix form:

$$AX=B,$$

where *A* is a square matrix of coefficients at unknown values; *X* is a matrix-column of unknown values; *B* is a matrix-column of constant terms of the equations.

$$\begin{vmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ R_1 & 0 & R_3 & -R_4 & 0 \\ 0 & -R_2 & -R_3 & 0 & -R_5 \end{vmatrix} \cdot \begin{vmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{vmatrix} = \begin{vmatrix} -J \\ 0 \\ 0 \\ E_1 \\ -E_2 \end{vmatrix}$$

or

$$\begin{vmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 10 & 0 & 20 & -25 & 0 \\ 0 & -15 & -20 & 0 & -30 \end{vmatrix} \cdot \begin{vmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 50 \\ -30 \end{vmatrix}$$

The solution of this system in the matrix form we search in the form

$$X = A^{-1}B\,.$$

As a result of calculation of this system of equations we define branch currents.

The currents in the branches of this electric circuit:

$$X = \begin{vmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{vmatrix} = \begin{vmatrix} 1,535 \\ -0,472 \\ 1,063 \\ -0,535 \\ 0,528 \end{vmatrix}$$

As the directions of currents were chosen arbitrarily, as a result of calculation some currents may have negative values. It means that the real branch currents have opposite directions.

The method of direct application of Kirchhoff's laws demands the working out of a large number of equations. Therefore other methods are developed for practical calculations.

3.9. Network Analysis by Mesh Currents

The word network is used synonymously with the term circuit and refers to any arrangement of passive and active circuit elements which form closed paths.

A mesh is the most elementary form of a loop. It is a property of a planar network diagram and must be so identified that it cannot be further divided into other loops.

By definition, a mesh current is that current which flows around the perimeter of a mesh Equations are written for mesh currents. After they are solved, the branch currents are found in terms of mesh currents.

Thus, the mesh-current method is a form of network analysis on a current basis. The number of equations involved is the same as written by Kirchhoff's voltage law.

We shall consider here the mesh method of solving problems in circuitry because it offers some advantages over the methods discussed so far. To illustrate, let us describe the method and then apply it to an example.

The essence of *mesh-current method* consists that the true currents flowing in circuit branches, are exchanged by the algebraic sum of mesh currents. Thus for branches which enter only into one contour, the value of current is equal to the algebraic value of a corresponding mesh current (taken with the "+" sign if the directions of a branch current in and a mesh current coincide, or with the "-" sign if they do not coincide). For adjacent branches which are a part of several contours, the value of current is equal to an algebraic sum of corresponding mesh currents. The current signs are chosen analogously. The branches containing current sources can enter only into one independent contour.{n of the complex electric circuits with a considerable quantity of nodes. It allows to expel the equations which are made by the first Kirchhoff's law. The number of unknown values in this method is equal to the number of the equations which would be necessary to make for a circuit by the Kirchhoff's voltage law.

Generally for a compound circuit (for example, with three contour) the simultaneous equations according to the mesh-current method can be written as

$$\begin{cases} R_{11}\cdot I_{11}+R_{12}\cdot I_{22}+R_{13}\cdot I_{33}=E_{11} \\ R_{21}\cdot I_{11}+R_{22}\cdot I_{22}+R_{23}\cdot I_{33}=E_{22} \\ R_{31}\cdot I_{11}+R_{32}\cdot I_{22}+R_{33}\cdot I_{33}=E_{33} \end{cases} \tag{3.34}$$

where in the first of equations (3.34) current I_{11} of the first mesh is multiplied by the self-resistance R_{11} of the same mesh, and current I_{22} by the resistance of the common (adjacent) branch, R_{12}, (or the mutual resistance) with a minus sign. This mutual resistance has the "-" sign if two mesh currents in the common branch flow against each other, and the "+" sign if they flow in the same direction.

E_{11} is the e.m.f. across the first mesh; when there are several e.m.f.s. in the loop it is equal to the algebraic sum of e.m.f.s. around that mesh. The e.m.f.s. which is in the direction of summation enter it with the plus sign. E_{22} is the e.m.f. across the second mesh; E_{33} is the e.m.f. across the third mesh; R_{22} is the self-resistance of the second mesh; R_{33} is the self-resistance of the third mesh; R_{13}, R_{23}, R_{31}, R_{32} are the resistances of the common branches (or the mutual resistances). A mutual resistance may be with the "minus" or the "plus" sign.

So that all the resistances may have the same sign, all the currents should be represented as circulating in the same direction (clockwise, for example). If the actual direction of the current in any mesh is not in the direction of the arrow, the computation will show the value of the corresponding current to be negative.

In branches which are not common to any two adjacent meshes the calculated mesh currents will be the actual currents. For adjacent branches the actual currents are determined from the mesh currents.

Illustrated in Fig. 3.13 is a typical network. This network is composed of nine circuit elements. There are six passive elements, namely resistances, and three active elements, namely the energy sources E_1 and E_2 and a current source J.

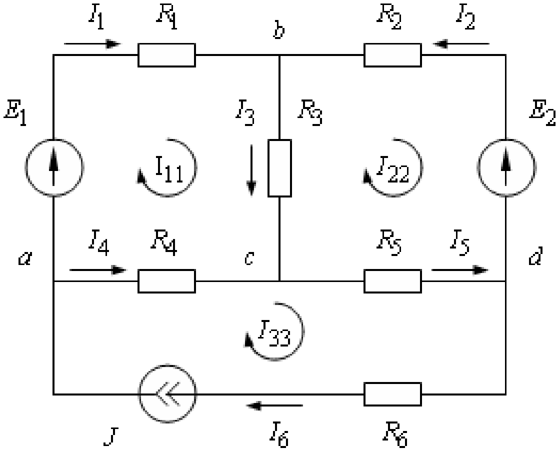


Fig.3.13

Example 3.4. Let's define currents in the branches of electric circuit with parameters:, $E_1 = 50\ V$, $E_2 = 30\ V$, $J = 1\ A$, and resistances: $R_1 = 10\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 20\ \Omega$, $R_4 = 25\ \Omega$, $R_5 = 30\ \Omega$, $R_6 = 35\ \Omega$.

The solution. Let mesh currents $I_{11,}$ $I_{22,}$ I_{33} flow through independent contours. The directions of these currents are chosen at will (for example, clockwise). The current in the sixth branch is spotted by the current source. On the other side, the sixth branch is a part only of the third contour, and the chosen direction of the current in the branch coincides with the chosen direction of the mesh current. That is why $I_6 = J$. Hence, $I_{33} = J$.

Thus, the quantity of unknown mesh currents is equal to the quantity of the equations which it would be necessary to compose for given circuit according to the second Kirchhoff's law.

Let write down equations according the Kirchhoff's voltage law for the first and second contours.

The left part of the equations contains voltage drops across resistive elements of a contour created by proceeding currents, the right part – the e.m.f.s acting in the contour.

The value of e.m.f. source is considered with "+" sign if its direction coincides with the direction of mesh current (the direction of summation round the mesh). A voltage drop which is created by a natural mesh current, is always considered with "+" sign, and voltage drops created in contour elements by the currents of adjacent contours, are considered with "+"signs if the currents of the adjacent contours proceed through the adjacent branches in the same direction, as the natural mesh current.

Now we apply Kirchhoff's voltage law to each mesh and write voltage equations for clockwise summation round the meshes:

$$(R_1 + R_3 + R_4)I_{11} - R_3 I_{22} = E_1 + JR_4;$$
$$-R_3 I_{11} + (R_2 + R_3 + R_5)I_{22} = -E_2 + JR_5,$$

where $E_{11} = E_1 + R_4 J = 50 + 25 \cdot 1 = 75\ V$ is the e.m.f. across the first mesh;

$E_{22} = -E_2 + R_5 J = -30 + 30 \cdot 1 = 0\,V$ is the e.m.f. across the second mesh;

$R_{11} = R_1 + R_3 + R_4 = 10 + 20 + 25 = 55\ \Omega$ is the self-resistance of the first mesh;

$R_{11} = R_2 + R_3 + R_5 = 15 + 20 + 30 = 65\ \Omega$ is the self-resistance of the second mesh;

$R_{12} = R_{21} = -R_3 = 20\ \Omega$ are mutual resistances of the first and the second meshes.

After substituting known values we have got the following system of equations:

$$\begin{cases} 55I_{11} - 20I_{22} = 85; \\ -20I_{11} + 65I_{22} = 0. \end{cases}$$

We express current I_{11} from the second equation of the system $I_{11} = 3{,}25 I_{22}$ and substitute it in the first equation. We have got

$$55 \cdot 3{,}25 I_{22} - 20 I_{22} = 85$$

or

$$78{,}75 I_{22} - 20 I_{22} = 85 .$$

Then the value of the second mesh current $I_{22} = 0{,}472\ A$.
And the value of the mesh current in the first loop $I_{11} = 1{,}535\ A$.

Actual currents in all branches of the electric circuit:

$$I_{11} = I_{11} = 1{,}535\ A, \quad I_2 = -I_{22} = -0{,}472\ A,$$
$$I_3 = I_{11} - I_{22} = 1{,}535 - 0{,}472 = 1{,}063\ A,$$
$$I_4 = I_{33} - I_{11} = J - I_{11} = 1 - 1{,}535 = -0{,}535\ A, ;$$
$$I_5 = I_{33} - I_{22} = J - I_{22} = 1 - 0{,}472 = 0{,}528\ A,$$
$$I_6 = I_{33} = J = 1\ A.$$

3.10. Node-analysis Method

The current in any branch of a network can be found by Ohm's law for a branch containing an e.m.f. This calls for knowledge of the potential difference across the terminals of the branch or, which is the same, across the nodes bounding the branch in question. The analysis of network in which the unknown quantities are the voltages across the branches of the network is known as the node-analysis method.

When making the equations by node-analysis method one must choose a basis node to be earthed without affecting the distribution of currents around the network. In other words, we consider that its potential is equal to zero. As a result, the number of unknown potentials becomes equal to the number of the independent equations which are made by the first Kirchhoff's law. On solving the system concerning these potentials, it is possible to define currents through known potentials.

This method is expedient to use for calculation of the complicated electric circuits with a small amount of nodes. It allows to expel the equations which are made by the second Kirchhoff's law.

Generally for any compound circuit it is necessary to make a system of equations of a following state:

$$\begin{cases} +G_{11}\varphi_1 \; -G_{12}\varphi_2 - ... \; -G_{1m}\varphi_m - ... \; -G_{1n}\varphi_n = J_{11}; \\ \quad ... \qquad ... \qquad ... \qquad ... \qquad ... \\ -G_{m1}\varphi_1 - G_{m2}\varphi_2 - ... + G_{mm}\varphi_m - ... - G_{mn}\varphi_n = J_{mm}; \\ \quad ... \qquad ... \qquad ... \qquad ... \qquad ... \\ -G_{n1}\varphi_1 \; -G_{n2}\varphi_2 - ... \; -G_{nm}\varphi_m - ... \; +G_{nn}\varphi_n = J_{nn}, \end{cases}$$

where φ_1, φ_2, …, φ_n are unknown nodal potentials; *m* is a variable number of a node; G_{11}, G_{22}, …, G_{nn} are self-conductances of nodes (the total conductance of branches which are joined to the appropriate node); $G_{km} = G_{mk}$ are mutual conductances of nodes (the total conductance of branches which join nodes *k* and *m);* J_{11}, J_{22}, …, J_{nn} are the nodal currents considering the presence of energy sources in branches which are joined to the appropriate node, $J_{kk} = \sum_{m=1}^{n} E_m G_m + \sum_{m=1}^{n} J_m$ is the nodal current of node *k*, where $\sum_{m=1}^{n} E_m G_m$ is the algebraic sum of multiplications of the branch e.m.f.s, joined to the node *k*, by the conductance of this branches. For all this, those ones which direct towards node enter the sum with a plus sign, and those which have the opposite directions enter the sum with a minus sum. $\sum_{m=1}^{n} J_m$ is an algebraic sum of the currents of current sources in branches which are joined to the node *k*, and those which direct towards node enter the sum with a plus sign, and those which direct from the node enter the sum with a minus sign.

Solving the system, we define nodal potentials φ_1, φ_2, …, φ_n, and then the actual currents in branches. Let consider this procedure as an example.

Example 3.5. Suppose, we have the direct current electric circuit (Fig. 3.14). with parameters of energy sources $E_1 = 50\ V$, $E_2 = 30\ V$, $J = 1\ A$ and resistances $R_1 = 10\ \Omega$, $R_2 = 15\ \Omega$, $R_2 = 20\ \Omega$, $R_4 = 25\ \Omega$, $R_5 = 30\ \Omega$, $R_6 = 35\ \Omega$, . Calculate currents in all branches of the electric circuit.

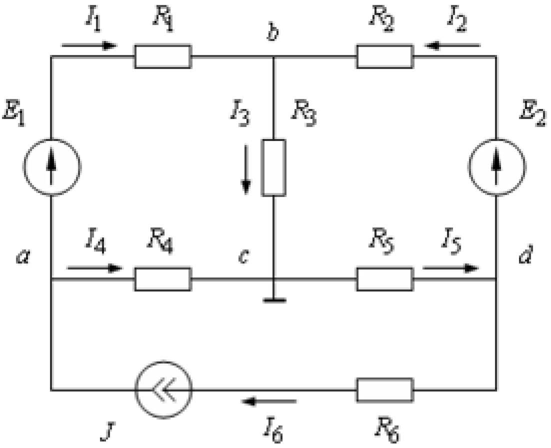


Fig.3.14

Solution. We will choose arbitrarily the directions of branch currents. Let's express currents in branches by Ohm's law through a circuit parameters and unknown nodal potentials.

$$I_1 = \frac{\varphi_a - \varphi_b + E_1}{R_1} = G_1(\varphi_a - \varphi_b + E_1)\,,$$

$$I_2 = \frac{\varphi_d - \varphi_b + E_2}{R_2} = G_2(\varphi_d - \varphi_b + E_2)\,,$$

$$I_3 = \frac{\varphi_b - \varphi_c}{R_3} = G_3(\varphi_b - \varphi_c)\,,$$

$$I_4 = \frac{\varphi_a - \varphi_c}{R_4} = G_4(\varphi_a - \varphi_c)\,,$$

$$I_5 = \frac{\varphi_c - \varphi_d}{R_5} = G_5(\varphi_c - \varphi_d)$$

where $G_m = 1/R_m$ is the conductance of the appropriate branch. The current in the sixth branch is defined by the current source $I_6 = J$.

The potential of node *c* is chosen to be earthed (at zero potential). So, we have to determine only the potentials of three nodes.

Let's work out the equations for ungrounded junctions *a*, *b*, *d* by the first Kirchhoff's law:

$$-I_1 - I_4 + I_6 = 0,$$

$$I_1 + I_2 - I_3 = 0,$$

$$-I_2 + I_5 - I_6 = 0$$

or taking into account written before the relationships for currents:

$$-G_1(\varphi_a - \varphi_b + E_1) - G_4(\varphi_a - \varphi_c) + J = 0;$$

$$+G_1(\varphi_a - \varphi_b + E_1) + G_2(\varphi_d - \varphi_b + E_2) - G_3(\varphi_b - \varphi_c) = 0;$$

$$-G_2(\varphi_d - \varphi_b + E_2) + G_5(\varphi_c - \varphi_d) - J = 0.$$

Let's regroup the elements of the equations concerning unknown nodal potentials and we will consider, that $\varphi_c = 0$

$$\begin{cases} (G_1 + G_4) \cdot \varphi_a - G_1 \cdot \varphi_b - 0 \cdot \varphi_d = J - G_1 \cdot E_1; \\ -G_1 \cdot \varphi_a + (G_1 + G_2 + G_3) \cdot \varphi_b - G_2 \cdot \varphi_d = G_1 \cdot E_1 + G_2 \cdot E_2; \\ -0 \cdot \varphi_a - G_2 \cdot \varphi_b + (G_2 + G_5) \cdot \varphi_d = -J - G_2 \cdot E_2, \end{cases}$$

or

$$\begin{cases} +G_{11} \cdot \varphi_a - G_{12} \cdot \varphi_b - G_{13} \cdot \varphi_d = J_{11}; \\ -G_{21} \cdot \varphi_a + G_{22} \cdot \varphi_b - G_{23} \cdot \varphi_d = J_{22}; \\ -G_{31} \cdot \varphi_a - G_{32} \cdot \varphi_b + G_{33} \cdot \varphi_d = J_{33}, \end{cases}$$

where the self-conductance of node *a* is

$$G_{11} = 1/R_1 + 1/R_4 = 1/10 + 1/25 = 0{,}14 \text{ Sm},$$

the self-conductance of node *b* is

$$G_{22} = 1/R_1 + 1/R_2 + 1/R_3 = 1/10 + 1/15 + 1/20 \approx 0{,}2167 \text{ Sm},$$

the self-conductance of node c

$$G_{33} = 1/R_2 + 1/R_5 = 1/15 + 1/30 = 0{,}1 \text{ Sm},$$

the mutual conductance of nodes *a* and *b*

$$G_{12} = G_{21} = 1/R_1 = 1/10 = 0{,}1 \text{ Sm},$$

the mutual conductance of nodes *a* and *c*

$$G_{13} = G_{31} = 0\ Sm,$$

(because the resistance of a branch containing a current source is infinitely large); the mutual conductance of nodes *b* and *c*

$$G_{23} = G_{32} = 1/R_2 = 1/15 = 0{,}0667 \text{ Sm},$$

the nodal current of the node *a*

$$J_{11} = J - E_1/R_1 = 1 - 50/10 = -4 \text{ A},$$

the nodal current of the node *b*

$$J_{22} = E_1/R_1 + E_2/R_2 = 50/10 + 30/15 = 7 \text{ A},$$

the nodal current of the node *c*

$$J_{22} = -J - E_2/R_2 = -1 - 30/15 = -3 \text{ A}.$$

Substitute known values into the last system of equations:

$$\begin{cases} 0{,}14 \cdot \varphi_a - 0{,}1 \cdot \varphi_b - 0 \cdot \varphi_d = -4; \\ -0{,}1 \cdot \varphi_a + 0{,}2167 \cdot \varphi_b - 0{,}0667 \cdot \varphi_d = 7; \\ -0 \cdot \varphi_a - 0{,}0667 \cdot \varphi_b + 0{,}1 \cdot \varphi_d = -3. \end{cases}$$

We solve this system by Cramer's rule.
The main determinant of the system is:

$$\Delta = \begin{vmatrix} 0{,}14 & -0{,}1 & -0 \\ -0{,}1 & 0{,}2167 & -0{,}0667 \\ -0 & -0{,}0667 & 0{,}1 \end{vmatrix} = 0{,}001411 \ Sm^2$$

Auxiliary determinants:

$$\Delta_a = \begin{vmatrix} -4 & -0{,}1 & -0 \\ 7 & 0{,}2167 & -0{,}0667 \\ -3 & -0{,}0667 & 0{,}1 \end{vmatrix} = \text{-}0{,}01889 \ A \cdot Sm^2;$$

$$\Delta_b = \begin{vmatrix} 0{,}14 & -4 & -0 \\ -0{,}1 & 7 & -0{,}0667 \\ -0 & -3 & 0{,}1 \end{vmatrix} = 0{,}02999 \ A \cdot Sm^2;$$

$$\Delta_d = \begin{vmatrix} 0{,}14 & -0{,}1 & -4 \\ -0{,}1 & 0{,}2167 & 7 \\ -0 & -0{,}0667 & -3 \end{vmatrix} = -0{,}02233 \ A \cdot Sm^2.$$

We calculate nodal potentials:

$$\varphi_a = -\frac{0{,}01889}{0{,}001411} = \text{-}13{,}39 \text{ V;}$$

$$\varphi_b = \frac{0{,}02999}{0{,}001411} = 21{,}25 \text{ V;}$$

$$\varphi_d = -\frac{0{,}02233}{0{,}001411} = \text{-}15{,}82 \text{ V.}$$

And the currents in all branches of the electric circuit:

$$I_1 = \frac{-13{,}39 - 21{,}25 + 50}{10} = 1{,}536 \ A, \quad I_2 = \frac{-15{,}82 - 21{,}25 + 30}{15} = 0{,}471 \ A$$

$$I_3 = \frac{21{,}25 - 0}{20} = 1{,}063 \ A, \qquad I_4 = \frac{-13{,}39 - 0}{25} = 0{,}536 \ A,$$

$$I_5 = \frac{0 + 15{,}82}{30} = 0{,}527 \ A, \qquad I_6 = J = 1 \ A.$$

3.11. Nodal Pairs or the Method of Two nodes

Sometimes an electric circuit may have only two nodes, as shown in Fig. 3.15.

The currents in such a network can be conveniently found by the nodal-pair method. In this method we at first define the voltage between two nodes (junctions), and the branch currents are found in its terms.

Let's consider junction *b* to be earthed ($\varphi_b = 0$).Then proceeding from the method of nodal potentials:

$$U_{ab} = \frac{\sum_{k=1}^{n} E_k G_k + \sum_{k=1}^{n} J_k}{\sum_{k=1}^{n} G_k} \tag{3.35}$$

where U_{ab} is the voltage between two nodes; $\sum_{k=1}^{n} E_k G_k$ is the algebraic sum of multiplications of the branch e.m.f.s joined to the node, by the conductances of these branches; $\sum_{k=1}^{n} J_k$ is the algebraic sum of the values of current sources in the branches which are joined to the node; $\sum_{k=1}^{n} G_k$ is the sum of branch conductances of the electric circuit.

After definition of voltage U_{ab} a current in any branch k which does not contain a current source, is defined according to Ohm's law for a circuit section:

$$I_k = G_k(\pm E_k - U_{ab}) \tag{3.36}$$

where the e.m.f.s which direct towards node a,, are considered with a plus sign, and those which direct from the node *a* - with a minus sign.

Example 3.6. We will observe the direct current circuit (Fig. 3.15). The parameters of energy sources $E_1 = 55\ V$, $E_2 = 45\ V$, $J = 1\ A$, and the resistances, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$, , $R_4 = 35\ \Omega$, $R_5 = 15\ \Omega$. One must define currents in all branches of the electric circuit.

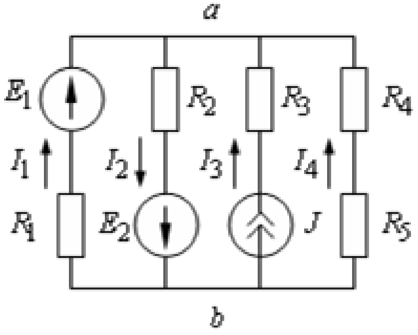


Fig. 3.15

Solution: The conductances of branches:

$$G_1 = \frac{1}{R_1} = \frac{1}{10} = 0{,}1\ Sm; \quad G_2 = \frac{1}{R_2} = \frac{1}{20} = 0{,}05\ Sm;$$

$$G_4 = \frac{1}{R_4 + R_5} = \frac{1}{35 + 15} = 0{,}02\ Sm.$$

The voltage U_{ab} between two nodes:

$$U_{ab} = \frac{E_1 G_1 - E_2 G_2 + J}{G_1 + G_2 + G_4} = \frac{55 \cdot 0{,}1 - 45 \cdot 0{,}05 + 1}{0{,}1 + 0{,}05 + 0{,}02} = 25 \text{ V}$$

As the voltage U_{ab} being known, we can find from Eq. (3.36) the currents in branches:

$$I_1 = G_1(E_1 - U_{ab}) = 0{,}1 \cdot (55 - 25) = 3 \ A;$$

$$I_2 = G_2(E_2 + U_{ab}) = 0{,}05 \cdot (45 + 25) = 3{,}5 \ A;$$

$$I_3 = J = 1 \ A; \ \ I_4 = -G_4 U_{ab} = -0{,}02 \cdot 25 = -0{,}5 \ A.$$

3.12. The Superposition Theorem

This theorem has to do with the presence of two or more energy sources acting within a network. Accordingly, in the interest of establishing the proper background, let us return temporarily to the circuit of Fig. 3.16 in which appear a voltage source and a current source. The expression for the currents existing in this circuit can be written in terms of the effect which each energy source produces.

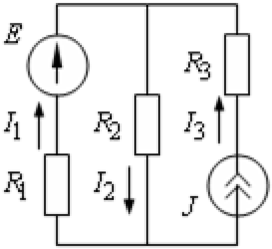


Fig. 3.16. The circuit with voltage and current sources

The superposition theorem (or the *method of superposition*) immediately follows from superposition principle.

At calculation of any electric circuit by the method of superposition one must leave only one energy source in it and define partial currents in all its branches.

For this purpose all energy sources in turn, except one, are considered to be absent. Circuit e.m.f sources are replaced by short-circuited cross connections, and branches with current sources are broken. Then we remove this energy source and define partial currents from the action of another energy source.

Accordingly, on the basis of the foregoing discussion the *superposition theorem* may be stated as follows: if some partial current is the response to the first energy source, and another partial current is the response to the second energy source, then in any *linear* electric circuit the response to the combine forcing function is equal to the algebraic sum of these partial currents.

The current in the *k*-th branch can be found by the following equation:

$$I_k = E_1 g_{k1} + E_2 g_{k2} + E_3 g_{k3} + \ldots$$

$$\ldots + E_k g_{kn} + \sum_{p=1}^{m} k_{I_{kp}} J_p \ , \tag{3.37}$$

where $E_1, E_2, E_3, \ldots E_k$ are given electromotive forces, $g_{k_1}, g_{k_2}, g_{k_3}, \ldots g_{kn}$ are the conductances of corresponding branches, J is the value of current source in the branch, $k_{I_{kp}}$ is a current transfer function of the *k*-th branch.

Eq. (3.37) is mathematical expression of the theorem which may be also stated as follows: *The current in any branch of a network is the algebraic sum of the currents due to each source separately with all other sources removed and the internal resistances of all sources left in the circuit*.

The superposition theorem holds for all linear electric circuits, that is, for the circuits U/I characteristics of which are straight lines. It is expedient for applying, if the amount of energy sources is insignificant.

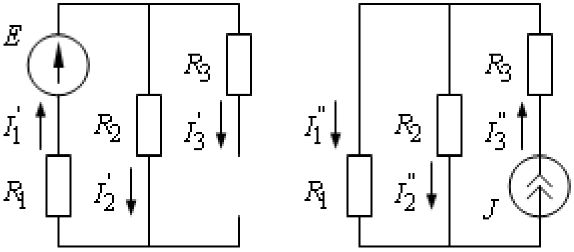
Let's observe the use of the method on an instance of the electric circuit (see Fig 3.16) with two energy sources.

Example 3.7. We will observe the d. c. electric circuit (Fig. 3.16). The parameters of energy sources $E = 75\ V$, $J = 3\ A$, the value of resistances are $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$ Define currents in all branches of this electric circuit.

Solution. We make up by the positive directions of currents (Fig. 3.16). Now one must calculate currents in all branches of a circuit from each energy source separately in turn.

At first we break the current source and define partial currents only from the action of the e.m.f. source (Fig. 3.17, *a)* (there is an open circuit in the branch instead of the current source, as its internal resistance is equal to infinity):

$$I_1^{'} = I_2^{'} = \frac{E}{R_1 + R_2} = \frac{75}{10 + 20} = 2{,}5\ A, \qquad I_3^{'} = 0\ A$$



a) b)

Fig. 3.17

Let's remove the e.m.f. source and define partial currents from the current source (Fig. 3.17) (there is a short-circuited cross connection in the branch as its internal resistance is equal to zero). To calculate these partial currents it is convenient to use equations (3.32) and (3.33) which allow calculating currents in two parallel branches fulfilled so-called "resolving of the total current". So,

$$I_3^{''} = J = 3\ A;$$

$$I_1^{''} = J \cdot k_{I_1} = J\frac{R_2}{R_1 + R_2} = 3 \cdot \frac{20}{10 + 20} = 2\ A;$$

$$I_2^{''} = J \cdot k_{I_2} = J\frac{R_1}{R_1 + R_2} = 3 \cdot \frac{10}{10 + 20} = 1\ A.$$

We have got real required currents algebraically adding the appropriate partial currents created by separate energy sources, taking into account their directions:

$$I_1 = I_1^{'} - I_1^{''} = 2{,}5 - 2 = 0{,}5\ A;\ I_2 = I_2^{'} + I_2^{''} = 2{,}5 + 1 = 3{,}5\ A;$$

$$I_3 = I_3^{''} - I_3^{'} = 3 - 0 = 3\ A.$$

All the currents are positive. It means that at the beginning of solution we chose their directions correctly.

3.13. Active and passive two-pole unit

The two-pole unit (or two-terminal network) is a generalizing name of an electric circuit of any complexity or its part which has two terminals. The two-terminal network is conditionally represented by a rectangle in figures.

A two-terminal network containing a voltage or a current source, or both, is called *active* (marked by letter *A* in Fig. 3.18, *a*).

A two-terminal network containing neither a voltage nor a current source, is termed *passive* (marked by letter *P* in Fig. 3.18, *b*).

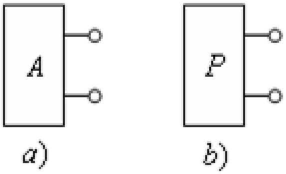


Fig. 3.18

The active two-pole unit is characterized by three interconnected parameters: an open circuit (or an idling) voltage $U_{o \cdot c.}$, a short circuit current I_k and an input resistance R_{in}.

$$R_{in} = \frac{U_{o.c.}}{I_k} \tag{3.38}$$

The passive two-pole unit is characterized by one parameter - an input resistance R_{in}. The open circuit (or idling) voltage $U_{o \cdot c.}$, and the short circuit current I_k are equal to zero here.

It is possible to replace an active two-pole unit either an equivalent real e.m.f. source, the e.m.f of which is equal to the idling voltage between two terminals, and its internal resistance is equal to the input resistance of a two-terminal network, or the real current source the current of which is equal to a short circuit current of a two-pole unit, and its internal resistance is equal to the input resistance of a two-terminal network too. The input resistance is defined when all energy sources are removed (branches with ideal current sources are broken off, and ideal voltage sources are replaced with short-circuited crosspieces). The short circuit current and the idling voltage are defined by any known method of calculation. The parameters of two-terminal network can be defined experimentally with the help of idling experience (one must connect the voltmeter to two terminals), and short circuit experience (one must connect the amperemeter to two terminals). The input resistance of a two-pole unit then is defined by Eq. (3.38).

3.14. Thevenin's Theorem

Situations sometimes occur in electric engineering in which it is desirable to find a particular branch current in a network as the resistance of that branch is varied while all other resistances and sources remain constant. Then we remove this branch, and the remaining two-terminal network (Fig. 3.19, *a*) can be replaced by an equivalent voltage generator (Fig. 3.19, *b*) whose e.m.f. is equal to the one appearing across the two terminals when the branch is open circuited, that is there is no load on it, and whose internal resistance is equal to the driving-point resistance of the two-terminal network between the branch terminals.

This method is known as *Thevenin's Theorem* or *equivalent-generator method*. The remaining two-terminal network can be also replaced by an equivalent current source (Fig. 3.19, *c*).

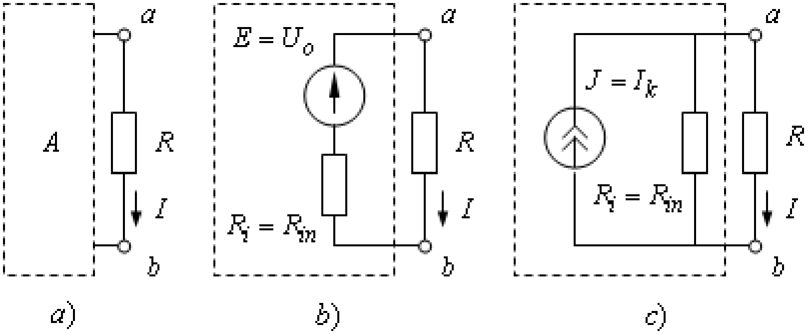


Fig. 3.19

Accordingly there are two alternatives of the method: the *equivalent-generator method* and a *method of the equivalent current source*.

The equivalent-generator method. For finding a current in any branch *ab* which resistance is *R*, it is necessary:

- break a branch *ab* and consider the rest part of the electric circuit as a two-terminal network;
- define the voltage between terminals *ab* of two-terminal network (open-circuit voltage $U_{o.c.}$);

- find input resistance R_{in} of the two-terminal network concerning terminals *ab* (in spite of this voltage sources are replaced by short-circuited connectors, and branches with current sources are broken);

- exchange the two-terminal network by the equivalent real e.m.f. source, whose e.m.f. is equal to the open-circuit voltage of two-terminal network $E = U_{o.c.}$, and its internal resistance is equal to the input resistance of two-terminal network $R_i = R_{in}$;

- calculate the current in the branch *ab* by Ohm's law:

$$I = \frac{U_{o.c.}}{R_{in} + R} = \frac{E}{R_i + R} \qquad (3.39)$$

Method of the equivalent current source. For finding a current in any branch with resistance *R* by this method, it is necessary:

- break the branch *ab* and consider the rest part of an electric circuit as the two-terminal network;

- exchange the branch *ab* with a short-circuited connector and find a current in this connector (a short-circuit current I_k);

- define the input resistance R_{in} of the two-terminal network concerning terminals *ab;*

- exchange the two-terminal network by an equivalent real current source whose current is equal to a short-circuit current $J = I_k$, and the internal resistance is equal to the input resistance of two-terminal network $R_i = R_{in}$;

- calculate the current in a branch *ab* using Eq. (3.32) for a parallel branch.

$$I = I_k \frac{R_{in}}{R_{in} + R} = J \frac{R_i}{R_i + R}$$

Let's consider the application of this method to a problem of the current definition in one branch of the electric circuit which has been considered in example 3.7.

Example 3.8. Consider d.c. electric circuit (Fig. 3.20 *a*). The parameters of energy sources $E_1 = 55\ V$, $E_2 = 45\ V$, $J = 1\ A$, and the resistances $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_3 = 30\ \Omega$, $R_4 = 35\ \Omega$, $R_5 = 15\ \Omega$. Define the current in the branch with the resistor R_2.

Solution: Let's convert the initial circuit removed the branch with R_2 how it is shown in Fig. 3.20, *b*.

We find the voltage between points *a* and *b* in an idling mode (Fig. 3.21, a).

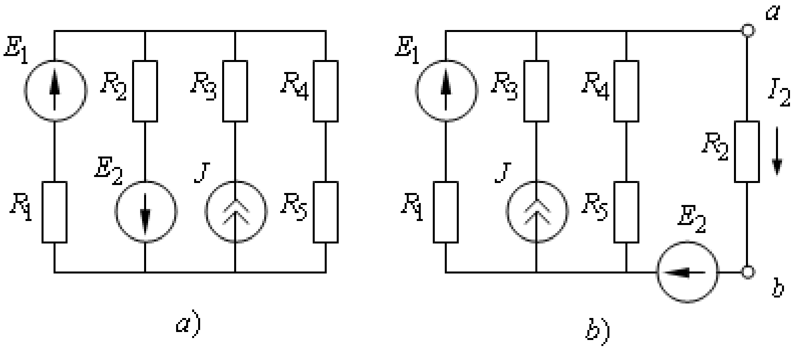


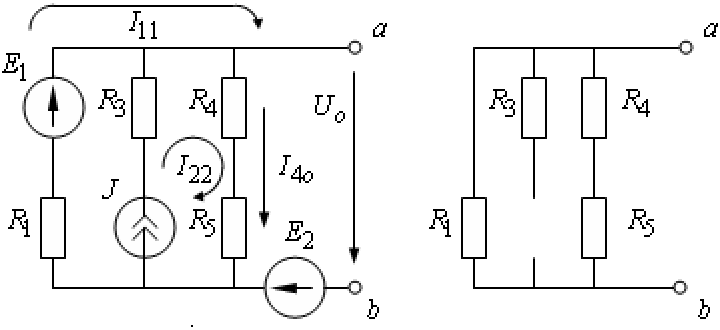
Fig. 3.20

For this purpose we use a mesh-current method. We will consider, that in the second contour the mesh current is defined by a current source $I_{22} = J = 1\ A$.

$$(R_1 + R_4 + R_5)I_{11} + (R_4 + R_5)I_{22} = E_1 .$$

$$I_{11} = \frac{E_1 - J(R_4 + R_5)}{R_1 + R_4 + R_5} = \frac{55 - 1(35 + 15)}{10 + 35 + 15} = 0{,}0833 \quad \text{A;}$$

$$I_{4o} = I_{11} + I_{22} = I_{11} + J = 0{,}0833 + 1 = 1{,}0833 \ \text{A.}$$



a) *b*)

Fig. 3.21

Then we can find the open-circuit voltage appearing across the terminals of the removed branch:

$$U_{o.c.} = E_2 + (R_4 + R_5) I_{4o} = 45 + (35 + 15) \cdot 1{,}0833 = 99{,}17 \text{ V.}$$

We determine the resistance that the two-terminal network presents to the terminals *a* and *b* of the removed branch, with the voltage sources E_1 and E_2 short-circuited and all current sources open-circuited *J* (Fig.3.21, *b*):

$$R_{in} = \frac{R_1 (R_4 + R_5)}{R_1 + R_4 + R_5} = \frac{10 \cdot (35 + 15)}{10 + 35 + 15} = 8{,}333 \ \Omega.$$

Find the current in the removed branch by the equation:

$$I_2 = \frac{U_o}{R_{in} + R_2} = \frac{99{,}17}{8{,}333 + 20} = 3{,}5 \text{ A.}$$

We solve a problem by equivalent current source method.

We define the short circuit current using a mesh-current method (Fig. 3.22).

We will consider, that in the third contour the mesh current is defined by the known value of the current source $I_{33k} = J = 1\ A$.

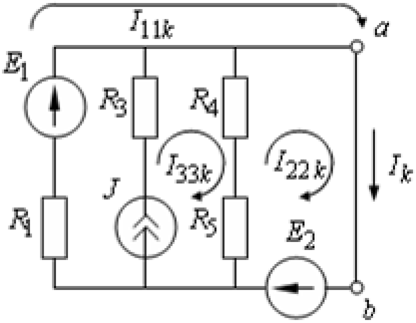


Fig. 3.22

The system of equations by the mesh-current method:

$$\begin{cases} R_1 I_{11k} = E_1 + E_2 \\ (R_4 + R_5) I_{22k} - (R_4 + R_5) I_{33k} = E_2 \end{cases}$$

The mesh currents from this system:

$$I_{11k} = \frac{E_1 + E_2}{R_1} = \frac{55 + 45}{10} = 10\ A;$$

$$I_{22k} = \frac{E_2 + (R_4 + R_5)J}{R_4 + R_5} = \frac{45 + (45 + 15) \cdot 1}{45 + 15} = 1{,}9\ A.$$

Then the short circuit current:

$$I_k = I_{11k} + I_{22k} = 10 + 1{,}9 = 11{,}9\ A.$$

The resistance that the two-terminal network presents to the terminals *a* and *b* of the removed branch is determined in the same way as in the previous case.

The current in the removed branch:

$$I_2 = I_k \frac{R_{in}}{R_{in} + R_2} = 11{,}9 \cdot \frac{8{,}333}{8{,}333 + 20} = 3{,}5\ A.$$

As we can see the current value I_2 found by two methods coincides with its value which was got as a result of the decision of this problem in the Example 3.6.

3.15. The Power Transfer from an Active Two-Terminal Network to a Load. Operating conditions of electric circuits

For research of the energy transfer from an active two-terminal network to the load we consider the electric circuit consisting of a real source and a resistive element (Fig. 3.23).

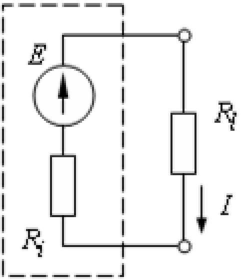


Fig.3.23

We consider, that the parameters of e.m.f. source are invariable, and a load resistance is varied. The current in the electric circuit by Ohm's law, taking into account the series connection of the source and the load:

$$I = \frac{E}{R_i + R_l}\,. \tag{3.40}$$

The voltage drop inside the voltage source:

$$U_i = R_i I = E\frac{R_i}{R_i + R_l} \tag{3.41}$$

The voltage across output terminals of the source and across the load:

$$U_l = E - U_i = E - E\frac{R_i}{R_i + R_l} = E\frac{R_l}{R_i + R_l} \tag{3.42}$$

The power generated by the e.m.f. source:

$$P = EI = \frac{E^2}{R_i + R_l} \tag{3.43}$$

The power of losses in the e.m.f. source:

$$P_d = I^2 R_i = \frac{E^2 R_i}{(R_i + R)^2} \tag{3.44}$$

The power actually dissipated in the load:

$$P_l = I^2 R = \frac{E^2 R_l}{(R_i + R_l)^2} \tag{3.45}$$

The efficiency is

$$\eta = \frac{P_l}{P} = \frac{R_l}{R_i + R_l} \tag{3.46}$$

Given equations connect the values which characterize an energy transfer from an energy source to a load.

The most important operating modes of any electric circuit are: *idling (or open circuit), the short circuit, coordinated, and nominal.*

The *idling (or open circuit) rate (o.c.)* is the mode at which the load is disconnected from an energy source ($R_{l_o} \to \infty$). Then the current in an external circuit is absent $I_o = 0$. Thus, the source does not give energy to the external circuit $P_o = 0$, and load does not consume it $P_{l_o} = 0$. The efficiency in this mode aspires to 1 (if there are no energy losses in a source in the idling mode). From Eq. (3.42) follows, that in this mode the idling voltage across the terminals is equal to the value of e.m.f. ($U_{l_o} = E$).

Short circuit is a condition which appears when source input terminals are short-circuited ($R_{l_k} \to 0$). Then a short-circuit current in a circuit is restricted only to an internal source resistance R_i, ($I_k = \frac{E}{R_i}$), that is the extremely dangerous, as usually this resistance has very little value, and the current in a circuit can reach such a high value at which the source can be put out of action. Therefore the short circuit rate in the most cases is emergency. But some kinds of the electrotechnical equipment (electroslag remelting furnaces, welding transformers, etc.) work in the rates which are close to a short circuit regime. The voltage across the input source terminals is equal to zero $U_{l_k} = 0$. The power which oscillated by an ideal e.m.f. source attains the maximum value and is completely outlaid for losses in the source: $P_{\max} = P_k = P_{d_k} = \frac{E^2}{R_i}$. The load does not consume the energy $P_{l_k} = 0$. Therefore, the efficiency is equal to zero: $\eta_k = 0$.

The idling and short circuit conditions are the boundary ones.

The coordinated load regime is the rate at which the power given out by a source into a load attains the maximum value. It is possible at a certain coordination of parameters of an electric circuit. That is why this rate has such a name.

From Eq. (3. 45) follows, that at a certain load resistance, the load power reaches a maximum value, as it is equal to zero in short circuit mode ($R_{l_k} = 0,\ I_k = \frac{E}{R_i}$) and idling one ($R_{l_o} \to \infty,\ I_o = 0$). On taking derivative $\frac{dP_l}{dR_l}$ and equating it to zero, we will define the value of the load resistance corresponding to a maximum power:

$$\frac{dP_l}{dR_l} = E^2 \frac{(R_i + R_l)^2 - 2R_l(R_i + R_l)}{(R_i + R_l)^4} = E^2 \frac{R_i^2 - R_l^2}{(R_i + R_l)^4} = 0.$$

From this equation we define the value of the load resistance corresponding to a maximum power: $R_l = R_i$. The last condition corresponds to the current: $I_y = \frac{E}{2R_i} = \frac{I_k}{2}$. The power of the receiver in this mode reaches a maximum value and is equal to the power of losses inside a source: $P_{l_y} = P_{l_{\max}} = P_{d_y} = \frac{E^2}{4R_i}$. The power input from the equivalent generator is $P_y = \frac{E^2}{2R_i}$. As one can see from the received expressions in the coordinated load mode, the efficiency of a source $\eta_y = 0{,}5$. Such a mode is characteristic for radio-electronic devices where transferring low-power signals one must aspire to receive the maximum power in the receiver not looking at the low value of efficiency. In power electrotechnical circuit transferring electric energy of large power one must aspire to provide the greatest possible efficiency. In this case one must observe the condition $R_l >> R_i$, for what it is necessary either to increase a load resistance or to reduce the internal resistance of a source.

The nominal condition is the rate at which the elements of an electric chain work in the conditions of corresponding to the design. For elements of electric circuits the nominal parameters providing a nominal operating mode, are current, voltage and power which are specified in reference books, the engineering specifications or on the element. The choice of wires and devices for electric circuits is made taking into account nominal voltages and currents of sources. Usually under a nominal voltage U_{nom} is meant an idling voltage $U_{o.c.}$, instead of the voltage at which a nominal current I_{nom} is provided.

Power installations usually work in the modes at which currents and powers do not exceed nominal values, and voltages are close to nominal ones.

For more evident representations about the received interrelations of values one can use characteristics - functions in which as the argument a current is used:

For more evident representations about the received interrelations of values one can use characteristics - functions in which a current I is used as the argument: $U_l(I_l)$; $U_i(I_l)$; $P(I_l)$; $P_d(I_l)$; $P_l(I_l)$; $\eta(I_l)$.

In order to receive the generalized characteristics we will express the given values in relative units, using as base-line values - a short circuit current $I_k = \frac{E}{R_i}$ - for a current , the e.m.f. source E - for a voltage ; the maximum power of e.m.f. source $P_{max} = \frac{E^2}{R_i}$ - for power.

As the argument in all functions it is used the relative value of current $I^* = \frac{I}{I_k} = \frac{E}{R_i + R_l} \cdot \frac{R_i}{E} = \frac{R_i}{R_i + R_l}$. Taking into account the last parity we will receive dimensionless characteristics from equations (3.41) - (3.46).

The voltage drop inside in a source:

$$U_i^* = \frac{U_i}{E} = \frac{R_i}{R_i + R_l} = I^* \tag{3.47}$$

The voltage across output source terminals and across a load:

$$U_l^* = \frac{U_l}{E} = \frac{E - U_i}{E} = 1 - \frac{R_i}{R_i + R_l} = 1 - I^* \tag{3.48}$$

The power generated by an e.m.f. source:

$$P^* = \frac{P}{P_{max}} = \frac{E^2}{R_i + R_l} \cdot \frac{R_i}{E^2} = \frac{R_i}{R_i + R_l} = I^* \tag{3.49}$$

The power of losses inside the source:

$$P_d^* = \frac{P_d}{P_{max}} = \frac{E^2 R_i}{(R_i + R)^2} \cdot \frac{R_i}{E^2} = \frac{R_i{}^2}{(R_i + R)^2} = (I^*)^2 \tag{3.50}$$

The power which is consumed by a load (the receiver power):

$$P_l^* = \frac{P_l}{P_{max}} = \frac{E^2 R_l}{(R_i + R_l)^2} \cdot \frac{R_i}{E^2} = \frac{R_i}{R_i + R_l} \cdot \frac{R_l}{R_i + R_l} = I^*(1 - I^*) \tag{3.51}$$

The efficiency of the e.m.f. source:

$$\eta^* = \frac{P_l}{P_{max}} \cdot \frac{P_{max}}{P} = \frac{R_l}{R_i + R_l} = I^* \tag{3.52}$$

The graphs of dependences $U_l^*(I^*)$ and $U_i^*(I^*)$ in the dimensionless form are presented in Fig. 3.24, *a* and the graphs of dependences $P^*(I^*)$, $P_l^*(I^*)$, $U_d^*(I^*)$, $\eta^*(I^*)$ are in Fig. 3.24, *b*.

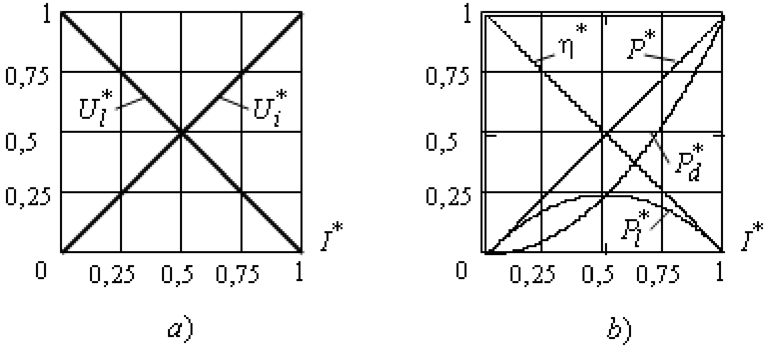


Fig. 3.24. The graphs of dependences

3.16. Energy Relation in Electric Direct Current Circuit

For electric dc circuit of any complication, it is possible to make *an energy balance* outflowing immediately from the law of energy conservation: the algebraic sum of all powers which are developed by electric energy sources in a circuit, should be equal to the total sum of the powers which the load converts into other kinds of energy.

The energy balance equation has the form:

$$\sum_{p=1}^{P} E_p I_p + \sum_{q=1}^{Q} U_{Jq} J_q = \sum_{k=1}^{K} I_k^2 R_k \tag{3.53}$$

where $\sum_{p=1}^{P} E_p I_p$ is the power which P e.m.f. sources develop; $\sum_{q=1}^{Q} U_{Jq} J_q$ is the power which Q current sources develop; U_{Jq} is the voltage across the terminals of q current source; $\sum_{k=1}^{K} I_k{}^2 R_k$ is the power which is diffused by K load resistances.

The power which is converted in the load $I_k{}^2 R_k$, can accept only positive values as a load (resistance devices) always works in the rate of electric energy consumption. Expressions $E_p I_p$ and $U_{Jq} J_q$ may be both positive (sources work in the rate of energy generation) and negative (sources work in the rate of power consumption). It depends on, whether the true current directs from a node with a smaller potential to a node with a more potential (the driving of charges is carried out at the expense of the operation of outside forces) or, on the contrary, from a node with a more potential to a node with a smaller potential (the driving of charges is carried out at the expense of electric field operation).

For example, the accumulator when charging works in a rate of electric energy consumption.

In Eq. (3.53) the powers of e.m.f. sources are taken into account with a plus sign if a current direction coincides with the direction of e.m.f. And the powers of current sources are taken into account with a plus sign if a voltage drop across the terminals of the current source is opposite to the current direction.

Let's consider electric circuits which consist of two energy sources (Fig. 3.25).

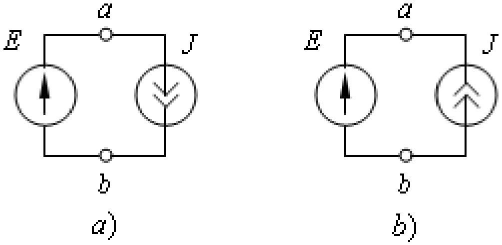


Fig. 3.25. Electric circuits with two energy sources

The node potential a is more than node potential b (as the e.m.f. source directs from the node b to the node a). The true current direction is determined by a current source connection. Therefore in the circuit in Fig. 3.25, *a* the e.m.f. source works as a generator, and the current source works as a receiver. And in the circuit in Fig. 3.25, *b*, on the contrary, the current source works as a generator, and the e.m.f. source works as a in a receiver.

3.17. Compensation Theorem

For any given set of circuit conditions, any resistance R in a network carrying a current I can be replaced in the network by a voltage generator of zero internal resistance and electromotive force $E = -IR$.

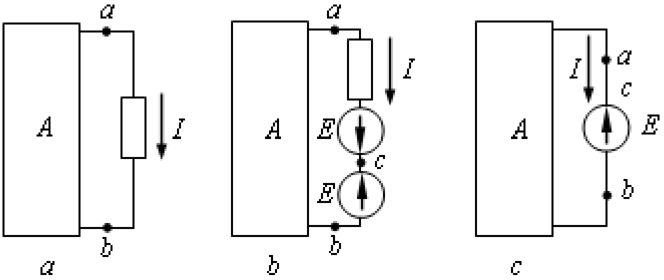


Fig. 3.26. Illustration of compensation theorem.

For a proof of a theorem, we remove a branch of resistance R, carrying a current I, from the network of Fig. 3.26, *a* and symbolize the remaining network by a box.

Then we place in this branch two voltage generators of e.m.f.s opposite in sign and equal in magnitude to the voltage drop across R due to I ($E = IR$, Fig. 3.26, *b*). It will be noted that there is no difference in the current I flowing around the network. It can be shown that the potential difference between points *a* and *c* in the network of Fig. 3.26, *b is zero.*

Indeed

$$\varphi_c = \varphi_a - I\,R + E = \varphi_a - I\,R + I\,R = \varphi_a .$$

But if $\varphi_c = \varphi_a$, the points *a* and *c* may be made common by shorting them which gives the circuit of Fig. 3.26, *c*. In this circuit R is replaced by E.

Example 3.9. Prove the identity of the circuits shown in Fig. 27, *a* and *b*.

Solution: In the circuit of Fig. 27, *a*, the current $I = E_1(R_1 + R_2)$. For the circuit of Fig. 27, *b*

$$I=\frac{E_1-E_2}{R_1}=\frac{E_1-E_2\dfrac{R_2}{R_1+R_2}}{R_1}=\frac{E_1}{R_1+R_2}$$

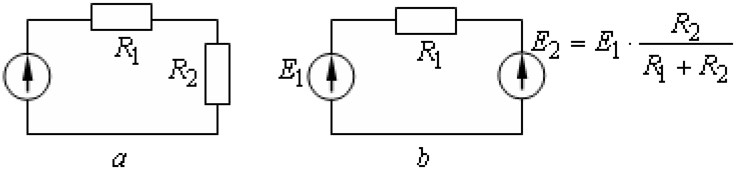


Fig. 3.27

cvTrue to the compensation theorem, the replacement of R_2 by E_2 in the circuit of Fig. 3.27 has not affected the current round the circuit.

3.18. The Potential Diagram

A potential diagram is a plot of potential distribution round a circuit, incomplete or complete, as a function of an independent variable, such as the circuit resistance. Resistances are then laid off as abscissa, and the potentials, as ordinate. For each point round a circuit there is a separate point in the potential diagram.

The sequence of construction of a potential diagram:

1) choose any closed mesh with several e.m.f.s;

2) assume a positive direction for summation round that mesh;

3) some node of the circuit is considered to be earthed (grounded), i.e. its potential is equal to zero;

4) mark every point of the mesh. We must have only one element between two points;

5) calculate the nodal potentials along the mesh.

Earthing a junction in a circuit will not affect the current distribution in this circuit. This is because no additional path is formed for a current to flow. The situation is different when two or more junctions with different potentials are earthed (or grounded). Then additional paths are formed through earth (or any conducting medium), so that the circuit arrangement or configuration changes, and the current distribution becomes different.

If we have an e.m.f. between two points, and its direction coincides with the direction for summation round, we take it with a plus sign. In the opposite case its sign will be minus.

If a current through resistor has the same direction as the direction for summation round, the voltage drop from this current will have a minus sign. If a current doesn't coincide with the direction for summation round, we will take the voltage drop with a plus sign. We must remember that current always flows from the point with a higher potential to the point with a lower potential.

Example 3.9. For dc circuit (Fig. 3.28) with parameters, $E_1 = 50\ V$, $E_2 = 30$, $J = 1\ A$, $R_1 = 10\ \Omega$, $R_2 = 15\ \Omega$, $R_3 = 20\ \Omega$, , $R_4 = 25\ \Omega$, $R_5 = 30\ \Omega$, $R_6 = 35\ \Omega$, draw a potential diagram for a contour with two e.m.f. sources.

Solution: The currents in the branches were found in examples 3.5 – 3.7.

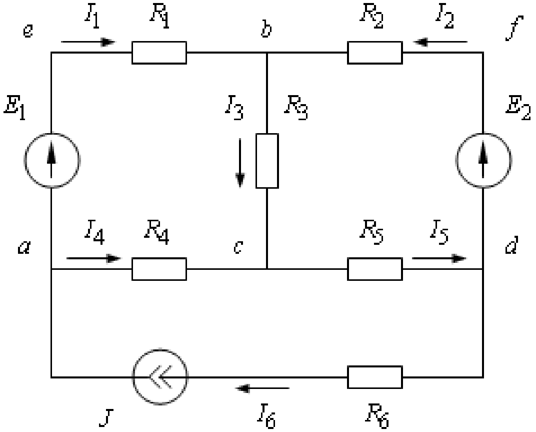


Fig. 3.28

Let's consider the mesh which consists of following elements: R_4, E_1, R_1, R_2, E_1, R_5. We choose the junction *c* to be at zero that is $\varphi_c = 0\ V$. We choose scales for the x- and y-axis, and set the potential at, say, junction *c* at zero. So, we consider that $\varphi_c = 0$. In the diagram of Fig. 3.28 this will be the origin of coordinates. Now we can calculate the potentials of nodes along the outside mesh:

$$\varphi_c = 0 \text{ V};$$

The potential at *a* will be:

$$\varphi_a = \varphi_c + R_4 I_4 = 0 + 25 \cdot (-0{,}536) = -13{,}4 \text{ V}.$$

The coordinates of this junction are $x = 25,\ \ y = -13{,}4.$

The potential at *e*

$$\varphi_e = \varphi_a + E_1 = -13{,}4 + 50 = 36{,}6 \text{ V}.$$

The coordinates of this junction are $x = 25,\ \ y = 36{,}6$.

The potential at *b*:

$$\varphi_b = \varphi_e - R_1 I_1 = 36{,}626 - 10 \cdot 1{,}536 = 21{,}24 \text{ V};$$

The coordinates of this junction are $x = 10\ \ \ y = 21{,}24.$

The potential at *f*:

$$\varphi_f = \varphi_b + R_2 I_2 = 21{,}24 + 15 \cdot (-0{,}471) = 14{,}18 \text{ V};$$

The coordinates of this junction are $x = 15\ \ y = 14{,}18.$

The potential at *d*:

$$\varphi_d = \varphi_f - E_2 = 14{,}18 - 30 = -15{,}82 \text{ V}.$$

The coordinates of this junction are $x = 15,\ y = -15{,}82.$

Now we must return to junction *c*:

$$\varphi_c = \varphi_d + I_5 R_5 = -15{,}82 + 30 \cdot 0{,}527 = -0{,}01 \approx 0 \text{ V}.$$

Hence, the zero result shows that the calculation of nodal potentials for a contour is fulfilled correctly.

Now we draw a potential diagram. Resistances are then laid off an abscissa, and the values of point potentials, as ordinate. All values are put in chosen scale. For each point round a circuit, incomplete or complete, there is a separate point in the potential diagram.

The potential diagram represents definite practical interest as it gives obvious representation about the distribution of voltages between separate points of a contour.

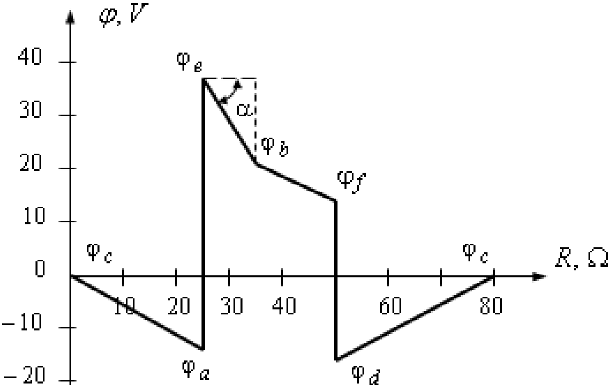


Fig. 3.28. The potential diagram for contour *caebfdc*.

We have got a sloping line if there is a resistor in this part of the mesh. If we have the e.m.f. source in the part of the mesh there will be a perpendicular to y-axis in the diagram. For any contour in the network the potential diagram shows us the change of potentials when we go round the mesh.

Using the potential diagram, it is possible to define a current on the separate circuit sections which belong to the given contour. For example, we will consider a part of the circuit which characterizes the change of a potential along resistive element R_1. One cathetus of rectangular triangle is equal to the resistance of a resistive element, and the other cathetus is equal to a voltage drop across it. Then tangent of the declination angle is proportional to a current:

$$tg\alpha = \frac{m_U}{m_R} \cdot \frac{\varphi_e - \varphi_b}{R_1} = \frac{m_U}{m_R} \cdot I_1 = m_I \cdot I \qquad (3.54)$$

where m_U, m_R, m_I are scales for a voltage, a resistance and a current correspondently.

Thus, one can judge about the magnitude of the current flowing through the given part of the circuit, by the declination of a diagram section. In case of a series connection a current is the same through all elements, therefore for all resistive elements the declination of corresponding segments of the broken line must be also identical.

Summary review questions

1. What is the test for determining whether or not two circuit elements are in series? In parallel?

2. Which has the greater equivalent resistance; two equal resistors in series or in parallel? Explain.

3. Distinguish among a mesh current, a branch current, and a loop current.

4. How many equations can we write by Kirchhoff's current law?

5. Distinguish between a node and an independent node.

6. Distinguish between a mesh and a loop of a circuit.

7. Formulate Kirchhoff's current law and Kirchhoff's voltage law. What circuits are they conveniently applied to?

8. How will the equivalent resistance of several identical series elements change, if their number increases in two times?

9. How will the equivalent resistance of several identical parallel elements change, if their number increases in three times?

10. Formulate the basic circuit law upon which the mesh method of circuit analysis is based.

11. Formulate the main principle upon which the nodal method of circuit analysis is based.

12. Formulate the main principle upon which the superposition method is based.

13. Describe the basic idea that underlies the Thevenin equivalent circuit.

14. Let we have a potential diagram. If a broken line for one section of a circuit is parallel to an abscissa axis, what is a value of the current on this part of the circuit?

15. Describe the principle of conversation of charge and its relationship to Kirchhoff's current law.

16. Describe the principle of conversation of energy and its relationship to Kirchhoff's voltage law.

17. Describe the most important operating modes of any electric circuit. Explain the difference among them.

18. What is a potential diagram? Explain the sequence of construction of a potential diagram.

19. Describe and illustrate the voltage-divider principle.

20. Describe and illustrate the current-divider principle.

21. Distinguish among a mesh current, a branch current, and a loop current.

22. Can complete knowledge of the mesh currents in a network permit the evaluation of the current that flows in any branch of the network? Explain.

23. How is the number of independent equations needed to completely solve a circuit problem determined? Illustrate.

24. Name the basic circuit law upon which the mesh method of circuit analysis is based.

25. Name the basic circuit law upon which the nodal method of circuit analysis is based. 20//Is the number of independent equations needed to solve a given circuit's problem fixed? Explain.

26. In a given network it is possible to identify four meshes and four independent nodes. Which method of circuit analysis is to be preferred? Why?

27. A network is found to have a total of seven branches and four junction points (i.e., independent nodes). Which method is to be preferred in analyzing the circuit? Explain.

28. When is the delta-wye transformation useful in network reduction? Illustrate.

29. Explain the principle of conversation of charge and its relationship to Kirchhoff's current law. Explain the principle of conversation of energy and its relationship to Kirchhoff's voltage law.

30. In a given network it is possible to identify four meshes and four independent nodes. Which method of circuit analysis is to be preferred? Why?

Problems

3.1. Define the equivalent resistance of resistors connected in parallel, if their resistances $R_1 = 20\ \Omega$; $R_2 = 10\ \Omega$; $R_1 = 20\ \Omega$.

3.2. Some resistive elements are connected in series. Their resistances $R_1 = 20\ \Omega,\ R_2 = 15\ \Omega,\ R_3 = 25\ \Omega$. A circuit current is equal to $I = 0.5$ A. What is the voltage across the terminals of the circuit?

3.3. Four resistances of values 50 kΩ, 250 kΩ, 1 MΩ, and 500 kΩ are placed in parallel. Compute the equivalent parallel resistance.

3.4. The capacitance values of three capacitors are 10, 20, and 40 μF. When these are placed in parallel across a 200 V source, find:

(a) The equivalent capacitance. (b) The total charge residing on the capacitors. (c) The charge on each capacitor.

3.5. The three capacitors of Prob. 3-4 are placed in series across a 350 V source. (a) Compute the equivalent capacitance. (b) Find the charge on each capacitor. (c) Find the voltage drop across each capacitor.

3.6. Three series-connected inductor coils have voltages of 20, 30, and 50 V appearing across their terminals when the circuit current is changing at a rate of 100 A/s. Determine the equivalent series inductance.

3.7. In the circuit configuration of Fig. P3.1 determine the voltage drop across each circuit element as well as the power dissipated. All resistances are in ohms.

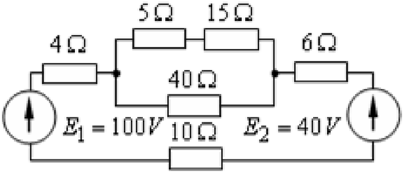


Fig. P3.1

3.8. For the circuit of Fig. P3.2, find the current *I*. All resistance values are in ohms.

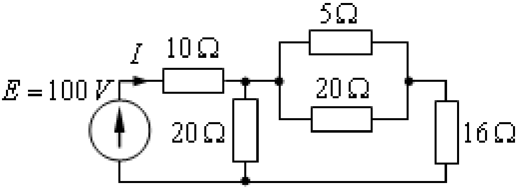


Fig. P3.2

3.9. Determine the voltage drop across each circuit element (Fig. P3.2) as well as the power dissipated. All resistances are in ohms.

3.10. In the configuration of Fig. P3.3, find the value of *E* which permits a power dissipation of 180 W in the 20-th resistor. All values are expressed in ohms.

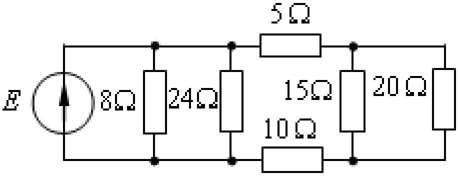


Fig. P3.3

3.11. Determine the equivalent resistance appearing at the battery terminals in the circuit of Fig. P3.3.

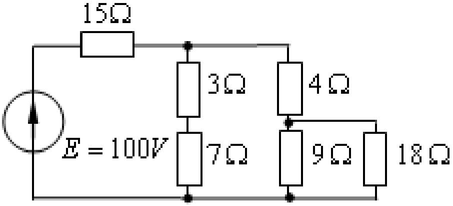


Fig. P3.4

3.12. Find the voltage drop appearing across the 18-th resistor in the network of Fig. P3.4. All values are in ohms.

3.13. Determine the equivalent resistance between points *ab* in the circuit of Fig. P3.5. All resistances are in ohms.

3.14. In the circuit of Fig. P3.5, determine: (a) The voltage needed across *ab* so that the voltage drop across the 15-Ω resistor is 45 V. (b) The corresponding voltage across the 8-Ω resistor for this condition.

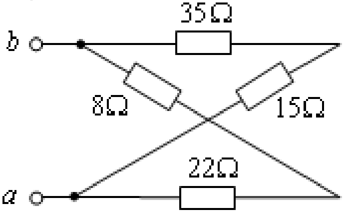


Fig. P3.5

3.15. Find the equivalent resistance between terminals *ab* for the circuit of Fig. P3.5. All resistance values are in ohms.

3.16. In Fig. P3.5 compute the voltage required between terminals *ab* so that a voltage drop of 45 *V* occurs across the 15-Ω resistor.

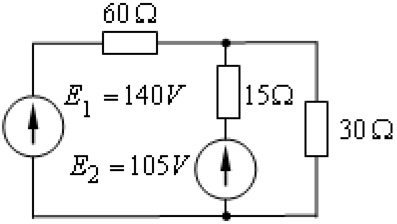


Fig. P3.6

3.17. Find the current which flows through the 15-Ω resistor in the circuit of Fig. P3.6 by using the mesh method. Check your solution by using the superposition theorem.

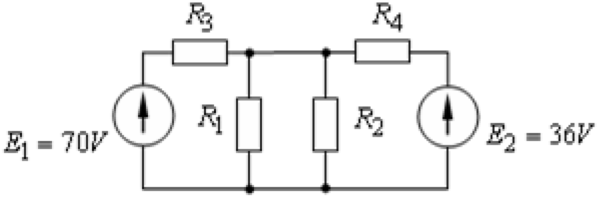


Fig. P3.7

3.18. In the circuit of Fig. P3.7 find the current through R_1 when it is made to take on the following values: $R_1 = 90\ \Omega$; $R_2 = 180\ \Omega$; $R_3 = 40\,\Omega$; $R_4 = 60\,\Omega$. Use the method requiring the least effort to yield the answers.

3.19. A circuit has an arrangement of circuit elements as depicted in Fig. P3.8. The circuit parameters: $E_1 = 60V$; $E_2 = 70V$; $E_3 = 40V$ and $R_1 = 10\,\Omega$; $R_2 = 20\,\Omega$; $R_3 = 5\,\Omega$.

(a) Find the Thevenin equivalent circuit considering R_4 as the variable load resistance.

(b) Find the current through R_4 when it has values of 0.5 and 4.5.

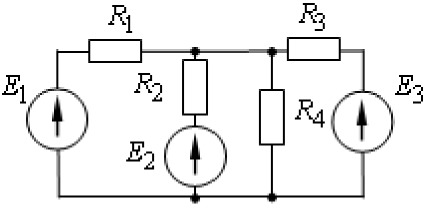


Fig. P3.8

3.20. In the circuit of Fig. P3.8 it is desirable to obtain information about each of the branch circuits. The parameters of the circuit must be taken from the previous problem.

(a) By using the mesh method write (but do not solve) the necessary independent equations which will allow all branch currents to be determined.

(b) By using the nodal method, write (but do not solve) the independent equations from which all the branch currents can be found.

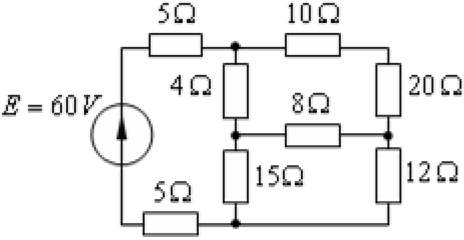


Fig. P3.9

3.21. In the circuit of Fig. P3.9 find the current delivered by the battery. Use the method requiring the least effort to yield the answers.

3.22. Calculate the currents in branches of Fig. P3.9 by mesh-current method.

3.23. The capacitors are connected across a 100-V source as shown in Fig. P3.9. (a) Find the charge on each capacitor. (b) Compute the total stored energy. (c) Compute the energy stored in each capacitor.

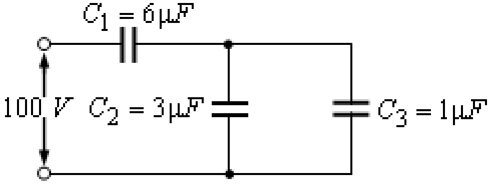


Fig. P3.10

3.24. In a configuration of three series-connected capacitors the voltage drops across their terminals are 20, 30, and 50 V and the charge on each capacitor is 300 μC. (a) Find the equivalent capacitance. (b) Find the charge on each capacitor. (c) Find the voltage across each capacitor.

3.25. The capacitors of Prob. 24 are disconnected in the charged condition and then placed in parallel. However, one of the capacitors is connected with its polarity reversed from that of the other two. (a) Find the resultant voltage of the parallel combination. (b) Determine the charge on each capacitor. (c) Compute the stored energy.

3.26. The equivalent inductance of two inductors placed in parallel is 4 H. When series-connected, the equivalent inductance is 20 H. Find the values of the individual inductances.

3.27. Solve for the power delivered to the 10-Ω resistor in the circuit shown in Fig. P3.8. All resistances are in ohms.

3.28. Use Thevenin's theorem to find the power delivered to the 3 Ω resistance in the circuit of Fig. P3.11. All resistances are in ohms.

3.29. Use Thevenin's theorem to replace the three-loop equivalent circuit of Fig. P3.11 by a single-loop equivalent circuit in which calculate the internal resistance and the open circuit voltage across the 5-Ω resistor when this branch is open circuited.

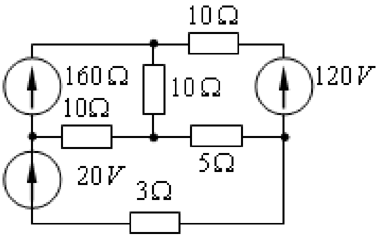


Fig. P3.11

3.30. Use Thevenin's theorem to replace the three-loop equivalent circuit of Fig. P3.12 by a single-loop equivalent circuit in which identity of R_L is preserved. All resistances are expressed in ohms.

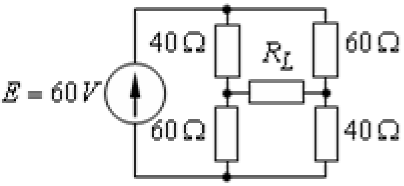


Fig. P3.12

3.31. The parameters of the circuit Fig.3.13: $I_1 = 10A$, $I_2 = 4A$, $R_1 = 1\Omega$, $R_2 = 2.5\Omega$, $R_3 = 0.5\Omega$.

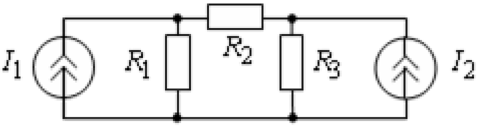


Fig. P3.13

Answer the following questions as they relate to the mesh method of analysis. (a) How many meshes does this circuit have? (b) What are the mesh currents for the inner and outer meshes? (c) Find all mesh currents. (d) Determine voltages across resistances R_1 and R_3.

Chapter 4

SINGLE-PHASE SINUSOIDAL ALTERNATING CURRENT CIRCUITS

The theory of the sinusoidal steady-state response of circuits occupies a position of pre-eminence in electric-circuit theory. The analysis of many circuits and devices throughout all branches of electrical engineering is accomplished by the techniques embodied in the sinusoidal theory. Particularly impressive in this regard is the fact that the sinusoidal circuit theory is applicable not only in situations involving sinusoidal forcing functions but equally as well in those situations where the forcing functions are of a non-sinusoidal character.

It is not an accident that the bulk of the electric power generated in power plants throughout the world and distributed to the consumer appears in the form of sinusoidal variations of voltage and current. There are many technical and economical advantages associated with the use of sinusoidal voltages and currents. A significant appreciation of this statement will be gained upon the completion of the study of this book. It will be learned, for example, that the use of sinusoidal voltages applied to appropriately designed coils results in a revolving magnetic field which has the capacity to do work. As a matter of fact it is this principle which underlies the operation of almost all the electric motors found in home appliances and about 90% of all electric motors found in commercial and industrial applications. Although other waveforms can be used in such devices, none leads to an operation which is as efficient and economical as that achieved through the use of sinusoidal functions.

In addition to these practical aspects, however, the sinusoidal function offers some very important and significant advantages in a mathematical sense.

Recall that by Euler's theorem the sine as well as the cosine function can be represented quite simply by the exponential function. Thus, $e^{j\omega t} = \cos\omega t + j\sin\omega t$. The equations which govern the behaviour of electric circuits frequently involve derivative and integral terms; this exponential character of the sinusoid is of prime importance. The reason is that it permits a simplification of mathematical analysis which cannot be achieved by any other function. The exponential function is the only mathematical function, the original form of which is preserved even though such operations as differentiation and integration are performed on it. Consequently, as described in this chapter, it is possible to treat sinusoidal time functions entirely in terms of complex numbers. Knowledge of the sinusoidal theory has another notable advantage in the mathematical sense. By means of the Fourier series it is possible to represent any periodic function of whatever form in terms of an infinite series of sinusoids. Accordingly, the steady-state response of electric circuits to non-sinusoidal forcing functions can be obtained through a repeated application of the sinusoidal theory followed by a summation of the individual sinusoidal response.

4.1. Sinusoidal Functions - Terminology

In dealing with sinusoidal functions we must become familiar with the nomenclature before proceeding with the sinusoidal steady-state analysis of circuits. This makes it easier to describe and to interpret the results. Appearing in Fig. 4.1 are two sinusoids - one denoting voltage and the other current.

The voltage sinusoid is a sine function which is represented mathematically as

$$u = U_m \sin(\omega t + \varphi_u) \tag{4.1}$$

where $(\omega t + \varphi_u)$ is called the argument of the sine function. The *amplitude* of the sine function is U_m and it denotes the maximum value of the sine function. Equation (4.1) is often referred to as the expression for the *instantaneous voltage* because by insertion of any particular value for t the corresponding value of u is determined.

The equation which describes the instantaneous value of the current wave must include the angular displacement (or phase angle) existing between the two sinusoids. A little thought reveals that the current expression is

$$i = I_m \sin(\omega t + \varphi_i)\,. \tag{4.2}$$

This current is termed a *sinusoidal alternating current*. In this equation and also in the graph of Fig. 4.1, i is the *instantaneous current* at the time t. The value I_m is the *peak value* or *amplitude* of the alternating current (*ac* in abbreviated form), and T is the time occupied by one complete cycle of change, or the *period*. A word of caution is appropriate at this point concerning the units of the argument $(\omega t - \varphi_i)$. The form of Eq. (4.2) demands that the unit for this total quantity be radians.

However, in engineering usage of this equation it is customary to express ωt in radians and φ in degrees. The reason lies in the fact that in engineering calculations it is the phase angle which is important, not the total angular displacement. Therefore, when Eq. (4.2) is occasionally seen written as $i = I_m \sin(\omega t - 45^o)$, the inconsistency in the units of the argument should be accepted in light of the foregoing comment.

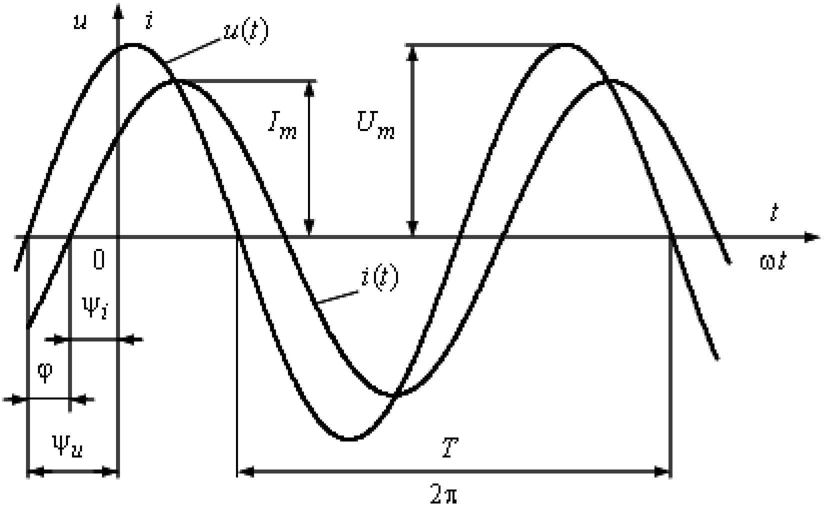


Fig. 4.1. Sinusoidal voltage and current waves of the same frequency

There exist alternate forms with which to express sinusoids. These originate with Euler's identity, that is,

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

It should be apparent from this expression that a cosine and a sine function can be expressed in terms of the exponential notation as follows:

$$\cos \omega t = \mathrm{Re}\left\lfloor e^{j\omega t} \right\rfloor$$

$$\sin \omega t = \mathrm{Im}\left\lfloor e^{j\omega t} \right\rfloor$$

where $\mathrm{Re}\left\lfloor e^{j\omega t} \right\rfloor$ denotes the real part of the expression in brackets and

$\mathrm{Im}\left\lfloor e^{j\omega t} \right\rfloor$ the imaginary part. This representation is used on various occasions throughout the book when it is convenient to do so.

The values of the argument of the sine function that lie between 0 and 2π radians, different and distinct values of the function result. However, as the argument of the function is made to take on values between 2π and 4π there occurs a repetition of the first set of values obtained. Any complete set of positive and negative values of the function is called a *cycle*.

Another characteristic of the sinusoid is its *frequency*, which is denned as the number of cycles of the function which is traversed in 1 *s*. Accordingly, the frequency is measured in units of cycles per second, which has recently come to be called the hertz and abbreviated *Hz*. In radio and radar transmission work where high frequencies are involved it is customary to use units of kilohertz, megahertz, and gigahertz. The time duration of one cycle is called the period of the function. It follows then that if f denotes the frequency of the periodic function, the period is related to the frequency by

$$T = \frac{1}{f} \tag{4.3}$$

Moreover, a glance at Fig. 4.1 indicates that each cycle spans 2π radians. Hence, if this quantity is divided by the period, there results the *angular velocity* (or angular frequency) of the sine function. It is denoted by ω and has units of rad/second.

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4.4}$$

The second form of ω is obtained from Eq. (4.3).

The equation to be used to describe the second sinusoid appearing in Fig. 4.1, i.e., the current wave, obviously cannot be identical to that used for the voltage wave. The reason is that the two sinusoids are displaced from one another by some angles and so cannot have the same instantaneous value. Accordingly, the current wave is said to be *lagging* behind the voltage wave by the relative phase angle $\varphi = (\omega t + \psi_u) - (\omega t + \psi_i) = \psi_u - \psi_i$. *Relative phase* is the term used to denote the angular displacement between two sinusoids of the same frequency. Thus in Fig. 4.1 the current sinusoid can be described as having a phase lag of $\varphi = \psi_u - \psi_i$ degrees relative to the voltage sinusoid. Alternatively we can say that the voltage wave has a *phase lead* of φ degrees relative to the current wave.

4.2. Average and Effective Values of Periodic Functions

The energy sources throughout the treatment of Chapter 3 are all of the nonvarying, constant type - often called the direct kind. Thus when a voltage source of fixed magnitude is applied to a network, the forced solution is found to be a constant quantity too. This constant character of the response makes it a simple matter to identify the number of amperes flowing in the circuit and thereby to describe the energy-transferring capability of the circuit. Moreover, the computation of the power absorbed by each circuit element is accomplished in a direct manner through the use of the equation $P = U \cdot I$ by inserting the constant values of voltage and current.

Note that the sinusoidal current is an *alternating* current, i.e., one which has positive and negative values. In such a case the manner of describing the energy-transferring capability of the current is not at all obvious, as it is when direct sources are used; in the latter case the average current flow is identical to the direct (or constant) value.

In view of the fact that the average current serves as a useful criterion in determining the energy transfer in circuits involving direct sources, let us investigate its usefulness in situations involving periodic driving functions.

The term periodic is used rather than sinusoidal in order that the treatment is general. The sinusoid is only one example of a periodic function. Any function whose cycle is repeated continuously irrespective of waveform is called a periodic function.

The *average* (or *mean*) value of sinusoidal quantity over one cycle, or an integral number of cycles, is obviously zero. A significant mean will therefore be the average of the values prevailing during one positive (or negative) half-cycle. Thus, the average current during one half-cycle is

$$I_{av} = \frac{1}{T/2} \cdot \int_0^{T/2} I_m \sin \omega t \cdot dt = \frac{2}{\pi} \cdot I_m = 0.636\, I_m \tag{4.5}$$

that is, the average current during one positive (or negative) half-cycle is $2/\pi = 0.636$ times the peak current. Similarly

$$E_{av} = \frac{2}{\pi} \cdot E_m; \; U_{av} = \frac{2}{\pi} \cdot U_m \tag{4.6}$$

Thus the average value of either the positive or negative half of a sine function can be found simply by multiplying the amplitude of the wave by 0.636. When taken over a full cycle the equal and opposite average values cancel out.

On the basis of the foregoing results it should be clear that, although the criterion of the average value of current works well in describing the energy-transferring capacity for direct sources, it is a meaningless criterion for symmetrical periodic functions because its value is always equal to zero.

Therefore we must search for a more suitable criterion to measure the effectiveness of a periodic function. Preferably the chosen standard should in some way be related to the energy or power capability associated with the periodic function. Herein lies a clue about the procedure to follow in establishing such a criterion. Consider that the sinusoidal current of Eq. (4.2) is made to flow through a resistor having *R* ohms. Then by Joule's law the instantaneous power absorbed by the resistor and converted to heat is

$$p = i^2(t)\,R \tag{4.7}$$

Since the current in this expression varies usually from a positive maximum through zero to a negative maximum, the insertion of any one value of the current is of little usefulness in determining the actual power absorbed by the resistor.

Rather a much more meaningful approach is to sum the instantaneous power consumption over one full cycle. Such a formulation cannot lead to a zero result because the power dissipation is real whether the current flows clockwise (positive) or counterclockwise (negative) in this circuit.

Therefore a nonzero value must result. This conclusion is also borne out by the presence of the second power of the current in Eq. (4.7). Thus even if the current is negative the contribution to the power expression is positive. Proceeding on this basis, we find that the expression for the average power absorbed by the resistor becomes

$$P_{av} = \frac{1}{T}\int_0^T i^2(t)\,R\,dt = \left[\frac{1}{T}\int_0^T i^2(t)\;dt\right]R$$

A study of the quantity in brackets reveals some interesting points. Note first that this quantity is the average value of the function

$f(t) = i^2(t)$. A comparison with Eq. (4.5) makes this obvious. We are still dealing with average effects - but with one important difference: instead of working with the periodic function *i(t)* directly we are dealing with the squared function. This not only gives significance to the formulation in terms of a description involving the power capability of the current, but it also eliminates the possibility of dealing with a zero average value because the square of a periodic function always yields a positive contribution.

Another point of interest is that the unit of the bracketed quantity is amperes squared. Accordingly we can interpret this quantity as the square of an effective current, which, when multiplied by *R*, yields the average power. Expressing this mathematically, we can write

$$I_{ef}{}^2 = \frac{1}{T}\int_0^T i^2(t)\,dt = average \quad i^2(t)$$

From this it follows that the effective current is the root mean square value. Thus

$$I_{ef} = \sqrt{average \quad i^2(t)} = \sqrt{\frac{1}{T}\int_0^T i^2(t)}$$

For the practical purposes a convenient mean value is the *effective* value of current. It is the *square root of the mean of the currents squared* (which are all positive). It is the magnitude of that direct current which has the same heating effect in a given resistive circuit as the alternating current in question.

Hence the root mean square (r.m.s. or effective) value of current

$$I_{ef} = \sqrt{\frac{1}{T}\cdot\int_0^T i^2 dt} = \sqrt{\frac{1}{T}\cdot\int_0^T I_m{}^2 \sin^2 \omega t \cdot dt} = \frac{I_m}{\sqrt{2}} = 0.707\cdot I_m \qquad (4.8)$$

Similarly,

$$E_{ef} = \frac{E_m}{\sqrt{2}} = 0.707\cdot E_m,\ U_{ef} = \frac{U_m}{\sqrt{2}} = 0.707\cdot U_m \qquad (4.9)$$

Although the effective current may be denoted by either of the subscript notations used in the last equation, it is frequently written without a subscript. The understanding is that reference is always to the effective value of a periodic function as defined in Eq. (4.8) unless otherwise specified.

Finally, as a third point of interest, let us make an analogy with the direct-current case. A direct voltage source applied to the resistor *R* causes an average power dissipation of I^2R to take place. Here *I* denotes the direct or average current. The flow of a periodic function of current through the same resistor yields an average power dissipation of $I^2_{ef}R$. Comparing this result with that of the direct-current case yields another interpretation of effective current: it is that current which produces the same heating effect as the direct current. Most electrical measuring instruments are constructed to indicate the effective values of currents and voltages being measured.

4.3. Complex Representation of Sinusoidal Quantities

Often in determining the sinusoidal steady-state response of circuits it is necessary to perform algebraic operations such as addition, subtraction, multiplication, and division on two or more sinusoidal quantities of the same frequency. Usually the sinusoids differ in amplitude and phase. Specifically consider the matter of adding two sinusoidal currents whose equations are

$$i_1 = I_{m_1} \sin \omega t, \quad i_2 = I_{m_2} \sin(\omega t + \psi_2) \tag{4.10}$$

The current i_2 leads i_1 by the relative phase angle ψ. The resultant current i_3 can obviously be written as

$$i_3 = I_{m_1} \sin \omega t + I_{m_2} \sin(\omega t + \psi_2) \tag{4.11}$$

Since the addition of two sinusoids of the same frequency always results in another sinusoid, it is desirable to express in this equation in terms of a resultant appropriate amplitude and phase. Keep in mind that any sinusoid at a given frequency is completely specified once its amplitude and phase are known. Of course one obvious way of obtaining the result called for in the last equation is to plot each sinusoid and then make a point-by-point summation of the two sine waves. The amplitude and phase of the resultant sinusoid can then be measured, thus allowing i_3 to be written in the more useful form

$$i_3 = I_{m_3} \sin(\omega t + \psi_3) \tag{4.12}$$

where ψ_3 is the phase angle measured with respect to the same reference point used for ψ_2. Needless to say, such a procedure is laborious and time-consuming.

An alternative to the graphical solution is an analytical one which simplifies the procedure through the use of trigonometric identities

Although this analytical procedure requires less effort than the graphical one, the method is still too cumbersome to be practical. This is particularly so when situations arise where more than two sinusoidal quantities must be summed.

Furthermore, multiplication and division present additional complications. Clearly, then, we need a simpler and more direct method of treating sinusoidal quantities. In 1893 such a method was introduced, when Charles P. Steinmetz advanced the idea of using a constant-amplitude line rotating at a frequency ω to represent a sinusoid. Let us now see how such an idea is effective in simplifying the algebraic operations involving sinusoidal quantities.

Attention is first directed to the expression for i_1, as given by Eq. (4.10). By employing the notation of $\sin \omega t = \mathrm{Im}\left[e^{j\omega t}\right]$ we can write

$$i_1 = I_{m_1} \sin \omega t = \mathrm{Im}\left[I_{m_1} \ e^{j\omega t} \right] \tag{4.13}$$

Keep in mind that the exponential function $e^{j\omega t}$ may be treated as a rotational operator. Its amplitude is always unity, but the cosine and sine components vary as time progresses. This is illustrated in Fig. 7.2 (a). As ωt moves through one full period of 2π radians (i.e., one complete cycle) the line *OA* makes one complete traversal of the circle in a counter clockwise direction. Line *OA* is fixed in value to the amplitude of the sine function it represents. Note that the vertical component of line *OA* is the sine function.

As a matter of fact this is the meaning of the notation $I_m\left[\ \right]$ - it refers to the values generated by taking the projections of a rotating line on a pre-established reference line (the vertical in this case). Accordingly, if we plot the vertical components of *OA* as it makes one complete revolution, the sine function shown in Fig. 7.2 (b). is generated. When $\omega t = 0$ the position of *OA* is on the horizontal axis directed towards the right. Its vertical component at this instant is zero, as it should be for the sine function.

The current i_2 can be represented in a similar fashion. Thus in terms of the exponential notation, we have

$$i_2 = I_{m_2} \sin(\omega t + \psi_2) = \mathrm{Im}\left[I_{m_2} \ e^{j\psi_2 t} \ e^{j\omega t} \right] \tag{4.14}$$

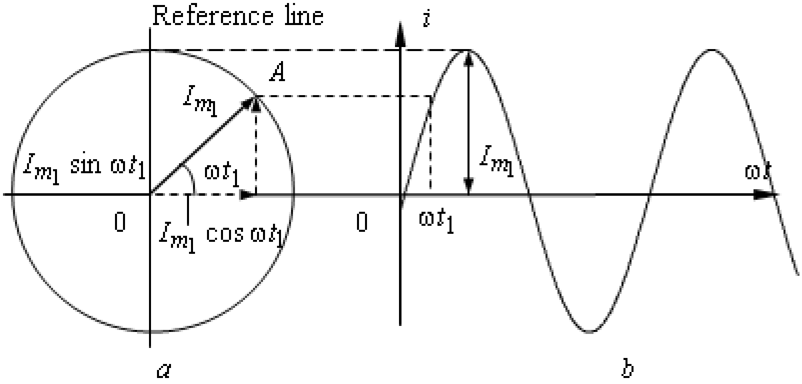


Fig. 4.2. Generating the sine function from the vertical component of the rotating line

To simplify the notation we next define

$$\underline{I}_{m_2} = I_{m_2} e^{j\psi_2} = I_{m_2}\ (\cos\psi_2 + j\sin\psi_2) \qquad (4.15)$$

and denotes it as line *OB* in Fig. 4.3, *a*. It is this quantity - I_{m2} - which is called the *phasor* of the sinusoidal function of Eq. (4.14).

In general the phasor can be represented by a *complex number* which is a result of locating a line in a plane. Thus the phasor *OB* can be located by specifying its magnitude, I_{m2}, and its displacement from the horizontal axis ψ_2.

The student certainly recognizes these quantities as the polar coordinates of line *OB*. However, note that *OB* can also be located in terms of a horizontal (i.e., real-axis) component and a vertical (i.e., imaginary or j-axis) component as indicated by the second form of $\underline{I}_{m_2}$ in Eq. (4.15). For a given ψ_2 the real part of the complex number is I_{m2} cos ψ_2 and the imaginary or j part is I_{m2} sin ψ_2.

It is important to understand that the position of *OB* in Fig. 4.3, *a* corresponds to time $t = 0$ in Eq. (4.14). Hence the corresponding value of i_2 at this instant must be the projection of *OB* on the vertical reference line, which clearly is $I_{m2}\sin\psi_2$.

The phasor of current i_1 is $\underline{I}_{m1} = I_{m1} e^{j0^o} = I_{m_1}$. Hence it has no vertical {or j) component and so initially (i.e., at $t = 0$) must lie along the horizontal axis as shown by line *OA* in Fig. 4.3, *a*.

Note too in this figure that the phasors $\underline{I}_{m1}$ and $\underline{I}_{m2}$ are displaced from one another by ψ_2. This is entirely consistent with Eqs. (4.10).

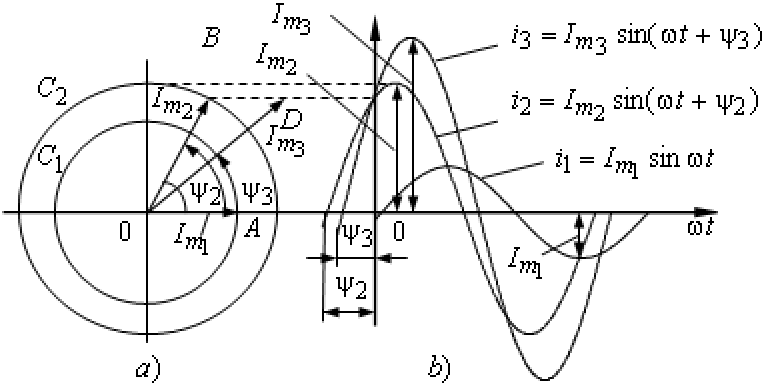


Fig. 4.3. Illustrating the phasor representation and addition of two sinusoids, *a*) position of the phasor for $t = 0$; *b*) sinusoidal function for increasing time

Now, as time elapses, both these phasors revolve at a constant frequency of ω radians per second, and a continuous plot of the vertical components of both phasors results in the sinusoids shown in Fig. 4.3, *b*. In this way a sinusoid can be represented by a line fixed at one end and rotating at a frequency equal to the angular velocity of the sinusoid. The magnitude of the line must be equal to the amplitude of the sinusoid.

How does the phasor representation of sinusoids simplify the procedure for adding two sinusoidal quantities?

To understand this, let us find $i_3 = i_1 + i_2$ by rewriting Eq. (4.11) in exponential form. Thus

$$i_3 = i_1 + i_2 = \mathrm{Im}\left[\underline{I}_{m_1} e^{j\omega t}\right] + \mathrm{Im}\left[\underline{I}_{m_2} e^{j\omega t}\right] = \mathrm{Im}\left[\left(\underline{I}_{m_1} + \underline{I}_{m_2}\right) e^{j\omega t}\right] \quad (4.16)$$

Expressed in words, Eq. (4.16) states:

To add two sinusoidal quantities it is necessary merely to add the corresponding phasors.

Accordingly, if we call the phasor quantity of the resultant $\underline{I}_{m3} = I_{m3} e^{j\psi_3}$, we may write

$$\underline{I}_{m_3} = \underline{I}_{m_1} + \underline{I}_{m_2} \tag{4.17}$$

$$I_{m_3} e^{j\psi_3} = I_{m_1} + I_{m_2} e^{j\psi_2} = (I_{m_1} + I_{m_2} \cos\psi_2) + jI_{m_2} \sin\psi_2 \tag{4.18}$$

The right side of the last equation involves all known quantities. Moreover, since it is complex number having a horizontal and vertical components, it follows that the magnitude of the component is

$$I_{m_3} = \sqrt{(I_{m_1} + I_{m_2} \cos\psi_2)^2 + (I_{m_2} \sin\psi_2)^2} \tag{4.19}$$

And the angle is

$$\psi_3 = \tan^{-1} \frac{I_{m_2} \sin\psi_2}{I_{m_1} + I_{m_2} \cos\psi_2} \tag{4.20}$$

The resultant phasor I_{m3} is depicted in Fig. 4.3 as line OD. When the value of I_{m3} has been established, the corresponding time expression for i_3 readily follows from Eq. (4.16). Hence

$$i_3 = \operatorname{Im}\left[\underline{I}_{m_3} e^{j\omega t}\right] = \operatorname{Im}\left[I_{m_3} e^{j\psi_3} e^{j\omega t}\right] = \operatorname{Im}\left[I_{m_3} e^{j(\omega t + \psi_3)}\right] =$$

$$= \operatorname{Im}\left[I_{m_3} \cos(\omega t + \psi_3) + jI_{m_3} \sin(\omega t + \psi_3)\right] =$$

$$= I_{m_3} \sin(\omega t + \psi_3) \tag{4.21}$$

The set of vectors of electrical magnitudes of the same frequency represented in uniform axes is called a vector diagram. If there are some vectors of various electrical magnitudes (usually a current and a voltage) in a diagram, then such a figure is called a combined diagram Fig. (4.4).

Vector diagrams are widely applied at circuit analysis of a sinusoidal current. Their application does a circuit design more evident and simple.

At calculation of sinusoidal current circuit one must constantly carry out the operations of addition (or subtraction) of instantaneous values of currents, voltages and e.m.f.s.

The use of vector diagrams allows replacing addition and subtraction of instantaneous values by addition and subtraction of corresponding vectors.

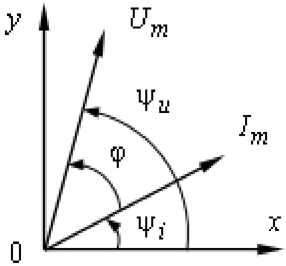


Fig. 4.4. A combined diagram

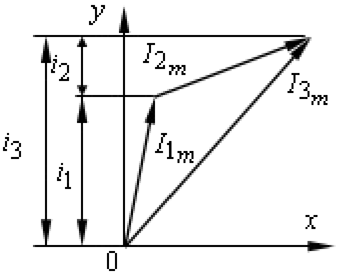


Fig. 4.5. The additional of vectors

The vector which represents a total sinusoid is equal to the sum of the vectors which represent composed items (Fig. 4.5).

The subtraction of vectors can be substituted by the operation of vector addition by the way of change of the vector direction by 180° that corresponds to the modification of its sign.

Example 4.1. Two sinusoidal currents are described as follows: $i_1 = 10\sqrt{2}\sin\omega t$ and $i_2 = 20\sqrt{2}\sin(\omega t + 60^o)$. Find the expression for the sum of these currents.

Solution: The solution is found by performing phasor addition in the manner indicated by Eq. (4.17).

In passing, note that i_1 and i_2 have effective values of 1o and 20 *A*, respectively. The phasor quantities are

$$\underline{I}_{m_1} = I_{m_1} = 10\sqrt{2}$$

$$\underline{I}_{m_2} = 20\sqrt{2}(\cos 60^o + j\sin 60^o) = 20\sqrt{2}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 10\sqrt{2} + j10\sqrt{6}$$

.

Hence

$$\underline{I}_{m_3} = \underline{I}_{m_1} + \underline{I}_{m_2} = 10\sqrt{2} + 10\sqrt{2} + j10\sqrt{6} = 20\sqrt{2} + j10\sqrt{6}\ .$$

So that

$$I_{m_3} = \sqrt{800 + 600} = 37.4\ ,\ \psi_3 = \tan^{-1}\frac{10\sqrt{6}}{20\sqrt{2}} = 41^o\ .$$

Therefore,

$$\underline{I}_{m_3} = 37.4e^{j41^o} \text{ and } i_3 = 37.4\sin(\omega t + 41^o)\ .$$

As we have already known the most important and useful, quantity of a sinusoidal function is its effective value. Although knowledge of the peak value of a sinusoid can be useful, it is not nearly so useful as the effective value.

In this Example it is much more significant to speak in terms of the effective current than in terms of the peak value, because the former conveys information about its energy-transferring capability per cycle whereas the latter does not.

Therefore in finding the sum of two currents we are often really interested in the resultant effective current. Of course, from Eq. (4.8) we know that the effective current can readily be obtained by dividing the peak value by $\sqrt{2}$.

In view of the far greater importance of effective values it is therefore customary to use a phasor diagram in which the phasors are expressed as effective values rather than maximum values. Accordingly, this procedure shall be followed in the remainder of the book.

To express the addition of two sinusoids in terms of the effective values of the phasors, it follows readily from Eq. (4.18) upon dividing each term in the expression by $\sqrt{2}$. Thus

$$\frac{\underline{I}_{m3}}{\sqrt{2}} = \frac{\underline{I}_{m1}}{\sqrt{2}} + \frac{\underline{I}_{m2}}{\sqrt{2}}$$

or, more simply,

$$\underline{I}_3 = \underline{I}_1 + \underline{I}_2 . \qquad (4.22)$$

We take note of one other convention which is used in phasor diagrams and the algebraic manipulations associated with them. Thus the phasor quantity for a current such as i can be expressed in terms of its effective value $Ie^{j\psi}$. The angle ψ is always measured relative to the horizontal axis counter clockwise. For negative values of the angle the direction is taken as clockwise.

Let us consider *multiplication and division of complex quantities*. Up to now attention has been directed exclusively to the problem of adding sinusoidal quantities which are out of phase with one another. The use of phasor representation of the sinusoids simplified the procedure considerably. The method of solution was reduced to one of dealing with complex numbers as illustrated in following.

In dealing with the sinusoidal steady-state response of electric circuits the need frequently arises to multiply and divide complex numbers. The complex numbers, however, do not always represent sinusoidal functions. In the interest of illustrating how the product of two complex numbers is obtained, consider the following two complex numbers. One is denoted by the phasor $\underline{I} = Ie^{j\psi}$ the other is represented by the operator $\underline{Z} = Ze^{j\varphi}$. The product is desired.

As the first step in the procedure, formulate the product in terms of the exponential form. The exponential form is preferred initially because as a legitimate part of the language of mathematics we are familiar with the rules governing its manipulation.

Accordingly, we can write

$$\underline{I} \cdot \underline{Z} = Ie^{j\psi} Ze^{j\varphi} = IZ\, e^{j(\psi+\varphi)} \tag{4.23}$$

Therefore the product of two complex numbers is found by taking the product of their magnitudes and the sum of their angles.

The division of one complex quantity by another is treated in a similar fashion. To illustrate, let it be required to divide the complex quantity $\underline{U} = U\, e^{j\psi}$ into the value $\underline{Z} = Ze^{j\varphi}$. We shall call the quotient $\underline{I}$. Thus

$$\underline{I} = \frac{\underline{U}}{\underline{Z}} = \frac{Ue^{j\psi}}{Ze^{j\varphi}} = \frac{U}{Z} e^{j(\psi-\varphi)} . \tag{4.24}$$

Therefore the division of one complex number by another involves the division of their magnitudes to yield the magnitude of the quotient and the difference of their phase angles to yield the phase of the quotient. For the sake of completeness we give the corresponding time expression for the phasor of equation (4.24), which is

$$i = \sqrt{2}\, \frac{U}{Z} \sin\,(\omega t + \psi - \varphi) .$$

The angular frequency ω is the same as that for u .

4.4. The Receipt of the Sinusoidal Electromotive Force

Sinusoidal electromotive forces in modern technical devices are obtained by various methods in electric machines, electronic oscillators and other devices.

Let's observe a cross-section of the simplified model of a monophase sinusoidal current generator (Fig. 4.6).

The right-angled conductor frame (or convolution) with cross -section *S*, is uniformly revolving with a constant angular frequency *ω* in a magnetic field between poles *N* and *S* of a constant magnet. We will assume that the excited magnetic field is constant and uniform in the area of rotation. The value of a magnetic induction is equal to *B*.

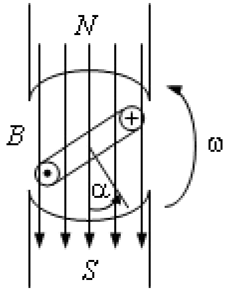


Fig. 4.6. A cross-section of a generator

The basic energy sources working at an alternating current are the electromechanical generators which transform the energy of a rotatory movement into electrical one.

The principle of action of such generators is based on the effect of excitation of vortical electric field when a magnetic field changes.

This effect leads to origin of the induced electromotive force in a closed loop at a magnetic flux modification through the surface restricted with this contour.

When the conductor frame revolves the value of the magnetic flux, passing through its plane, changes. This magnetic flux is maximum when the frame is perpendicular to the magnetic lines of the field.

$$\Phi_m = B \cdot S$$

In the process of rotation of the frame from this position, it decreases and becomes zero when the frame plane settles down along field lines. Then the flux direction changes its sign, and the flux starts to increase. Thus, the magnetic flux piercing the frame, changes depending on the angle of its rotation by the law:

$$\Phi = \Phi_m \cos \alpha$$

where α is an angle between the direction of magnetic field lines and a normal to the frame plane.

Hence the magnetic flux varies in a time according to the equation

$$\Phi = \Phi_m \cos(\omega t + \psi_e) = B \cdot S \cos(\omega t + \psi_e)$$

where **Φ** is the magnetic flux through an area *S*, and *B* is a magnetic induction.

According to the law electromagnetic induction discovered by Michael Faraday around 1831, a change in a magnetic flux linking a closed electric circuit produces an e.m.f. in the circuit, irrespective of the cause of the flux change.

Experiments show that the induced e.m.f. is directly proportional to the rate of change of flux linkages:

$$e(t) = -\frac{d\Phi}{dt} = B \cdot S \cdot \omega \sin(\omega t + \psi_e) = E_m \sin(\omega t + \psi_e)\,.$$

The e.m.f. in time mounts to an amplitude value when the frame plane is parallel to the lines of magnetic induction. At such position of the frame the flux piercing it, is equal to zero, and the velocity of intersection of magnetic lines is maximum.

This e.m.f. through the contact rings rotating together with the frame, and through the motionless brushes which slide on the surfaces of slip-rings, is transmitted to output terminals of a generator where it creates the sinusoidal voltage which is equal to e.m.f. If some load is connected to the brush terminals there will be an electric current in this formed electric circuit.

4.5. A Sinusoidal Current Through a Resistance

The response of each circuit parameter to a sustained sinusoidal source function is found individually - for two reasons. First, it provides an opportunity to illustrate the manner in which the response can be found easily and directly by the use of the phasor representation of sinusoids. Second, it affords the opportunity to establish once and for all the phase-angle relationships existing between the current and voltage for each circuit parameter. It is seen that these relationships are fixed and must always be satisfied irrespective of whether a given circuit element is in a series or parallel arrangement with other circuit elements. Let us start by treating the simplest of the three parameters - resistance.

Assume a sinusoidal forcing function which can be described by

$$u = U_m \sin \omega t = \mathrm{Im}\left\lfloor U_m e^{j\omega t} \right\rfloor \tag{4.25}$$

Because the circuit is linear we know that the steady-state response must also be a sinusoid having the same frequency as the source function. However, in general, the response sinusoid (i.e., the current) differs in two respects - amplitude and phase. Therefore, on this basis we can say that the form of the response must be

$$i = I_m \sin(\omega t + \psi) = \operatorname{Im}\left[I_m e^{j\psi} e^{j\omega t}\right] \tag{4.26}$$

where I_m and ψ are respectively the amplitude and phase of the response which must be determined to achieve a solution of the problem. The equation which makes available the amplitude and phase information in this instance is Kirchhoff 's voltage law applied to the circuit of Fig. 4.7. Thus

$$u = R\,i = RI_m \sin \omega t = R \times \operatorname{Im}\left[I_m e^{j\psi} e^{j\omega t}\right] \tag{4.27}$$

Note that this expression is nothing more than Ohm's law with the modification that sinusoidal quantities are involved for the variables u and i. The exponential form is preferred because it leads to the solution in a direct and easy fashion as is apparent from the material which follows. But first let us rewrite the last equation in a simpler form by dropping the notation "imaginary part of…". *This is permissible because such equations reduce to an identity for all values of time whenever the coefficients of* $e^{j\omega t}$ *on one side equal the coefficients of* $e^{j\omega t}$ *on the other side*. There is a precaution to be observed, however, when using such a procedure. After completion of the algebraic manipulations which lead to the solution, it must be remembered that the actual solution is found by taking only the j part of the resulting exponential solution. On this basis Eq. (4.27) can be written as

$$R\,I_m e^{j\psi} e^{j\omega t} = U_m e^{j\omega t}$$

Since the time factor $e^{j\omega t}$ is common to both sides of the equation, it may be suppressed, thus leading to

$$I_m e^{j\psi} = \frac{U_m}{R} U_m\, e^{j0^o} \tag{4.28}$$

On the left side of this equation appear the two unknown quantities I_m and ψ = 0, while on the right side appear the known quantities. Although in general both sides of this equation represent complex numbers, it is clear from the right side that in this case we have just a real number. This is characteristic of purely resistive circuit. A comparison of amplitudes and angles of the left and right sides of Eq. (4.26) leads to

$$I_m = \frac{U_m}{R} \text{ and } \psi = 0 \tag{4.29}$$

Substituting this information into Eq. (4.26) yields the final expression for the solution. Hence

$$i = \frac{U_m}{R} \sin \omega t \tag{4.30}$$

Comparing Eq. (4.30) with Eq. (4.25) reveals that for a resistive circuit the current and voltage are in time phase, i.e., the peak values and the zero values occur at the same time instants. The instantaneous values of current, voltage and power are illustrated by plots in Fig. 4.7, *a*.

The instantaneous power is

$$p = U_m I_m \sin \omega t \sin \omega t = \frac{U_m I_m}{2}(1 - \cos 2\omega t) \tag{4.31}$$

Thus, the instantaneous power has an unvarying component $(U_m I_m / 2)$ and a varying component $(U_m I_m \cos 2\omega t/2)$, varying at frequency 2ω. The energy input during a time dt is pdt.

As noted in the discussion of phasor representation of sinusoids, it is more frequently desirable to deal in terms of the effective value of voltages and currents rather than their maximum values. Accordingly, to express the solution as represented by Eqs. (4.29) in terms of effective values, use is made of Eq. (4.8). Thus

$$\mathrm{I}_m = \sqrt{2} I = \frac{\sqrt{2} \cdot U}{R}$$

where I and U denote effective values. Therefore, one can very simply describe the current response caused by a sinusoidal voltage of effective value U applied to a circuit of resistance R as

$$\underline{I} = \frac{\underline{U}}{R} \tag{4.32}$$

Referring to Fig. 4.7, *b*, the complex current $\underline{I}$ and the complex voltage $\underline{U}$ are in phase.

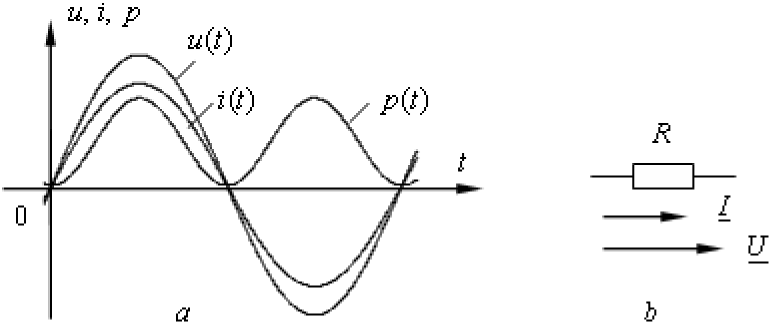


Fig. 4.7. Resistive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex current and voltage

Appearing in Fig. 4.7, *b*, is the phasor diagram for the resistive circuit. The reference phasor is arbitrarily taken to be the applied voltage and is placed along the horizontal. By Eq. (4.32) the current phasor has an angle of 0° which means that the complex current $\underline{I}$ and the complex voltage $\underline{U}$ are in phase. They are located on the horizontal axis. It is customary to show only effective values of voltage and current in the phasor diagrams. Note that the relationship depicted in Fig. 4.7, *b*, is consistent with that shown in Fig. 4.7, *a*.

4.6. A Sinusoidal Current Through an Inductance

The same method of analysis is used to find the response of a purely inductive circuit to a sinusoidal source function in the steady state. Assume that the potential difference appearing across the inductor terminals is given by Eq. (4.25). Because the circuit is assumed linear, the response can again be represented by Eq. (4.26). However, for the inductive circuit the defining equation is

$$u = L\frac{di}{dt} \tag{4.33}$$

Substituting Eqs. (4.25) and (4.26) gives

$$\operatorname{Im}\left[U_m e^{j\omega t}\right] = L\frac{di}{dt}\operatorname{Im}\left[I_m e^{j\Psi} e^{j\omega t}\right] \tag{4.34}$$

where I_m and ψ in general will be different from those values found for the resistive circuit. This is to be expected, since the defining equation involves a derivative of the current. Performing the differentiation and suppressing $e^{j\omega t}$ leads to

$$I_m e^{j\psi} = \frac{U_m}{j\omega L} \tag{4.35}$$

Again note that the term involving the unknown amplitude and phase angle is isolated to the left side of the equation. The quantities appearing on the right side are known. Examination of the right side in this instance shows that it is a number which is located along the vertical axis. In mathematical language this is described as the imaginary number. An alternative way of writing the right side follows upon recalling that the factor j is a rotational operator defined as

$$j = e^{j90^o} \tag{4.36}$$

Any real number which is multiplied by j changes its position from the horizontal axis to the vertical axis. Thus $j10$ means that a line of 10 units is now measured on the vertical axis. Accordingly, Eq. (4.35) can be rewritten

$$I_m e^{j\psi} = \frac{U_m}{\omega L} e^{-j90^o} \tag{4.37}$$

Therefore,

$$I_m = \frac{U_m}{\omega L} \text{ and } \psi = -90^o\ . \tag{4.38}$$

Expressed in terms of effective values, the solution for the response can be found conveniently by writing Eq. (4.37) as

$$I = \frac{U}{\omega L} = \frac{U}{X_L} \quad \text{and } \psi = -90^{\circ} \tag{4.39}$$

where $X_L = \omega L$ and is called the *inductive reactance* of the coil. The term reactance is used to distinguish it from resistance. A resistive circuit element causes no phase shift between voltage and current.

However, an *inductive element causes a current to lag behind a voltage by* 90°. Any circuit element which exhibits this property in the sinusoidal steady state is said to have inductive reactance.

The phasor diagram for the purely inductive circuit is illustrated in Fig. 4.8, *b*. The location of the current phasor is drawn consistent with the results of Eq. (4.39), which calls for the current phasor to lag behind the voltage phasor by angle equalled = -90°.

It is always true that for sinusoidal source functions *the current flowing through an inductor always lags behind the potential difference across the inductor terminals by* 90°. In the interest of completeness keep in mind the fact that the phasors of Fig. 4.8, *a* are actually rotating counterclockwise (the assumed positive direction) at the angular frequency *ω* radians/second.

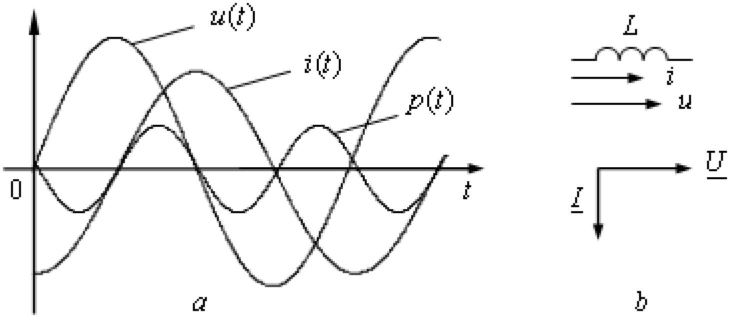


Fig. 4.8. Inductive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex voltage and current

Equation (4.39) contains all the useful information needed for the solution of the problem. However, if the complete time solution for the current response is desired, it readily follows from this equation by recalling the meaning of phasor quantities. This leads to

$$i = \sqrt{2}\,\frac{U}{X_L}\sin(\omega t + \psi) = \frac{U_m}{X_L}\sin(\omega t - 90^o) \qquad (4.40)$$

A plot of Eq. (4.40) appears in Fig. 4.8, *a*. Note that the response sinusoid in the case of an inductive circuit is displaced 90° behind the potential difference across the inductor.

Note also that just as Eq. (4.39) is a simpler and more direct way of representing the solution of Eq. (4.40) so too is Fig. 4.8, *b* a simpler and more convenient way of expressing the results shown in Fig. 4.8, *a*. The instantaneous values of current, voltage and power are given by plots of Fig. 4.8, *a*.

The instantaneous power

$$p = ui = -\frac{U_m I_m \sin 2\omega t}{2}.$$

It is zero when either voltage or current is zero. During the first quarter-cycle, when the voltage and the current are positive, the power is also positive. The area enclosed by the curve *p* and the axis of abscissa during this period of time represents the energy taken from the source and spent to establish a magnetic field in the inductance.

During the second quarter-cycle, when the current in the circuit decreases from a maximum to zero, the energy of the magnetic field is returned to the supply, and the instantaneous power is negative. During the third quarter-cycle, the energy is again taken from the supply and is returned to it during the fourth quarter-cycle, and so on.

Thus, an inductance alternately draws energy from the supply and gives it back.

4.7. A Sinusoidal Voltage Across a Capacitance

The current which flows through the capacitor is related to the potential difference *u* by

$$i = C\frac{du}{dt}$$

For an assumed sinusoidal source function of $u = U_m \sin \omega t$ the general form of the current response can be represented by $i = I_m \sin(\omega t + \psi)$. Inserting the exponential forms of voltage and current into the last expression yields

$$I_m e^{j\psi} e^{j\omega t} = C\frac{d}{dt}\left[U_m e^{j\omega t}\right] \tag{4.41}$$

Performing the differentiation leads to

$$I_m e^{j\psi} = j\omega C U_m \tag{4.42}$$

Or

$$I_m e^{j\psi} = \frac{U_m}{1/\omega C} e^{j90^o} \tag{4.43}$$

Equating the magnitudes and angles of the right and left sides then yields

$$I_m = \frac{U_m}{1/\omega C} \text{ and } \psi = 90^o \tag{4.44}$$

Once Eq. (4.44) is obtained by using the exponential forms of voltage and current in the governing differential equation, it can then be rewritten in terms of effective values. It is helpful at this point to employ effective values, because from here on the solution of capacitive circuits can be found by applying Ohm's law in the modified form dictated by Eq. (4.44).

Accordingly, we can write

$$\underline{I} = \frac{\underline{U}}{1/j\omega C} \tag{4.45}$$

This states that the phasor current response $\underline{I}$ is equal to the phasor potential difference $\underline{U}$ across the capacitor divided by the quantity $1/j\omega C$. It is important to understand that the last quantity arises from the exponential formulation as revealed by Eq. (4.42).

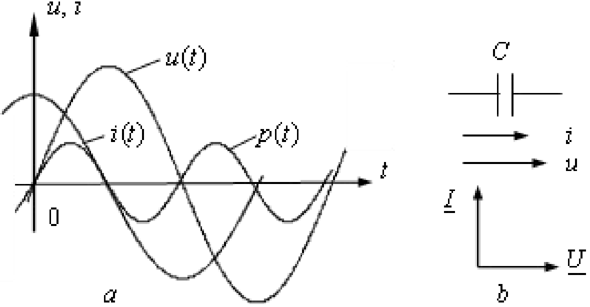


Fig. 4.9. Capacitive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex current and voltage

Moreover, the unit of this value is volt/ampere, which is ohm. This is evident from Eq. (4.45). Therefore, $1/j\omega C$ may be looked upon as a kind of resistance, but it is not called this because it involves the rotational operator *j*. For this reason it is called a reactance; more specifically, it is called a capacitive reactance because a capacitor is involved. Equation (4.45) often appears in the form

$$\underline{I} = \frac{\underline{U}}{X_C e^{-j90^o}} = \frac{\underline{U}}{X_C} e^{j90^o} \tag{4.46}$$

Where X_C is the capacitive reactance in ohms.
Therefore, for the capacitive circuit

$$I = \frac{U}{X_C} = U\omega C \text{ and } \psi = 90^o . \tag{4.47}$$

By the information contained in Eq. (4.47) the phasor diagram of a purely capacitive circuit is readily drawn. Figure 4.9,*b*, depicts the result. The corresponding time-domain description of the current response is illustrated in Fig. 4.9, *a*. The time-domain equation for the response is obtained in the usual way and found to be

$$i = \sqrt{2}\,\frac{U}{X_C}\sin(\omega t + \psi) = \frac{U_m}{X_C}\sin(\omega t + 90^o) \tag{4.48}$$

A study of Eq. (4.47) as well as Fig. 4.9 reveals an important characteristic of the capacitive circuit: *the current flowing through a capacitor in the sinusoidal steady state always leads the potential difference across the capacitor by 90°.*

4.8 The Series *RL* Circuit

In this section we direct attention to the sinusoidal steady-state analysis of a circuit configuration which involves the series connection of a resistive and an inductive element. This leads to the concepts of complex impedance and reactive power.

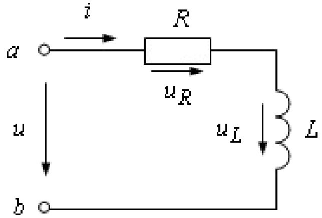


Fig. 4.10. The series *RL* circuit

Appearing in Fig. 4.10 is a diagram of the series *RL* circuit. Our objective is to find the steady-state current response assuming that *R*, *L*, and the forcing function are known. The differential equation for the circuit is

$$u = iR + L\frac{di}{dt}$$

This expression can be written in terms of effective values as

$$\underline{U} = \underline{I}(R + j\omega L) \tag{4.49}$$

Equation (4.49) is Kirchhoff's voltage law for the *RL* circuit written in terms of phasor quantities. This expression can be written directly without going through the preceding steps. It is obtained by summing the voltage drops across each circuit element. However, it is important to attach the rotational operator *j* to the inductive reactance. Upon transposing Eq. 4.49 the solution for the current becomes

$$\underline{I} = \frac{\underline{U}}{R + j\omega L} \tag{4.50}$$

where *R* is a resistance and ωL is the *inductive reactance*.

Information about the magnitude and relative phase angle of the response then readily follows when Eq. (4.50) is rewritten as

$$I = \frac{U}{\sqrt{R^2 + \omega^2 L^2}} \text{ and } \psi = \tan^{-1}(\omega L / R) \tag{4.51}$$

where ψ is the angle between a voltage and a current.

Examination of the solution for the relative phase angle reveals that for finite resistance the angle must be less than 90°. Note too that it is a lag angle, which means the current sinusoid is behind the voltage in time. The potential difference across the resistive element is clearly

$$\underline{U}_R = \underline{I}R \tag{4.52}$$

Because there is no phase shift associated with a resistor, it follows that this voltage drop must be in phase with the current. Hence the voltage drop across a resistance is drawn on the same line as the current. Equation (4.52) is represented by in the phasor diagram in Fig. 4.11.

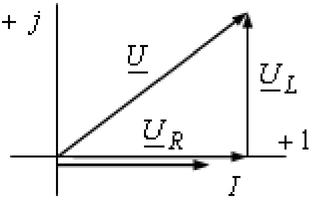


Fig. 4.11 *A* vector diagram

Similarly, the potential difference across the inductive element is shown by the second term of the right side of Eq. (4.49) to be expressed as

$$\underline{U}_L = j\omega L\underline{I} \tag{4.53}$$

The presence of the rotational operator *j* conveys the meaning that this quantity is to be rotated in the positive (or counter clockwise) direction through 90°. Accordingly, this value is disposed perpendicularly to the current in Fig. 4.11. Of course the sum of these two voltages must be equal to the input voltage.

4.9. The Series *RC* Circuit

The principles underlying the method of solution are presented in detail in the two preceding sections. Accordingly, in the interest of brevity as well as to illustrate the method to be employed from here on in solving such problems, we proceed directly with Kirchhoff's voltage law for the circuit of Fig. 4.12 by using effective quantities throughout.

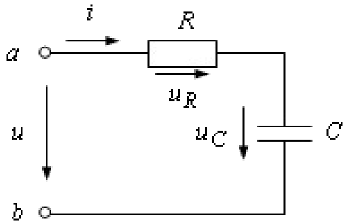


Fig. 4.12. The series *RC* circuit

Thus

$$\underline{U} = \underline{I}R + \underline{I}\frac{1}{j\omega C} \tag{4.54}$$

The effective potential difference appearing across the capacitor is expressed always in terms of the effective current passing through the capacitor multiplied by its capacitive reactance, which is written in conjunction with the rotational operator *j* as called for by Eq. (4.45). Rewriting Eq. (4.54) gives

$$\underline{U} = \underline{I}\left(R + \frac{1}{j\omega C}\right) = \underline{I}\left(R - j\frac{1}{\omega C}\right) \tag{4.55}$$

where *R* is a resistance and $1/\omega C$ is the capacitive reactance.

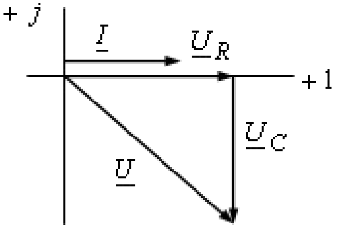


Fig. 4.13 *A* vector diagram

Equation (4.55) is Kirchhoff's voltage law for the *RC* circuit written in terms of phasor quantities. Upon transposing the equation the solution for the current becomes

$$I = \frac{U}{\sqrt{R^2 + 1/\omega^2 C^2}} \text{ and } \psi = \tan^{-1}\frac{1}{\omega RC} \tag{4.56}$$

The phasor diagram for the series *RC* circuit is depicted in Fig. 4.13. The current now leads the voltage in time - a result which is consistent with Eq. (4.56) showing a positive value for the phase angle. Again note that the effective potential difference across the resistor is in phase with the current, while that across the capacitor is 90° behind the position of the current phasor in Fig. 4.13. This rotation (ψ = -90°) is brought about in the diagram by the attachment of *-j* to the quantity which is equal to the voltage drop across capacitance.

4.10. The Series *RLC* Circuit. Ohm's Law for A.C. Circuits

The circuit configuration for the series combination of the three parameters is shown in Fig. 4.14; the terminals are lettered in order to facilitate the discussion of the phasor diagram.

For the circuit in Fig. 4.14 the instantaneous voltage is given by

$$u = u_R + u_L + u_C \tag{4.57}$$

or

$$iR + L\frac{di}{dt} + \frac{1}{C}\int i dt = u \tag{4.58}$$

But *ac* circuit problem are most solved by the method of complex numbers. The basis of this method is that for a sinusoidal varying current the equations based on instantaneous values, which are in fact differential equations (see Eq. 4.59), may be replaced by algebraic equations in complex currents and voltages.

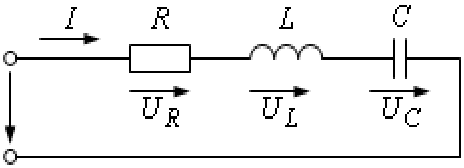


Fig. 4.14. The series *RLC* circuit

More specifically, in any Kirchhoffian equation written for a steady state, the instantaneous current *i* is replaced by the complex peak current $\underline{I}_m$; the instantaneous voltage across a resistance (*RI*) by the complex voltage drop $R\underline{I}_m$ which is in phase with the current $\underline{I}_m$; the instantaneous voltage across an inductance (u_L) with the complex quantity $\underline{I}_m j\omega L$ which leads the current by 90°; the instantaneous voltage across a capacitance (u_C) with the complex quantity $\underline{I}_m(-j/\omega C)$ lagging behind the current by 90°; and the instantaneous input voltage *u* with the complex value $\underline{U}_m$.

Indeed, it is shown in previous sections that the peak voltage across an inductance is given by the product of the peak current and the inductive reactance. The operator *j* indicates that the voltage across an inductance leads the current in it by 90°. Similarly, it follows that the peak voltage across a capacitor is the product of the peak current and the capacitive reactance. The fact that the voltage across a capacitor lags the current by 90° brings in the operator *-j*.

In complex notation,

$$\underline{I}_m R + \underline{I}_m j\omega L + \underline{I}_m \frac{-j}{\omega C} = \underline{U}_m$$

Putting before the brackets gives

$$\underline{I}_m \left(R + j\omega L - \frac{j}{\omega C} \right) = \underline{U}_m \tag{4.59}$$

Thus for the current of Fig. 4.14

$$\underline{I}_m = \frac{\underline{U}_m}{R + j\omega L - \dfrac{j}{\omega C}} \tag{4.60}$$

The above equation gives the complex peak current in terms of the complex peak input voltage and the operators $R, \ \omega L, \ 1/\omega C$.

The complex-number method is also called the *symbolic method* for the reason that the currents and voltages are represented by their complex transforms or symbols.

The operator in the denominator of Eq. (4.60) is symbolized by the letter $\underline{Z}$, the unit being the *ohms*. In all English-speaking countries $\underline{Z}$ is called the *complex impedance* of the circuit. Thus

$$\underline{Z} = R + j\omega L - \frac{j}{\omega C}. \tag{4.61}$$

The value

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{4.62}$$

is called the *impedance* of an electric circuit.

Then Eq. (4.60) may be written thus

$$\underline{I}_m \cdot \underline{Z} = \underline{U}_m$$

Dividing the last equation by $\sqrt{2}$ and substituting the complex effective values of the current and voltage for the complex peak values $\underline{I}_m$ and $\underline{U}_m$, respectively, we get

$$\underline{I} = \frac{\underline{U}}{\underline{Z}} \tag{4.63}$$

Eq. (4.63) is *Ohm's law* for alternating current circuits.

Writing Kirchhoff's voltage law expressed in terms of effective values we have

$$\underline{U} = \underline{I}\left(R + j\omega L + \frac{1}{j\omega C} \right)$$

From here

$$\underline{I} = \frac{\underline{U}}{R + j\omega L - \dfrac{j}{\omega C}}. \tag{4.64}$$

Therefore the magnitude and phase of the effective current are

$$I = \frac{U}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \text{ and } \psi = \tan^{-1}\frac{\omega L - 1/\omega C}{R} \tag{4.65}$$

A study of the last equation reveals that the current phasor may either lead or lag the voltage phasor, depending upon the relative values of the inductive and capacitive reactance.

Whenever the inductive reactance is more then the capacitive one, i.c. $\omega L > 1/\omega C$, the *RLC* circuit essentially behaves as an inductive circuit insofar as the current is concerned. It is interesting to note that this condition can be satisfied either by having a large inductance or else by operating at a high frequency.

On the other hand, whenever $\omega L < 1/\omega C$, the current leads the voltage, thereby indicating that the *RLC* circuit behaves as a capacitive circuit as far as the current is concerned. However, there are some differences; these are discussed presently.

Appearing in Fig. 4.15, *a*, *b*, is the phasor (or vector) diagram for the circuit of Fig. 4.14.

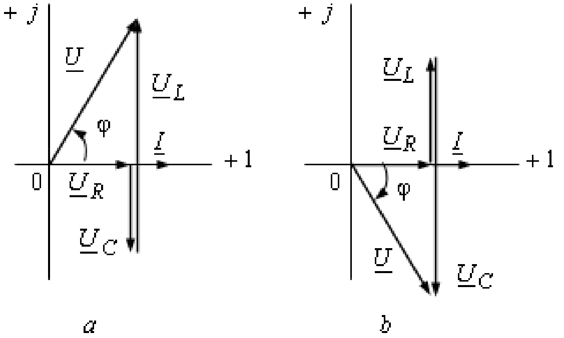


Fig. 4.15 *a*) The phasor diagram for a case $X_L > X_C$; *b*) the phasor diagram for a case $X_L < X_C$

Fig. 4.15, *a* is drawn for the case where $\omega L > 1/\omega C$. Hence the current phasor must lag the voltage phasor. The component values of the effective potential difference appearing across each circuit element are also depicted. Note that the voltage drop across the resistor terminals $\underline{U}_R$ must be in phase with the current $\underline{I}$. It is represented by the line, which is parallel to phasor current. In vector diagrams any line drawn parallel to another line means that the quantities represented by the two lines are in phase. The current $\underline{I}$ leads the voltage across the capacitor terminals $\underline{U}_C$ by 90° or the capacitive voltage lags behind the current by 90°, as it always must for sinusoidal forcing functions. Finally, note that the effective potential difference appearing across the inductor terminals $\underline{U}_L$ leads the current $\underline{I}$ by 90° as expected.

Although the phasor diagram depicted in Fig. 4.15, *a*, is for an *RLC* circuit in which the inductive reactance predominates, note that the circuit does behave differently than the straight *RL* circuit in two respects. One, the potential difference across the inductor contains a component which is equal and opposite to the total voltage across the capacitor terminals. This leaves a net reactive voltage, as "seen'" by the source. Two, the voltage across the inductor can be several times greater in magnitude than the source voltage $\underline{U}$. This cannot occur in the simple *RL* circuit. The large value of $\underline{U}_L$ should not be disturbing, however, because Kirchhoff's voltage law continues to be satisfied. Keep in mind that with *ac* circuits it is the phasor sum that is important. An algebraic sum is meaningless except in those instances where only elements of the same type appear in the circuit.

The angle φ is positive at inductive character of the circuit. Thus for the first vector diagram the current lags the input voltage in phase, and the angle φ is read off on an abscissa axis to the left from the current to the voltage.

Fig. 4.15, *b* is drawn for the case where $\omega L < 1/\omega C$. The angle φ is negative at the capacity character of the circuit. Thus voltage lags behind the current in phase, and the angle φ is read off on a real axis to the right from the current to the voltage.

Now we will pass from vector diagrams (Fig. 4.15, *a*, *b*) to impedance triangles. One can see the case of active-inductive resistance of an electric circuit in Fig. 4.16, *a*; and the case of active-capacitive resistance of an electric circuit in Fig. 4.16, *b*.

As it follows from the formula (4.62) the impedance, the resistance and the reactance may be depicted by a rectangular triangle, similar to the triangle of the vectors of voltages in Fig. (4.15, *a*, *b*). The impedance triangle graphically interprets the relationship between the impedance magnitude *Z* and the associated resistance and reactance of the circuit.

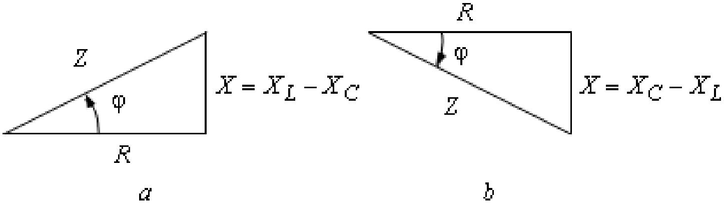


Fig. 4.16. The impedance triangles

As follows from impedance triangles resistance and reactance are connected with impedance by relationships

$$R = Z\cos\varphi, \quad X = Z\sin\varphi$$

4.11. The Voltage Resonance

Resonance is identified with engineering situations which involve energy-storing elements subjected to a forcing function of varying frequency. Specifically, resonance is the term used to describe the steady-state operation of a circuit or system at that frequency for which the resultant response is in time phase with the source function despite the presence of energy-storing elements.

Resonance cannot take place when only one type of energy-storing element is present, e.g., inductance or mass. There must exist two types of independent energy-storing elements capable of interchanging energy between one another - for example, inductance and capacitance or mass and spring.

Although attention here is confined to electric circuits, resonance is a phenomenon found in any system involving two independent energy-storing elements be they mechanical, hydraulic, pneumatic, or whatever. In the material which follows, the parallel as well as the series arrangement of the energy-storing elements - L and C - are treated as functions of frequency, with particular emphasis focused on the performance at resonance. When a source of a.c. is connected across a two-terminal network containing one or several inductances and one or several capacitances in additional to a resistance, phenomena arise under certain conditions, referred to as *resonance*. When an inductive reactance and a capacitive reactance are joined in series (as in Fig. 4.14) conditions may be obtained under which *voltage resonance* (also known as *series resonance*) will take place.

The expression for the effective current flow caused by a sinusoidal forcing function is

$$\underline{I} = \frac{\underline{U}}{R + j\omega L - \dfrac{j}{\omega C}} = \frac{\underline{U}}{\underline{Z}} \tag{4.66}$$

What is the effect on $\underline{I}$ of increasing the frequency to of the source function from zero to infinity? A glance at Eq. (4.66) indicates that a change in frequency means a change in the magnitude and phase angle of the complex impedance. Note that for ω close to zero the inductive reactance is almost zero but the capacitive reactance approaches infinity. Hence the current magnitude approaches zero. As ω increases, the reactance part of $\underline{Z}$ decreases, thus causing an increase in current. As ω continues to increase, a point is reached where the reactance term is zero. The condition for voltage resonance is thus

$$\omega_o L = \frac{1}{\omega_o C} \tag{4.67}$$

from where

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{4.68}$$

The frequency ω_o is called the *resonant frequency* of the circuit. Its value is specified entirely in terms of the parameters of the two energy-storing elements of the circuit in the manner called for by Eq. (4.68).

At resonance the impedance of the circuit is a minimum, and specifically it is equal to R:

$$Z_o = \sqrt{R^2 + (X_{Lo} - X_{Co})^2} = R \tag{4.69}$$

Consequently, when a series RLC circuit is at resonance, the current is a maximum and is also in time phase with the voltage. The power factor is unity. The complete expression for the current phasor is then

$$\underline{I}_o = \frac{\underline{U}}{\underline{Z}} = \frac{\underline{U}}{R} \tag{4.70}$$

where $\underline{I}_o$ denotes the current at resonance.

The resistances of reactive elements at the resonant frequency are equal to each other

$$X_{Lo} = \omega_o L = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}},$$
$$X_{Co} = \frac{1}{\omega_o C} = \frac{\sqrt{LC}}{C} = \sqrt{\frac{L}{C}}. \qquad (4.71)$$

The value $\rho = \sqrt{\frac{L}{C}}$ is called a *wave resistance*. The wave resistance is measured in Ohms (Ω)

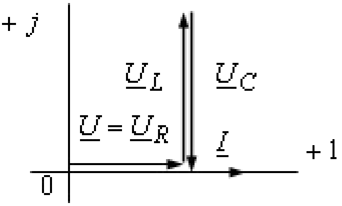


Fig. 4.17. The vector diagram for a series resonance

The vector diagram for a series resonance is shown in Fig. 4.17. Here $\underline{U}_L$ and $\underline{U}_C$ are equal to each other, and their vectors are in phase opposition (phase shift between them is equal π). These two values compensate each other. The voltage drop across the resistance R is equal to the input voltage U.

For operation at a frequency below ω_o the resultant *j* part of $\underline{Z}$ is capacitive so that the current leads the voltage.

When $\omega > \omega_o$ the inductive reactance prevails, so that the current then lags the voltage.

Note that when the ratio of inductive reactance to resistance in a series *RLC* circuit is high, a very rapid rise of current to the maximum value occurs in the vicinity of the resonant frequency. The ratio

$$\frac{\omega_o L}{R} = \frac{\sqrt{\frac{L}{C}}}{R} = \frac{\rho}{R} = Q \qquad (4.72)$$

is termed the *Q*-factor or the *quality factor* of a resonant circuit. It shows how many times the voltage across the reactive component exceeds the applied voltage at resonance. In practical circuits, *Q* is often relatively high.

Let's consider the frequency characteristics of an entering current and voltage across a resistance element (Fig. 4.18, *a)*, and the voltage across inductive and capacity elements (Fig. 4.18, *b)*. Such curve lines are called *resonance characteristics.*

They are expressed by the following equations:

$$
\begin{aligned}
I(\omega) &= \frac{U}{Z} = \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}};\\
U_R(\omega) &= \frac{UR}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}};\\
U_L(\omega) &= \frac{U\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}};\\
U_C(\omega) &= \frac{U}{\omega C\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.
\end{aligned}
\qquad (4.73)
$$

The graphic dependence of a current on frequency shows that the observed circuit possesses of selective properties. The contour possesses of the least resistance for the currents whose angular frequency is close to the resonance frequency.

Referring to Fig. 4.18, *a*, both the voltage across the inductance and the voltage across the capacitance are a maximum at frequencies other than the resonant frequency of the circuit.

The former is a maximum at a frequency higher than ω_0, and the latter at a frequency which is lower than the resonant frequency.

To describe the width of the resonance curve the term *bandwidth* is used. For the series *RLC* circuit bandwidth is defined as the range of frequency for which the power delivered to *R* is greater than or equal to $P_0/2$ where P_0 is the power delivered to *R* at resonance.

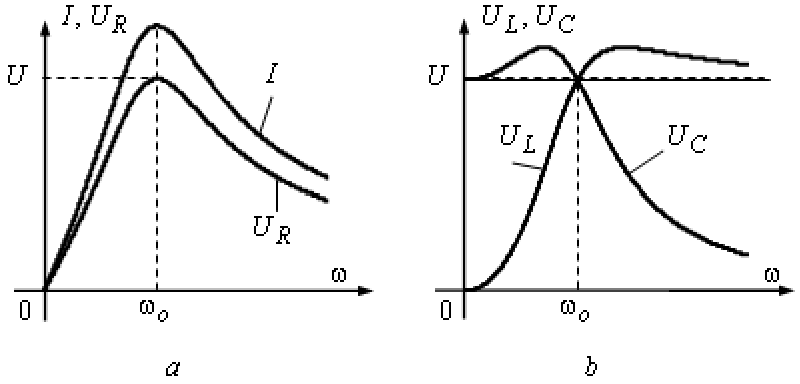


Fig. 4.18 *a*) The dependence of I and inductive U on frequency; *b*) the voltage across the inductance and the voltage across the capacitance

From the shape of the resonance curve it should be clear that there are two frequencies for which the power delivered to *R* is half the power at resonance.

For this reason these frequencies are referred to as those corresponding to the half-power points. The magnitude of the current at each half-power point is the same. It follows then that

$$I_1 = I_2 = \frac{I_o}{\sqrt{2}} = 0.707\ I_o \tag{4.74}$$

Accordingly, the bandwidth may be identified on the resonance curve as that range of frequency over which the magnitude of the current is equal to or greater than 0.707 of the current at resonance.

In Fig. 4.19 for Q_1, the frequency at the lower half-power point is denoted ω_1, and that of the upper half-power point is denoted ω_2. Hence the bandwidth is $\omega_2 - \omega_1$.

In view of the fact that the current at the half-power points is $I_o / \sqrt{2}$, it follows that the value of the impedance must be equal to $\sqrt{2}\ R$ to yield this current.

This information can now be used to obtain an expression for the bandwidth in terms of the parameters of the series circuit. Calling the reactance at the lower half-power frequency X_1, we have

$$X_1 = \omega_1 L - \frac{1}{\omega_1 C} = -R \tag{4.75}$$

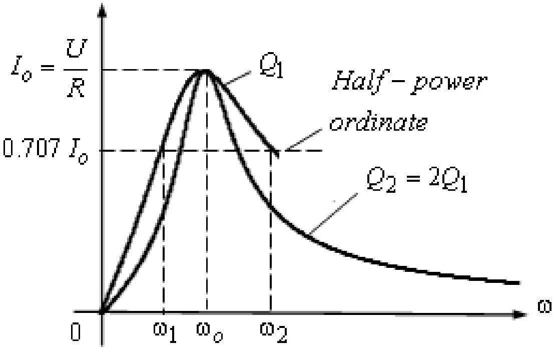


Fig. 4.19. Variation of current in a series *RLC* circuit with frequency

The minus sign appears on the right side of the equation because below resonance the capacitive reactance exceeds the inductive reactance. Rearranging Eq. (4.75) leads to

$$\omega_1^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

The roots are therefore

$$(\omega_1)_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 + \omega_o^2}$$

where $\alpha = R/2L$.

Although the previous equation provides two solutions, only one of these is physically realizable. The negative frequency is meaningless and so may be discarded.

Hence the expression for the lower half-power frequency is

$$\omega_1 = -\alpha \pm \sqrt{\alpha^2 + \omega_o^2} \tag{4.76}$$

The upper half-power frequency is found in a similar fashion.

In this instance we have

$$X_2 = \omega_2 L - \frac{1}{\omega_2 C} = +R$$

which leads to
$$\omega_2^2 + \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

The expression for the useful ω_2 is then

$$\omega_2 = \alpha \pm \sqrt{\alpha^2 + \omega_o{}^2} \tag{4.77}$$

Therefore, the expression for the bandwidth becomes, from Eqs. (4.76) and (4.77),

$$\omega_{bw} = \omega_2 - \omega_1 = 2\alpha = \frac{R}{L} \tag{4.78}$$

This expression is significant because it reveals that the bandwidth of the series *RLC* circuit depends solely upon the *R/L* ratio. Note that it is not the individual values of *R* or *L* but rather their ratio that is important. Note too that bandwidth depends not at all upon the capacitance parameter *C*.

By forming the ratio of the resonant frequency to the bandwidth we obtain a factor which is a measure of the selectivity or sharpness of tuning of the series *RLC* circuit. As this quantity is called the *quality factor* of the circuit and is expressed as

$$Q_o = \frac{\omega_o}{\omega_{bw}} = \frac{\omega_o}{R/L} = \frac{\omega_o L}{R} = \frac{X_{LO}}{R} \tag{4.79}$$

A glance at the curves appearing in Fig. 4.19 should make it clear as to why the quality factor Q_o is used as a measure of the selectivity or

sharpness of tuning. Note how much narrower the resonance curve is for Q_2 than for Q_1. Equation (4.78) shows that the value of *C* in no way influences the bandwidth but it does alter the value of ω_o.

Hence if *C* is changed, ω_o may be made to occur at a different position along the frequency scale in Fig. 4.19 without in any way affecting the sharpness of tuning. This feature of the high-*Q* circuit is used often in communication networks.

For example, in a radio the antenna may be considered in terms of an equivalent *RLC* circuit where *C* is adjusted by means of the dial tuning knob. If the dial is turned to the position which tunes in the frequency (ω_o) of a given radio transmitting station, the sharpness of tuning (i.e., small bandwidth) allows only signals from that station to produce large resonant current signals.

The signals from other broadcasting stations, although present in equal strength at the antenna, produce little or no signal strength in the circuit because the dial is tuned to a frequency considerably off resonance relative to their broadcasting frequencies.

4.12. The Parallel *RLC* Circuit

The circuit configuration for the study of parallel *RLC* connection appears in Fig. 4.20.

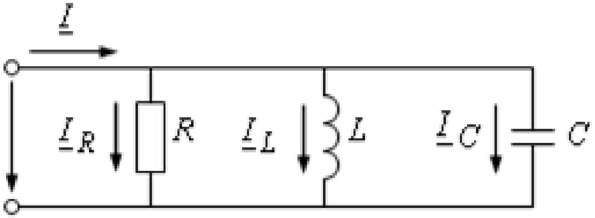


Fig. 4.20. The parallel *RLC* connection

If we apply sinusoidal voltage $u = U_m \sin \omega t$ to input terminals, in a non-ramified part it will appear the current whose instantaneous value according to the first Kirchhoff's law is equal to an algebraic sum of instantaneous values of branch currents

$$i = i_R + i_L + i_C \tag{4.80}$$

Thus the current through a resistance i_R coincides in phase with the voltage, the current through an inductance i_L lags behind the voltage by 90°, and the current through a capacitance i_C leads the voltage by 90°.

We can write this mathematically as

$$i_R = I_{Rm} \sin \omega t = \frac{U_m}{R} \sin \omega t,$$

$$i_L = I_{Lm} \sin\left(\omega t - \frac{\pi}{2}\right) = -\frac{U_m}{X_L} \cos \omega t,$$

$$i_C = I_{Cm} \sin\left(\omega t + \frac{\pi}{2}\right) = \frac{U_m}{X_C} \cos \omega t.$$

Then we can get the expression for the current in a non-ramified part of the circuit as follows:

$$\begin{aligned} i &= \frac{U_m}{R}\sin\omega t - \frac{U_m}{X_L}\cos\omega t + \frac{U_m}{X_C}\cos\omega t = \\ &= \frac{U_m}{R}\sin\omega t - \left(\frac{1}{X_L} - \frac{1}{X_C}\right)U_m\cos\omega t = \\ &= U_m G\sin\omega t - (B_L - B_C)U_m\cos\omega t = \\ &= U_m\left(G\sin\omega t - B\cos\omega t\right) = I_m\sin(\omega t - \varphi) \end{aligned} \tag{4.81}$$

where G is called a *conductance*, B_L is called the *inductive susceptance*, B_C is called the *capacitive susceptance*. The value B is called the *susceptance*.

$$B = B_L - B_C \quad \text{(4.82}$$

The *complex admittance* of the circuit

$$Y = G + j(B_L - B_C) \tag{4.83}$$

The equation (4.81) represents a trigonometrical notation of the first Kirchhoff's law for instantaneous values of currents. Rewrite Eq. (4.81) in complex notation,

$$\underline{I}_m = \underline{U}_m\left[G + j(B_L - B_C)\right] = \underline{U}_m Y \tag{4.84}$$

Dividing Eq. (4.84) through by $\sqrt{2}$ and substituting the complex effective values of the current and voltage for the complex peak values $\underline{I}_m$ and $\underline{U}_m$, respectively, we have got the equation which presents Ohm's law for parallel connection of *RLC*:

$$\underline{I} = \underline{U}\left[G + j(B_L - B_C)\right] = \underline{U}\cdot\underline{Y} \tag{4.85}$$

Appearing in Fig. 4.21, is the phasor (or vector) diagram for the circuit of Fig. 4.20.

Fig. 4.21, *a* is drawn for the case where $B_L > B_C$. Hence the current phasor must lag behind the voltage phasor.

The component values of the effective potential difference appearing across each circuit element are also depicted.

Note that the voltage drop across the resistor terminals $\underline{U}_R$ must be in phase with the current $\underline{I}_R$. It is represented by the line, which is parallel to phasor $\underline{I}_R$. In vector diagrams any line drawn parallel to another line means that the quantities represented by the two lines are in phase.

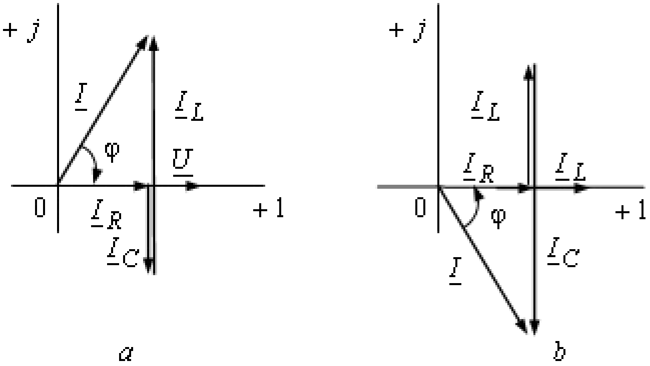


Fig. 4.21. Vector diagram for parallel *RLC* connection

The current $\underline{I}_C$ leads the voltage across the capacitor terminals $\underline{U}_C$ by 90° or the capacitive voltage lags behind the current by 90°, as it always must for sinusoidal forcing functions.

Finally, note that the effective potential difference appearing across the inductor terminals $\underline{U}_L$ leads the current $\underline{I}_L$ by 90° as expected.

From vector diagrams (Fig. 4.21) we will pass to triangles of conductivities.

Fig. 4.22, *a* shows us a right-angled triangle, called admittance triangle, for a case of is active-inductive load of an electric circuit, and Fig. 4.22, *b* -for a case of active-capacity load.

The magnitude of the complex admittance is

$$Y = \sqrt{G^2 + B^2} = \sqrt{G^2 + (B_L - B_C)^2} \tag{4.86}$$

The admittance too may be interpreted in terms of the role of an operator. It is that quantity which when multiplied by the voltage phasor yields the current phasor. In general, *Y* is a complex number which often is denoted as

$$\underline{Y} = G - jB \tag{4.87}$$

where *G* is the real part of the admittance and is called the *conductance*, and *B* is the quadrature component and is called *susceptance*.
Then we can write that for the parallel *RLC* case

$$G=\frac{1}{R} \text{ and } B=\frac{1}{X}, \tag{4.88}$$

where $B=B_L-B_C=1/(\omega L)-\omega C$ and $X=X_L-X_C=\omega L-1/\omega C$.

Therefore the same general result applies for the parallel arrangement of *R*, *L* and *C* as it does for the series combination: the resultant current lags behind the voltage.

In situations involving the sinusoidal steady-state analysis of circuits it is often helpful to replace a series combination of resistance and inductance by a corresponding parallel equivalent circuit which is expressed in terms of conductance and susceptance. We consider this subject matter for two reasons.

First, it is a useful technique to employ when three or more complex impedances appear in parallel because it is easier to add admittances than to deal with the reciprocal of impedances.

Second, it is important to understand the distinction between the equivalent admittance (i.e., parallel equivalence) of a series *RL* circuit and the admittance of an actual arrangement of *R*, *L* and *C*. The admittance of the parallel *R, L* and *C,* of course, is represented by Eqs. (4.87) and (4.88).

The active and reactive conductivities of a circuit are connected with admittance by the following relationships

$$G=Y\cos\varphi$$
$$B=Y\sin\varphi$$

Since the impedance of a series *RLC* circuit is $\underline{Z}=R+jX$, the corresponding equivalent admittance is expressed as the reciprocal of this quantity. Thus

$$\underline{Y}=\frac{1}{\underline{Z}}=\frac{1}{R+jX}$$

To express this in terms of an equivalent conductance and susceptance this equation is rationalized as follows:

$$\underline{Y}=\frac{1}{R+jX}\cdot\frac{R-jX}{R-jX}=\frac{R}{R^2+X^2}-j\frac{X}{R^2+X^2}$$

Recalling that $Y=G-jB$, we may rewrite the last equation as

$$G - jB = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

Equating the real and imaginary parts then yields

$$G = \frac{R}{R^2 + X^2} \text{ and } B = \frac{X}{R^2 + X^2} \qquad (4.89)$$

where G is the conductance and B is the susceptance of the equivalent parallel combination of the series RLC configuration.

On the other side, the corresponding equivalent impedance may be expressed as

$$\underline{Z} = \frac{1}{\underline{Y}} = \frac{1}{G - jB}$$

Express this in terms of an equivalent resistance and reactance as follows:

$$\underline{Z} = \frac{1}{G - jB} \cdot \frac{G + jB}{G + jB} = \frac{G}{G^2 + B^2} + j\frac{G}{G^2 + B^2}$$

Equating the real and imaginary parts then yields

$$R = \frac{G}{G^2 + B^2} \text{ and } X = \frac{G}{G^2 + B^2}. \qquad (4.90)$$

As it follows from the formula (4.86) the admittance, the conductance and the susceptance may be depicted by a rectangular triangle, similar to the triangle of the vectors of currents in Fig. (4.21, *a*, *b*).

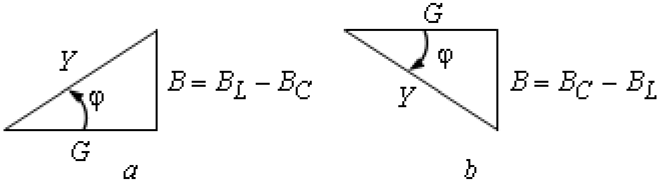


Fig. 4.22. Admittance triangles

The admittance triangle graphically shows the relationship between the magnitude of the admittance of a circuit, Y, and its in-phase and quadrature components.

4.13. A Current Resonance

Let's consider the circuit (see Fig. 4.23, *a*) where the conditions may be obtained under which current resonance (also known as parallel resonance) will take place.

Let one branch contain a resistance R_1 and an inductive reactance ωL, and the other branch, a resistance R_2 and a capacitive reactance $1/\omega C$. The complex current $\underline{I}_1$ in the first branch lags behind the complex voltage $\underline{U}$ and may be written thus

$$\underline{I}_1 = \underline{U} \cdot Y_1 = \underline{U} \cdot (G_1 - jB_1)$$

The complex current $\underline{I}_2$ in the second branch leads the voltage

$$\underline{I}_2 = \underline{U} \cdot Y_2 = \underline{U} \cdot (G_2 + jB_2)$$

The total current in the circuit

$$\underline{I} = \underline{I}_1 + \underline{I}_2 = \underline{U} \cdot \left[(G_1 + G_2) - j(B_1 - B_2) \right] = \underline{U}\,\underline{Y}$$

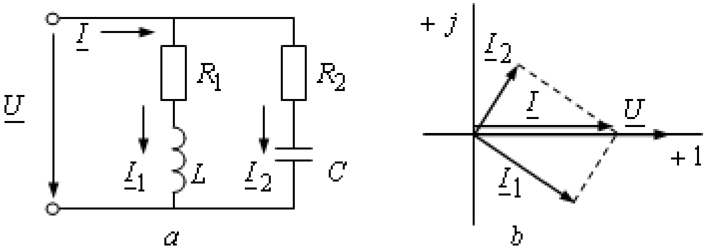


Fig.4.23. The circuit with parallel connection of elements

By definition, at resonance a current is in phase with a voltage. This can occur if inductive and capacitive susceptances are equal

$$B_L = B_C \tag{4.91}$$

or $B = B_L - B_C = 0$.

As we have active resistances in branches, there are the following equations for active conductances

$$G_1 = \frac{R_1}{Z_1^2} = \frac{R_1}{R_1^2 + \omega^2 L^2}; \quad G_2 = \frac{R_2}{Z_2^2} = \frac{R_2}{R_2^2 + \dfrac{1}{\omega^2 C^2}}; \tag{4.92}$$

and for reactive susceptances

$$B_L = \frac{X_L}{Z_1^2} = \frac{\omega L}{R_1^2 + \omega^2 L^2}; \quad B_C = \frac{X_{C2}}{Z_2^2} = \frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}} \tag{4.93}$$

From the resonance condition

$$\frac{\omega L}{R_1^2 + \omega^2 L^2} = \frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}} \tag{4.94}$$

Having solved Eq. (4.90), we will get the expression for a resonance angular frequency:

$$\omega_o = \frac{1}{\sqrt{LC}} \sqrt{\frac{\frac{L}{C} - R_1^2}{\frac{L}{C} - R_2^2}} = \omega_o \sqrt{\frac{\rho^2 - R_1^2}{\rho^2 - R_2^2}} \tag{4.95}$$

The analysis of Eqs (4.94) and (4.95) allows to note a number of features of the resonance in a parallel contour.

The first feature is that resonance can be obtained by varying angular frequency, inductance, and capacitance, or active resistances R_1 and R_2.In radio engineering, this is termed "tuning the circuit to resonance", and the circuits thus behaving are termed *resonant* or *tuned*. Although there is a circulating current round the closed *LC* circuit, the input current *I* becomes negligible in comparison with the currents in branches. In an idealized case, when $R_1 = R_2 = 0$, the circuit current vanishes altogether, and the input complex impedance of the network becomes infinity. The second feature is that the resonance is possible, if both resistances are more or less than a wave resistance (see Sec. 4.11). If this condition is not true, the resonance frequencies do not exist (as a radicand in Eq. (4.95) is negative in this case).

The vector diagram for a parallel resonance is shown in Fig. 4.24. Active and reactive current components are used at its construction

$$I_{a1} = U\,G_1; \quad I_{p1} = U\,B_L;$$

$$I_{a2} = U\,G_2; \quad I_{p2} = U\,B_C.$$

The reactive currents $\underline{I}_1$ and $\underline{I}_2$ are equal, and their vectors are in phase opposition (a phase shift between them is equal π) and compensate each other, the resonance in a parallel circuit is named a current resonance.

As a result the entering current is the sum of active components of branch currents. From a vector diagram one can see, that at a resonance the input current can be much less than branch currents.

This property allows using a parallel contour as a current amplifier.

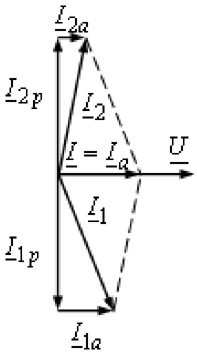


Fig. 4.24

The resistance of a parallel contour at a resonance

$$Z_o = R_o = \frac{R_1 R_2 + \rho^2}{R_1 + R_2}$$

The input current at a resonance

$$I_o = \frac{U}{R_0}$$

In an ideal parallel contour (see Fig. 4.25, *a*) any resistances are absent , i.e. $R_1 = R_2 = 0$. In this case $I_{a1} = I_{a2} = 0$. Therefore, at a resonance $I_1 = I_2 = I_{p1} = I_{p2}$, and a total (input) current $\underline{I} = \underline{I}_1 + \underline{I}_2 = \underline{I}_{p1} - \underline{I}_{p2} = 0$, what we can see in the vector diagram in Fig. 4.25, *b*.

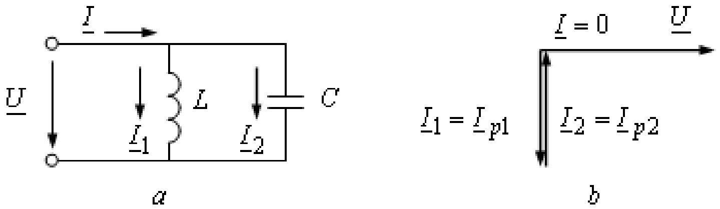


Fig. 4.25. Ideal parallel contour (*a*), and the vector diagram for it (*b*)

The current amplification in this case seeks to infinity.

Let's observe the frequency characteristics of branch susceptances and the input susceptance for an ideal parallel contour (Fig. 4.26, *a*), and also a resonance characteristic of an input current (Fig. 4.26, *b*). They are expressed by following formulas

$$B_L(\omega) = \frac{1}{\omega L}; \qquad B_C(\omega) = \omega C;$$

$$B(\omega) = \frac{1}{\omega L} - \omega C; \qquad I(\omega) = \left|\frac{1}{\omega L} - \omega C\right| \cdot U.$$

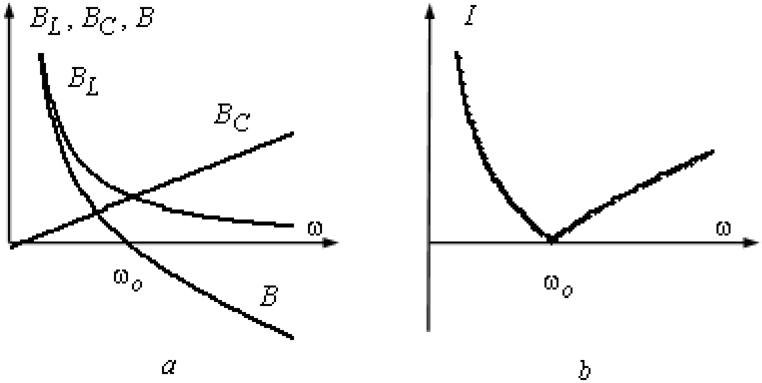


Fig. 4.26. Frequency characteristics for ideal parallel contour (*a*), and a resonance characteristic of an input current (*b*)

The current resonance in contrast to a series resonance is the safety condition for electropower installations. As currents in branches are mutually independent, their values are defined by input voltage and impedances of branches (according to Ohm's law).

4.14. Kirchhoff's Laws in Complex Notation

By Kirchhoff's first law, the algebraic sum of the instantaneous currents at any load of a network is zero, or $\sum i_k = 0$.

Substituting $\underline{I}_k e^{j\omega t}$ for i_k in the previous equation and putting $e^{j\omega t}$ before the brackets, we have

$$e^{j\omega t} \sum \underline{I}_k = 0$$

Since $e^{j\omega t} \neq 0$ for any *t*, it follows that

$$\sum \underline{I}_k = 0 \tag{4.96}$$

Equation (4.96) is Kirchhoff's current law (KCL) in complex form.

Consider Kirchhoff's voltage law in complex form.

Kirchhoff's voltage law gives the instantaneous values of currents, voltages and e.m.f.s for any closed loop or mesh in any electric network carrying an alternating sinusoidal current. Let a mesh has n branches and each (k-th) branch in the general case contains an electromotive force (e_k), a resistance (R_k), an inductance (L_k), and a capacitance (C_k), with a current i_k flowing through all of them. Then by Kirchhoff's voltage law

$$\sum_{k=1}^{n}\left(i_k R_k + L_k \frac{di_k}{dt} + \frac{1}{C_k}\int i_k dt\right) = \sum_{k=1}^{n} e_k$$

In accordance with Sec. 4.10, however, each term on the left-hand side may be replaced by $\underline{I}_k \underline{Z}_k$, and each term on the right-hand side may be replaced by $\underline{E}_k$. Then Kirchhoff's second law for the network with n branches may be written in complex form thus

$$\sum_{k=1}^{n} \underline{I}_k \underline{Z}_k = \sum_{k=1}^{n} \underline{E}_k \qquad (4.97)$$

Since Kirchhoff's current and voltage laws hold for a sinusoidal current as well, one might write down the equations for the values in complex form, solve them by Kirchhoff's laws and check up the result of calculation with the help of a vector diagram.

4.15. A Vector Diagram and its Application to Sinusoidal Circuit Problems

A vector diagram is a graphic representation of the vectors of sinusoidal quantities in a complex plane. The quantities are taken and in proper phase relationship with respect to one another.

As a rule, the currents and voltages of the various portions of a sinusoidal circuit are not in phase. Their phase relationships can be visualized by use of current and voltage vector diagrams. This is why it will be good practice to follow up analytical network calculations with vector diagrams as a qualitative check on the analytical results. The qualitative check provided by vector diagram consists in that the various vectors in the complex plane yielded by analytical calculations can be compared for their directions with those obtained graphically from purely physical considerations.

Thus, in a vector diagram the complex voltage across inductance should lead the current through it by 90^{o}, while the complex voltage across a capacitance should lag behind the current through it by 90^{o}. Should analytical calculations yield results which disagree with such obvious observations, there is a mistake in the calculation, which should be traced down and eliminated. Also vector diagrams offer a convenient technique in network analysis and synthesis by the proportional method. Let us consider some example.

Example 4.1. In the network of Fig. 2.6 the complex effective value of e.m.f. is $\dot{E} = 100V,$ and the characteristic parameters are $L = 0.0955H,\ \ R = 30\Omega,\ \ C = 53.08 \cdot 10^{-6}F.$ Find the current in and the voltages across the circuit elements.

Solution: At first we calculate inductive and capacitive reactances. Thus

$$X_L = 2\pi fL = 314 \times 0.0955 = 30\ \Omega,$$

$$X_c = \frac{1}{2\pi fC} = \frac{1}{314 \times 53.08 \cdot 10^{-6}} = 60\ \Omega..$$

Now one can determine the complex impedance of the circuit as

$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + j\,(X_L - X_C) =$$

$$= 30 + j\,(30 - 60) = 30 - j30 = 42.43\ e^{-j45^{o}}\ \ \Omega$$

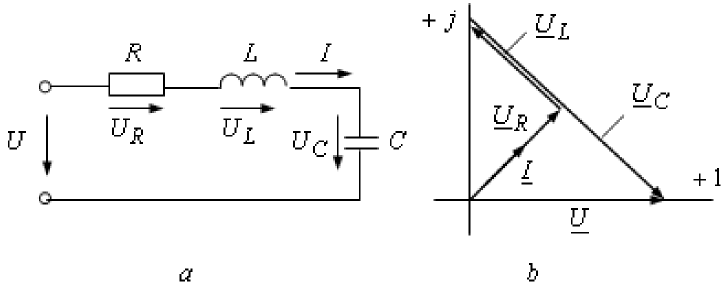


Fig. 4.27

We write down the equation by Kirchhoff's voltage law in complex form as $\underline{I}\,\underline{Z} = \underline{U}$.

Thus complex current in the loop

$$\dot{I} = \frac{\dot{E}}{Z} = \frac{100}{42.43e^{-j45^o}} = 2.36\ e^{j45^o} A\,.$$

The voltage across resistance

$$\dot{U}_R = R\dot{I} = 30 \cdot 2.36e^{j45^o} = 70.8e^{j45^o} V\ .$$

The voltage across inductance

$$\dot{U}_L = jX_L\dot{I} = 30e^{j90^o} \cdot 2.36e^{j45^o} = 70.8e^{j135^o} V.$$

The voltage across capacitance

$$\dot{U}_c = -jX_c\dot{I} = 60e^{-j90^o} \cdot 2.36e^{j45^o} = 141.6e^{-j45^o} V.$$

The vector diagram for this case is shown in Fig. 4.28, *b*. The vector of the input voltage $\underline{U}$ is placed along a positive real axis. The current leads the input voltage by 45$^{\text{o}}$ active - capacitive character of the load in this network.

4.16. Instantaneous and Average Power. Power Factor

Our interest in this section is to develop a general expression for the average power associated with a voltage and current in an a.c. circuit. The restrictive limitation of confining the treatment to resistive circuits is now dropped. In this connection then let $u(t) = U_m \sin \omega t$ represent the potential difference appearing across the branch terminals of a given circuit and let $i(t) = I_m \sin(\omega t - \varphi)$ denote the corresponding current flowing through that branch. The relative phase angle is given by φ.

The voltage and current sinusoid are shown in Fig. 4.28. It follows then that the expression for the instantaneous power is

$$p = ui = U_m \sin \omega t I_m \sin(\omega t - \varphi) = U_m I_m \sin \omega t \sin(\omega t - \varphi) =$$
$$= \frac{U_m I_m}{2}(\cos\varphi - \cos(2\omega t - \varphi)) = UI\cos\varphi - UI\cos(2\omega t - \varphi) \qquad (4.98)$$

A plot of Eq. (4.98) appears in Fig. 4.28.

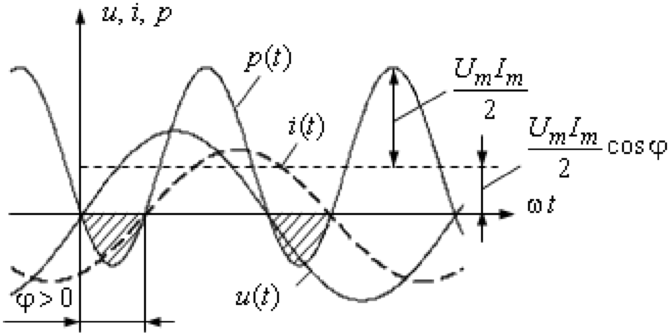


Fig. 4.28. Plot of instantaneous power as determine from the given voltage and current sinusoid

Note that for a fixed φ the instantaneous power consists of two components - a constant part and a time-varying part. Note, too, that the varying part has a frequency which is twice that of the voltage and current sinusoids. The shaded portions of the plot of $p(t)$ refer to those time intervals when the power is negative. In effect this means that the circuit is returning power to the source during these intervals. It should be apparent, then, that the branch circuit under consideration contains at least one energy-storing element.

A glance at Fig. 4.28 shows that the instantaneous power is negative whenever the voltage and current are of opposite sign. However, for the case plotted in Fig. 4.28 notice that the positive area under the $p(t)$ curve exceeds the negative area. Therefore, the average power is positive and finite, and specifically it is equal to the constant term of Eq. (4.94). As φ is made smaller, i.e., as i is brought more nearly in phase with u, the negative areas of the $p(t)$ curve of Fig. 4.28 become smaller and so the average power increases. This is equivalent to raising the $p(t)$ curve higher above the abscissa axis. When $\varphi = 0$ the current and voltage are in phase. There are no negative areas associated with the $p(t)$ curve; hence all the power is consumed between the circuit branch terminals. The circuit may then be called purely resistive. On the other hand, when φ is increased, the negative areas become larger and so less power is consumed between the terminals and more returned to the source. At the extreme value of φ, the $p(t)$ curve is dropped to that position which makes the negative and positive areas equal. In this instance there is no average power consumed between the circuit terminals.

The relative phase angle φ is determined by the values of the circuit parameters appearing between the circuit branch terminals across which $u(t)$ is assumed to exist. Because of the passive nature of these circuit parameters the value of φ is restricted to lie in the range expressed by $-\pi/2 \le \varphi \le \pi/2$.The general expression for the instantaneous power in an a.c circuit is described by Eq. (4.98). The really useful quantity in terms of the capability of the circuit to do work is the average value of the power over one cycle.

Since each cycle is continuously repeated, whatever is done for one cycle applies equally as well for each succeeding cycle. It has already been stated in connection with the discussion of Fig. 4.28 that this average power is given by the constant term of Eq. (4.98). A mathematical verification now follows. From the general definition of the average value over one cycle, we have

$$P = \frac{1}{T}\int_0^T uidt \tag{4.99}$$

Inserting Eq. (4.98) for $p(t)$ yields

$$P = \frac{1}{T}\left[\int_0^T UI\cos\varphi\, dt - \int_0^T UI\cos(2\omega t - \varphi)dt\right]$$

Since the second term on the right side involves the integration of a simple sine function over a time interval equal to two complete periods of the double frequency sine function, the value is always identically equal to zero. This leaves just the first term and, since φ is independent of t, it follows that the average power called *active* or *true* is

$$P = UI\cos\ \varphi \tag{4.100}$$

Although it is customary to drop the subscripts entirely when writing this equation, one of our reasons for leaving them here is to emphasize better that average power is determined in terms of the effective voltage and current values. Equation (4.96) points out in a general and significant fashion the usefulness of the root mean square value of a periodic function as a criterion to measure its effectiveness. The effective values of voltage and current play key roles in measuring the ability of a circuit to do work.

In the interest of introducing another term of electric-circuit theory rewrite Eq. (4.96) as follows

$$\cos\varphi = \frac{P}{UI} \tag{4.101}$$

The quantity P is the average power and is expressed in *watts* - a unit which conveys the capability to do work. However, note that the denominator of Eq. (4.101) involves a quantity whose units are represented by the product of volts by amperes. When we are dealing with direct sources this product is called watts, because it is real power which can be entirely converted to work. The same is not true when sinusoidal quantities are involved. For example, in Fig. 4.28, corresponding to $\varphi = \pi/2$ there is no useful (or work-producing) power in the circuit in spite of the large values which voltage U and current I may have. For this reason the product UI is called *apparent power*. This power is not always realizable in the circuit for doing work. The useful part depends upon the value of $\cos\varphi$, and because of this, $\cos\varphi$, is called the *power factor* of the circuit.

The closer angle φ to zero, the closer power factor cos φ to unity, and, hence, the more active-power is transmitted from a source to the two-terminal network.

Physically, the active power is the energy dissipated in unit time (assuming that a whole number of period T occurs within one second) as heat in a portion of a circuit of resistance R. Since $U\cos\varphi = IR$, it follows that

$$P = I^2R \tag{4.102}$$

The active-power can be determined by other formulas:

$$P = I^2Z\cos\varphi = U^2Y\cos\varphi = I^2R = U^2G\,. \tag{4.103}$$

The active power is measured in *watts*.

We enquire about the conditions of maximum energy transfer to the load. When the internal resistance of the active two-terminal network is equal to the load resistance, that is $R_L = R_i$, the power transferred to the load will be a maximum and given by

$$P = \frac{U^2_{ab_{o.c.}}}{4R_i} \tag{4.104}$$

where $U_{ab_{o.c.}}$ is the open-circuit voltage appearing across the terminals *ab* ((see Sec. 3.14).

If the circuit contains reactive (energy-storing) components, energy will circulate to-and-from between circuit and supply.

The expression for the instantaneous inductive power is

$$p_L = ui = \frac{U_m I_m}{2}\left[\cos\frac{\pi}{2} - \cos\left(2\omega t + \frac{\pi}{2}\right)\right] =$$

$$= UI \sin 2\omega t = I^2 \omega L \sin 2\omega t$$

This power p_L can also be expressed in terms of the effective values of current and voltage as well as the relative phase angle. The peak of circulating power is given by

$$Q = U\, I \, \sin\varphi \qquad (4.105)$$

Since $U \sin\varphi = IX$, it follows that

$$Q = I^2 X \qquad (4.106)$$

This equation may be rewritten for inductive reactance as

$$Q = I^2 X_L \qquad (4.107)$$

and for capacitive reactance

$$Q = I^2 (-X_C) \qquad (4.108)$$

In periods when input voltage and current have the same signs, i.e. they direct to one side, instantaneous power is positive; and energy arrives from the energy source to a two-terminal network. In the two-terminal network the energy dissipates in the form of heat energy in resistive elements and is reserved in the form of a magnetic field energy in inductive elements, and an electrical field energy in the capacitive elements. When voltage and current have different signs, instantaneous power is negative; the energy partially returns from the two-terminal network to the supply. It takes place at the expense of transformation of the energy which is reserved in magnetic and electric fields accordingly of inductive and capacitive elements

Equation (4.104) reveals that the product of the current and voltage of an inductor is a sinusoid having double the frequency of either the current or voltage sinusoids. It is important here to note that, unlike the case for resistance, the average value of this power over one cycle yields a result which is identically equal to zero. This means that over part of the cycle energy is delivered to the inductance where it is stored in the magnetic field, but in the next half-cycle it is returned to the source. The net transfer of energy in a pure inductance is thereby zero.

A comparison of the amplitude of the double-frequency sinusoid of Eq. (4.104) with Eq. (4.106) reveals the meaning of the latter quantity: it denotes the amplitude of the energy which is interchanged between the source and the energy-storing inductive element. Although the units of Eq. (4.106) are volt-amperes, the units of the power are not described in terms of watts. To distinguish this power from active one, it is called *reactive power* expressed in units of *reactive volt-amperes* (abbreviated *var*).

In general, the reactive power Q is proportional to the average energy over a quarter-cycle, delivered by the supply to the circuit to build up the variable component of the electric and magnetic fields of the inductance and capacitance.

Over a cycle of an alternating current, the energy is delivered twice by the supply to the circuit and is returned twice by the circuit to the supply.

If the r.m.s. (effective) voltage and the r.m.s. (effective) current are measured separately in an a.c. circuit, their product (UI) does not give the true power, except when $\cos\varphi = 0$ (see Eq. 4.100). In the general case, when $\varphi \neq 0$, and the circuit contains capacitance or inductance, or both, the product UI gives the *apparent* (*total*) power

$$S = UI \tag{4.109}$$

The total power makes no physical sense, but it can be defined as the maximum possible active power at given values of input current and voltage, i.e. an active power at $\cos \varphi = 1$.

The total or apparent power is measured in *volt-amperes*.

The true, reactive and apparent powers are related thus

$$P^2 + Q^2 = S^2 . \tag{4.110}$$

It may be graphically presented as a power triangle (Fig. 4.29).

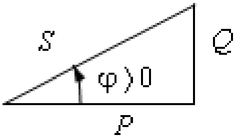


Fig. 4.29. A power triangle

The nameplate of any energy source bears the value of *S*, that is the energy that the source can deliver to the load, if the latter operates at $\cos\varphi = 1$, that is, if the circuit is a purely resistive one.

4.17. Power in Complex Form

If the complex voltage $U e^{j\psi_u}$ across a portion of a circuit and the corresponding complex current $Ie^{j\psi_i}$ are known, one can determine all kinds of powers for separate elements or for a whole electric circuit.

It can be shown that the product of these two values, as obtained by the ordinary rules for vector multiplication, would not give the apparent power, nor would the respective terms represent the true or the reactive power. The correct result is obtained if the complex voltage is multiplied by the conjugate of the complex current:

$$\begin{aligned} &\underline{S} = \tilde{S} = \underline{U}\,\underline{I}^* = U e^{j\psi_u}\left(Ie^{j\psi_i}\right)^* = U e^{j\psi_u} I e^{-j\psi_i} = \\ &= UI e^{j(\psi_u - \psi_i)} = UI e^{j\varphi} = UI\cos\varphi + jUI\sin\varphi = \\ &= S\cos\varphi + jS\sin\varphi = P + jQ, \end{aligned} \tag{4.111}$$

where the first term is *P* and the second term is *Q*, and the product is $\tilde{S}$, termed the vector power of an a.c. circuit. The "~" sign over *S* denotes that the product $\underline{U}\,\underline{I}^*$ gives the vector power and not its conjugate.

To determine a complex power of some passive element it is more convenient to take advantage of other equation

$$\underline{S} = \underline{Z}I^2 = (R + jX)I^2 = P + jQ \tag{4.112}$$

From Eq. (4.111) it follows that the active power is the real part, and the reactive power is the imaginary part of the product $\underline{U}\,\underline{I}^*$:

$$P = \mathrm{Re}\,\underline{U}\,\underline{I}^*;\;\; Q = \mathrm{Im}\,\underline{U}\,\underline{I}^* \tag{4.113}$$

Example 4.2. Consider sinusoidal alternating current circuit (Fig. 4.30). The instantaneous input voltage is u = 141 sin ωt where ω = 314 s^{-1}.The circuit resistances $R_1 = 10\ \Omega$, $R_2 = 40\ \Omega$, $R_3 = 20\ \Omega$; inductances and capacitances of reactive elements $L_1 = 15{,}9\ mH$, $L_2 = 95{,}5\ mH$, $C_1 = 212\ \mu F$, $C_3 = 79{,}6\ \mu F$.

One must find the branch currents, the voltages across the sections of the circuit, and write them in instantaneous forms.

Determine the powers of the supply and the load.

Draw a vector diagram of currents and voltages.

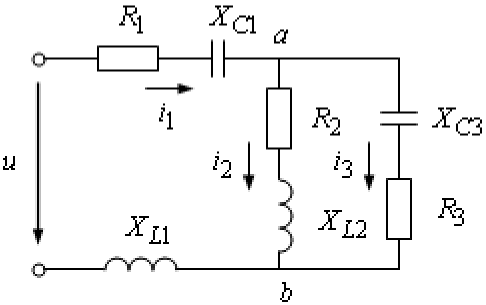


Fig. 4.30

Determine reactive susceptances of branches:

$$X_{L1} = \omega L_1 = 314 \cdot 15.9 \cdot 10^{-3} = 5\ \Omega;$$

$$X_{L2} = \omega L_2 = 314 \cdot 95{,}5 \cdot 10^{-3} = 30\ \Omega;$$

$$X_{C1} = \frac{1}{\omega C_1} = \frac{1}{314 \cdot 212 \cdot 10^{-6}} = 15\ \Omega;$$

$$X_{C3} = \frac{1}{\omega C_3} = \frac{1}{314 \cdot 79{,}6 \cdot 10^{-6}} = 40\ \Omega.$$

Use a symbolic method for calculation..

All complex values must be written both in algebraic and in exponential form.

The complex effective input voltage

$$\underline{U} = \frac{U_m}{\sqrt{2}} e^{j\psi_e} = \frac{141}{\sqrt{2}} e^{j0^o} = 100 e^{j0^o} = 100 + j0\ V.$$

The complex impedances of branches:

$$\underline{Z}_1 = R_1 + j(X_{L1} - X_{C1}) = 10 + j(5-15) = 10 - j10 = 14{,}1e^{-j45^o}\ \Omega;$$

$$\underline{Z}_2 = R_2 + jX_{L2} = 40 + j30 = 50{,}0e^{j37^o}\ \Omega;$$

$$\underline{Z}_3 = R_3 - jX_{C3} = 20 - j40 = 44{,}7e^{-j63^o}\ \Omega.$$

Now we pass from the initial circuit to the circuit with complex impedances (Fig. 4.31).

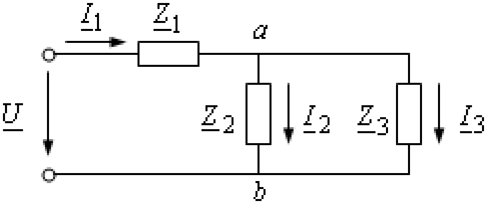


Fig. 4.31

For calculation of the circuit we use a method of equivalent transformations. The section *ab* with a parallel connection of branch impedances $\underline{Z}_2$ and $\underline{Z}_3$ is changed by an equivalent complex impedance

$$\underline{Z}_{ab} = \frac{\underline{Z}_2\underline{Z}_3}{\underline{Z}_2 + \underline{Z}_3} = \frac{(40 + j30)(20 - j40)}{40 + j30 + 20 - j40} =$$

$$= \frac{(40 + j30)(20 - j40)}{60 - j10} = 35{,}1 - j10{,}8 = 36{,}8e^{-j17^o}\ \Omega.$$

Then the equivalent impedance of the circuit

$$\underline{Z}_o = \underline{Z}_1 + \underline{Z}_{ab} = 10 - j10 + 35{,}1 - j10{,}8 =$$

$$= 45{,}1 - j20{,}8 = 49{,}7e^{-j25^o}\ \Omega.$$

The current in a non-ramified part of the electric circuit is found by Ohm's law:

$$\underline{I}_1 = \frac{\underline{E}_1}{\underline{Z}_o} = \frac{100 + j0}{45{,}1 - j20{,}8} = 1{,}83 + j0{,}84 = 2{,}01e^{j25^o}\ A.$$

The voltages across circuit sections:

$$\underline{U}_1 = \underline{I}_1 \underline{Z}_1 = (1{,}83 + j0{,}84)(10 - j10) = 26{,}7 - j9{,}9 = 28{,}5e^{-j20^o} \quad V;$$

$$\underline{U}_{ab} = \underline{I}_1 \underline{Z}_{ab} = (1{,}83 + j0{,}84)(35{,}1 - j10{,}8) = 73{,}3 + j9{,}7 = 73{,}9e^{j8^o} \; V.$$

The currents through the parallel branches

$$\underline{I}_2 = \frac{\underline{U}_{ab}}{\underline{Z}_2} = \frac{73{,}3 + j9{,}7}{40 + j30} = 1{,}29 - j0{,}72 = 1{,}48e^{-j29^o} \quad A;$$

$$\underline{I}_3 = \frac{\underline{U}_{ab}}{\underline{Z}_3} = \frac{73{,}3 + j9{,}7}{20 - j40} = 0{,}54 + j1{,}56 = 1{,}65e^{j71^o} \quad A.$$

The solution can be check up using Kirchhoff's voltage and current-laws

$$\underline{I}_1 = \underline{I}_2 + \underline{I}_3 = 1{,}29 - j0{,}72 + 0{,}54 + j1{,}56 = 1{,}83 + j0{,}84 = 2{,}01e^{j25^o} \quad A;$$

$$\underline{E} = \underline{U}_1 + \underline{U}_{ab} = 26{,}7 - j9{,}9 + 73{,}3 + j9{,}7 = 100{,}0 - j0{,}2 = 100{,}2e^{j0^o} \;\; V.$$

The relative error does not exceed one per cent.

Let's pass from complex currents and voltages to their instantaneous values:

$$u_1 = \sqrt{2}U_1 \sin(\omega t + \psi_{U_1}) = \sqrt{2} \cdot 28{,}5 \sin(314t - 20^0) =$$

$$= 40{,}3 \sin(314t - 20^o) \; V;$$

$$u_{ab} = \sqrt{2}U_{ab} \sin(\omega t + \psi_{U_{ab}}) = \sqrt{2} \cdot 73{,}9 \sin(314t + 8^0) =$$

$$= 104{,}5 \sin(314t + 8^o) \; V;$$

$$i_1 = \sqrt{2}I_1 \sin(\omega t + \psi_{I_1}) = \sqrt{2} \cdot 2{,}01 \sin(314t + 25^0) =$$

$$= 2{,}84 \sin(314t + 25^o) \;\; A;$$

$$i_2 = \sqrt{2}I_2 \sin(\omega t + \psi_{I_2}) = \sqrt{2} \cdot 1{,}48 \sin(314t - 29^0) =$$

$$= 2{,}09 \sin(314t - 29^o) \;\; A;$$

$$i_3 = \sqrt{2}I_3 \sin(\omega t + \psi_{I3}) = \sqrt{2} \cdot 1{,}65 \sin(314t + 71^0) =$$

$$= 2{,}33 \sin(314t + 71^o) \;\; A.$$

Draw a combined vector diagram of branch currents and voltages (in corresponding scale)

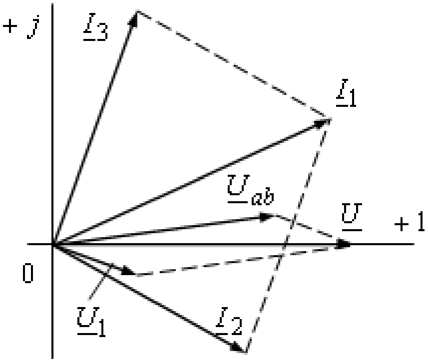


Fig. 4.32

The complexes of apparent powers of the energy source and the branches of this electric circuit:

$$\underline{S}_E = \underline{E} \cdot \overset{*}{\underline{I}_1} = 100 \cdot (1{,}83 - j0{,}84) = 183 - j84 = 201 e^{-j25^o} \ VA;$$

$$\underline{S}_1 = I_1^2 \underline{Z}_1 = 2{,}01^2 \cdot (10 - j10) = 40{,}4 - j40{,}4 = 57{,}1 e^{-j45^o} \ VA;$$

$$\underline{S}_2 = I_2^2 \underline{Z}_2 = 1{,}48^2 \cdot (40 + j30) = 87{,}6 + j65{,}7 = 109{,}5 e^{j37^o} \ VA;$$

$$\underline{S}_3 = I_3^2 \underline{Z}_3 = 1{,}65^2 \cdot (20 - j40) = 54{,}5 - j108{,}9 = 121{,}8 e^{-j63^o} \ VA.$$

Then we can write active powers for the branches of the electric circuit and for the supply:

$$P = 183 \ Wt; \ P_1 = 40{,}4 \ Wt; \ P_2 = 87{,}6 \ Wt; \ P_3 = 54{,}5 \ Wt.$$

The reactive powers for the branches of the electric circuit and for the supply:

$$Q = -84 \text{ var}; \qquad Q_1 = -40{,}4 \text{ var}$$

$$Q_2 = 65{,}7 \text{ var}; \qquad Q_3 = -108{,}9 \text{ var}.$$

Check up the balance of powers:

$$P = P_1 + P_2 + P_3 = 40.4 + 87.6 + 54.5 = 182.5 \approx 183 \ Wt$$

$$Q = Q_1 + Q_2 + Q_3 = -40.4 + 65.7 - 108.9 = -83.6 \approx -84 \text{ var}$$

4.18. Mutual Inductance in a Circuit

The coils of an electric circuit can place in space in such a way that the magnetic flux created by one coil partially links the other. The change in current in one coil is accompanied by a change in its magnetic field which induces an e.m.f. in the other.

Such an electromotive force is called an *e.m.f. of mutual inductance*. When two coils are mutually connected, they are also named *magnetically coupled coils*.

It was established experimentally that if coil cores are made of non-ferromagnetic materials or of ferromagnetic with a constant relative permeability, μ_r, the flux linkages are proportional to their currents. The coefficients of proportionality are symbolised by letter *M* with a suitable subscript.

Let two coils be in proximity to each other. The first carries a current i_1, and the second, a current i_2. Then we can write

$$M_{12} = \frac{\Psi_{12}}{i_2}; \qquad M_{21} = \frac{\Psi_{21}}{i_1}. \tag{4.114}$$

The coefficients M_{12} and M_{21} are numerically equal. The strict proof of this equation is possible only with the application of a electromagnetic field theory.

$$M_{12} = M_{21} = M$$

The coefficient *M* is termed the mutual inductance of circuits or coils and has the same dimensions as the inductance *L*.

Mutual inductance of circuit elements as well as inductance is measured in *Henry* (*H*). It is connected with the inductances of the coils by the following relationship

$$M = k\sqrt{L_1 L_2} \tag{4.115}$$

where *k* is a coefficient of inductive coupling. The coefficient of coupling *k* between two circuits of inductances L_1 and L_2, and of mutual inductance *M*, is defined as the ratio of *M* to the square root of the product of the inductances

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

It cannot be greater than unity $0 \le k < 1$. The coefficient of coupling will be equal to unity only if all of the flux produced by the first circuit links the second.

Travelling one coil concerning another one can change a coefficient of inductive coupling and mutual inductance of coupled coils.

When writing equations for a circuit with a mutual inductance one should take into account e.m.f.s of mutual inductance and know the relative directions of self and mutual inductance. These directions can be ascertained if one knows the direction in which the coils are wound on their cores and the positive direction of the current through them.

The coils in Fig. 4.33, *a* are so connected that their fluxes are aiding, and the coils in Fig. 4.33, *b* are connected so that their fluxes are bucking each other. The core symbols are crowd the diagrams. Instead, the convention is to mark the like terminals (stars of the coil) with an asterisk or a dot.

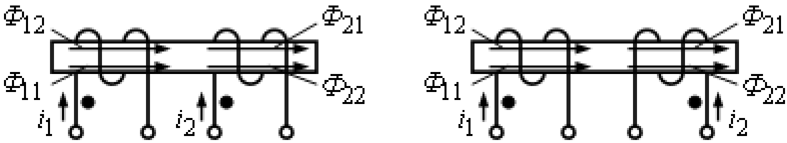


Fig. 4.33. Coupled coils

The dot convention is shown in the diagram of Fig. 4.34, *a* which is fully analogous with that of Fig. 4.33, *a*, and in the diagram of Fig. 4.34, *b* which is fully analogous with that of Fig. 4.33, *b*.

Fig. 4.34. Marking of coupled coils

If the currents in a mutual inductance are flowing towards (or away from) the like terminals (i.e., towards or away from the dot in the diagram), the corresponding coils are said to be connected *aiding*. If the current in one coil flows towards to the terminal marked with a dot, and that in the other coil flows away from the like terminal, the coils are said to be connected *bucking* or *in opposition*.

The circuits containing mutual inductances (or simply, *coupled circuits*) are analyzed by the method of complex numbers.

To define a position of similar terminals of two coils is possible on the basis of a simple experiment for which a direct current voltage source and an amperemeter of electro-magnetic system are necessary (see Fig. 4.35).

One of coils is joined to the amperemeter, another - to the voltage source. At closure of key Q it arrears a short-term current which relaxes the magnetic field created by the current.

Hence, at the moment of turning on of the power supply currents i_1 and i_2 direct to opposite sides concerning similar terminals.

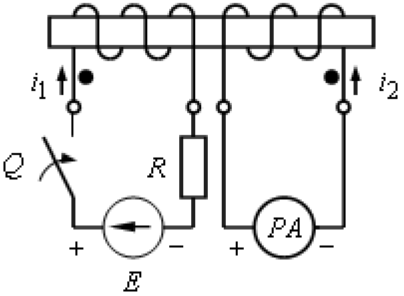


Fig. 4.35. The experiment circuit for defining a position of similar terminals

The direction of current i_1 is determined by the polarity of the voltage source. A short-term diversion of an arrow of the amperemeter defines the direction of the current.

If the arrow is declined towards a dial, the current i_2 directs to a positive terminal of the amperemeter. For this the terminals of coils which are joined to the terminals of the amperemeter and the voltage source are similar.

4.19. Mutual Inductance in Series

At aiding connection at any moment of time the currents in both coils of the circuit direct equally concerning the like terminals (Fig. 4.35), therefore, the fluxes of self-induction and mutual induction are summed up. So, it is a series aiding connection.

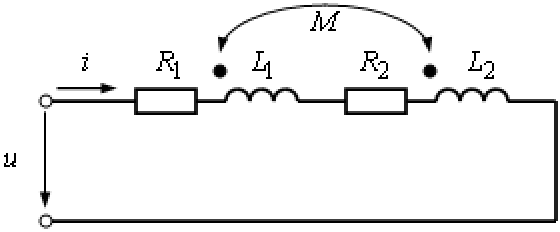


Fig. 4.35. The coils connected in series

When writing equations for a *coupled* circuit we assign positive directions to the currents in the branches of the circuit and take the direction for summation round them.

Let's work out the equation for this electric circuit by the second Kirchhoff's law for instantaneous values taking into account that the same current flows through both coils

$$R_1 i + L_1 \frac{di}{dt} + M \frac{di}{dt} + R_2 i + L_2 \frac{di}{dt} + M \frac{di}{dt} = u \;, \qquad (4.116)$$

or in complex notation

$$\underline{I}(R_1 + R_2 + j\omega(L_1 + L_2 + 2M)) = \underline{U} \;. \qquad (4.117)$$

The equivalent impedance of the circuit at aiding connection of two coils connected in series

$$\underline{Z}_{aid} = R_1 + R_2 + j\omega(L_1 + L_2 + 2M)\,. \qquad (4.118)$$

Fig. 4.37, *a*, shows a vector diagram for series aiding connection, where $\underline{U}_1 = \underline{I}(R_1 + j\omega L_1 + j\omega M)$ is the complex voltage across the first coil, and $\underline{U}_2 = \underline{I}(R_2 + j\omega L_2 + j\omega M)$ is the complex voltage across the second coil. Each of these complex voltages has three components.

At bucking connection at any moment of time the currents in both coils of the circuit direct equally concerning the different terminals

(Fig. 4.36), therefore, the fluxes of mutual induction are subtracted from the fluxes of self-induction. So, for a series opposition

$$R_1 i + L_1 \frac{di}{dt} - M \frac{di}{dt} + R_2 i + L_2 \frac{di}{dt} - M \frac{di}{dt} = u \tag{4.119}$$

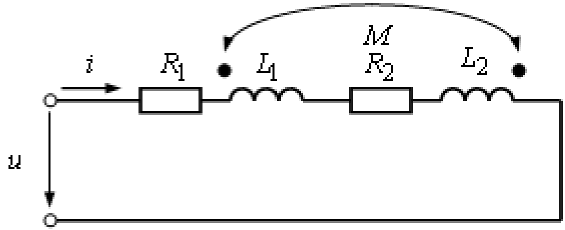


Fig. 4.36. A series opposition

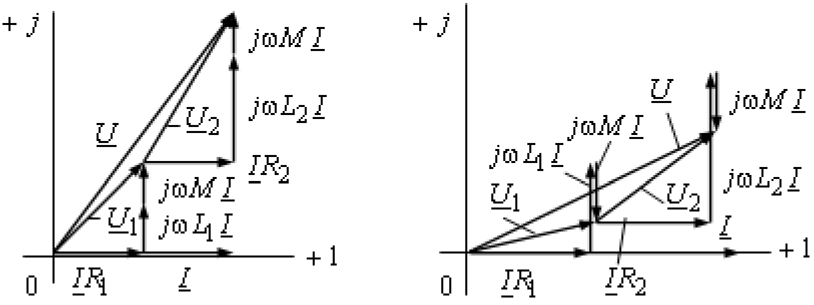
In complex notation

$$\underline{I}(R_1 + R_2 + j\omega(L_1 + L_2 - 2M)) = \underline{U} \tag{4.120}$$

where the equivalent impedance of the circuit at series opposition

$$\underline{Z}_{op} = R_1 + R_2 + j\omega(L_1 + L_2 - 2M) \tag{4.121}$$

Fig. 4.37, *b*, shows a vector diagram for series opposition.



a *b*

Fig. 4.37. A plot of vector diagram

4.20. Mutual Inductance in Parallel

Let's consider the electric circuit with parallel coupled coils (see Fig. 4.38).

Let's work out a system of equations by Kirchhoff's laws for instantaneous values of currents and voltages taking into account the chosen directions of currents

$$\begin{cases} i = i_1 + i_2 \\ e = R_1 i_1 + L_1 \dfrac{di_1}{dt} \pm M \dfrac{di_2}{dt} \\ e = R_2 i_2 + L_2 \dfrac{di_2}{dt} \pm M \dfrac{di_1}{dt} \end{cases} \tag{4.122}$$

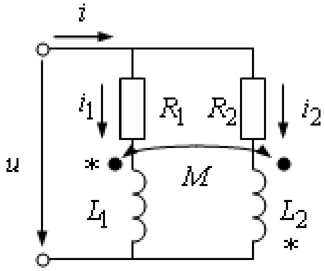


Fig.4.37

In Eq. (4.122) and other equations in this subsection the upper mark corresponds to aiding connection (the similar terminals are marked by dots • in Fig. 4.37), and the lower mark corresponds to the bucking connection (the similar terminals are marked by asterisk ∗ in Fig. 4.37).

Rewrite the system of equations in complex notation

$$\begin{cases} \underline{I} = \underline{I}_1 + \underline{I}_2; \\ \underline{E} = (R_1 + j\omega L_1)\underline{I}_1 \pm j\omega M \underline{I}_2; \\ \underline{E} = (R_2 + j\omega L_2)\underline{I}_2 \pm j\omega M \underline{I}_1. \end{cases} \tag{4.123}$$

On solving the system of equations concerning unknown currents, we will find

$$\begin{aligned} \underline{I}_1 &= \frac{\underline{Z}_2 \mp \underline{Z}_M}{\underline{Z}_1 \underline{Z}_2 - \underline{Z}_M^2} \underline{E}; \\ \underline{I}_2 &= \frac{\underline{Z}_1 \mp \underline{Z}_M}{\underline{Z}_1 \underline{Z}_2 - \underline{Z}_M^2} \underline{E}; \\ \underline{I} &= \frac{\underline{Z}_1 + \underline{Z}_2 \mp 2\underline{Z}_M}{\underline{Z}_1 \underline{Z}_2 - \underline{Z}_M^2} \underline{E}, \end{aligned} \tag{4.124}$$

where $\underline{Z}_1 = R_1 + j\omega L_1$; $\underline{Z}_2 = R_2 + j\omega L_2$ are complex impedances of branches, and $\underline{Z}_M = jX_M = j\omega M$ is the complex impedances of mutual inductance.

The input complex impedance of considered circuit

$$\underline{Z} = \frac{\underline{E}}{\underline{I}} = \frac{\underline{Z}_1 \underline{Z}_2 - \underline{Z}_M^2}{\underline{Z}_1 + \underline{Z}_2 \mp 2\underline{Z}_M}. \tag{4.125}$$

4.21. Experimental Determination of Mutual Inductance

The difference of inductive reactance at aiding and bucking connections allows measuring their mutual inductance.

The first method. We will make two experiments. In one of them we connect two coils in series aiding and measure the current and voltage at the input and the active power in the electric circuit. In the other, we connect two coils in series bucking and also measure the current, voltage and the active power in the circuit. If both experiments are spent at the same input voltage the smaller current will correspond to aiding connection. It allows marking the similar terminals of the coils.

We can find the reactances of the coils by the results of measurements

$$X_{aid} = \omega(L_1 + L_2 + 2M) = \frac{U}{I_{aid}} \sin\arccos\left(\frac{P_{aid}}{UI_{aid}}\right);$$

$$X_{op} = \omega(L_1 + L_2 - 2M) = \frac{U}{I_{op}} \sin\arccos\left(\frac{P_{op}}{UI_{op}}\right).$$

The difference of these two reactances $X_{aid} - X_{op} = 4\omega M$ allows finding the mutual inductance

$$M = \frac{X_{aid} - X_{op}}{4\omega}. \tag{4.126}$$

The second method. We will connect the first coil to a source of sinusoidal electromotive force through an amperemeter (see Fig. 4.38), and the other coil is connected across a high-impedance voltmeter. Then we measure the current I_1 and the voltage U_2

The effective value of voltage across the second coil

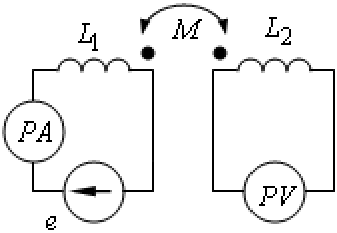


Fig. 4.38

$$U_2 = \omega M I_1$$

Hence, the mutual inductance is found as

$$M = \frac{U_2}{\omega I_1} \tag{4.127}$$

4.22. The calculation of complex electric circuits with mutual inductance

One can calculate complex electric circuits with mutual inductance by working out the equations by Kirchhoff's current and voltage laws as they are true for any circuits. In the equations by the second Kirchhoff's law one must add the voltage of mutual induction.

For calculation of such circuits we can apply mesh-current method as it is based by Kirchhoff's voltage law which takes into account the e.m.f. of a mutual induction. In the equations by mesh-current method it is necessary to remember that in the presence of inductive coupling between contours, their mutual impedance is not equal to zero even if they have no common branches.

One can also use the superposition theorem or Thevenin's theorem on condition that a current is determined through a branch without a mutual inductance.

Example 4.3. Consider ramified electric circuit (see Fig. 4.39) with known parameters. Work out the system of equations by Kirchhoff's current and voltage laws. It must consist of three equations: one - by the first Kirchhoff's law, and two - by the second Kirchhoff's law. Choose the directions of currents in branches and the summation round two meshes as shown in Fig. 4.39.

For the equation by Kirchhoff's voltage law we choose the clockwise direction of summation

$$-i_1+i_2+i_3=0;$$

$$R_1 i_1+\frac{1}{C_1}\int i_1 dt+L_2\frac{di_2}{dt}+M\frac{di_3}{dt}+R_2 i_2=e_1+e_2;$$

$$-\left(R_2 i_2+L_2\frac{di_2}{dt}+M\frac{di_3}{dt}\right)+L_3\frac{di_3}{dt}+M\frac{di_2}{dt}+R_3 i_3=-e_2+e_3$$

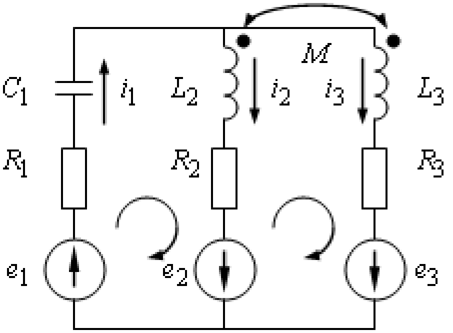


Fig. 4.39

In complex notation

$$\begin{cases}-\underline{I}_1+\underline{I}_2+\underline{I}_3=0;\\ R_1\underline{I}_1-j\dfrac{1}{\omega C_1}\underline{I}_1+j\omega L_2\underline{I}_2+j\omega M\underline{I}_3+R_2\underline{I}_2=\underline{E}_1+\underline{E}_2;\\ -\left(R_2\underline{I}_2+j\omega L_2\underline{I}_2+j\omega M\underline{I}_3\right)+j\omega L_3\underline{I}_3+j\omega M\underline{I}_2+R_3\underline{I}_3=\\ =-\underline{E}_2+\underline{E}_3.\end{cases}$$

Further the problem can be solved by any known method.

Example 4.4. Consider the same ramified electric circuit (see Fig. 4.39) and work out the set of equations by mesh-current method. We choose the directions of mesh currents clockwise the same way as the direction of summation round the meshes.

When calculating self-impedances ($\underline{Z}_{11}$; $\underline{Z}_{22}$) and mutual impedances $\underline{Z}_{12}$; $\underline{Z}_{21}$ of independent contours of the electric circuit it is necessary to take into account the presence of inductive couplings between the coils.

Then self-impedances of meshes would be as follows

$$\underline{Z}_{11} = R_1 + R_2 + j\left(\omega L_2 - \frac{1}{\omega C_1}\right);$$

$$\underline{Z}_{22} = R_2 + R_3 + j\left(\omega L_2 + \omega L_3 - 2\omega M\right).$$

The mutual impedances of contours look like

$$\underline{Z}_{12} = \underline{Z}_{21} = -\left(R_2 + j\omega L_2\right) + j\omega M$$

In the last equation the complex impedance $(R_2 + j\omega L_2)$ is taken with a minus sign by a general rule for mutual impedances of loops as two mesh currents direct in the opposite directions through the given impedance. The value $j\omega M$ is taken with a plus sign because the mesh current $\underline{I}_{11}$ in the second coil, and the mesh current $\underline{I}_{22}$ in the third coil are equally oriented concerning the similar terminals of two coils.

As the network consists of two meshes, the set of equations would be as follows

$$\begin{cases} \left(R_1 + R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_1}\right)\right)\underline{I}_{11} - \left(R_2 + j\left(\omega L_2 - \omega M\right)\right)\underline{I}_{22} = \underline{E}_1 + \underline{E}_2; \\ -\left(R_2 + j\left(\omega L_2 - \omega M\right)\right)\underline{I}_{11} + \left(R_2 + R_3 + j\left(\omega L_2 + \omega L_3 - 2\omega M\right)\right)\underline{I}_{22} = \\ = -\underline{E}_2 + \underline{E}_3. \end{cases}$$

On solving the given set by any method, we can define currents in all branches of the electric circuit as

$$\underline{I}_1 = \underline{I}_{11};\ \underline{I}_2 = \underline{I}_{11} - \underline{I}_{22};\ \underline{I}_3 = \underline{I}_{22}$$

4.23 Decoupling of Coupled Circuits

Sometimes it is necessary to decouple coupled circuits to simplify the calculation of a circuit with a mutual inductance.

The elimination of inductive couplings and their replacement with the electric ones allows passing to an equivalent circuit of replacement and carrying out calculation by any method which is applicable for the calculation of linear circuits.

This is done by placing additional inductances or modifying those already present in the original coupled circuits so that there is no coupling between any inductances in the transformed circuit.

Generally decoupling of any two coupled coils which are connected in one junction (Fig. 4.40, *a)*, one can realise by means of passage to the equivalent circuit which presented in Fig. 4.40, *b*.

Thus we will simultaneously consider two cases: when two coupled coils are connected by similar terminals and when they are connected by different terminals. Since all transformations are based on the Kirchoffian equation written for the original circuit, both the transformed and the original circuit are fully identical from the view-point of design.

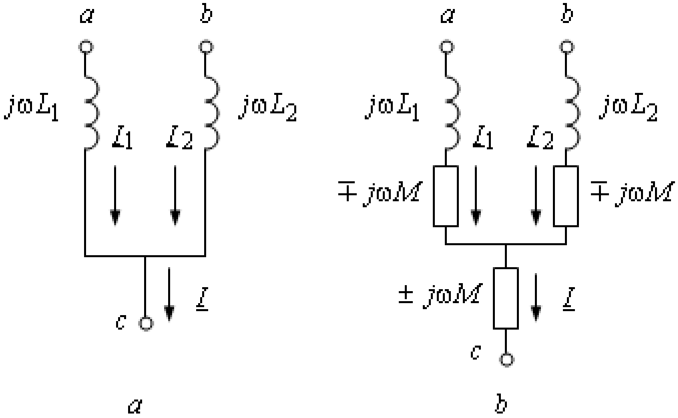


Figure 4.40

For the circuit with inductive coupled coils (Fig. 4.40, *a)*, we have the following equations

$$\begin{cases} \underline{U}_{ac} = j\omega L_1 \underline{I}_1 \pm j\omega M \underline{I}_2 \\ \underline{U}_{bc} = j\omega L_2 \underline{I}_2 \pm j\omega M \underline{I}_1 \end{cases} \tag{4.128}$$

For the circuit without coupled coils (Fig. 4.40, *b)*, we get

$$\begin{cases} \underline{U}_{ac} = j\omega (L_1 \mp M) \underline{I}_1 \pm j\omega M \underline{I} \\ \underline{U}_{bc} = j\omega (L_2 \mp M) \underline{I}_2 \pm j\omega M \underline{I} \end{cases} \tag{4.129}$$

The signs of additional inductances are determined only by the way of joint of inductive coupled elements and do not depend on the chosen directions of currents.

Comparing the circuits of Fig. 4.40, *a* and Fig. 4.40, *b*, we note that L_1 has been replaced by $L_1 \mp M$ and L_2 by $L_2 \mp M$, while the other arm has been extended to include an inductance $\pm M$. (Note that a negative inductance cannot be possibly realized in a circuit with linear elements). In Eq. (4.129) upper marks correspond to a case when two coupled coils are connected by similar terminals in a common junction, and the lower marks correspond to the case for different terminals.

4.24. Power Transfer to an A. C. Load

Let us consider how the energy is transmitted between two coupled circuits. In any linear a. c. circuit the sum of the active powers due to the voltage sources is equal to the sum of the active powers dissipated by the loads, and the sum of the reactive powers due to the voltage sources is equal to the sum of the reactive powers stored by the loads.

The sum of the reactive powers in this theorem is taken to be the sum of the squared branch currents times the branch reactances computed neglecting the mutual inductances, plus the algebraic sum of the power transferred by the magnetic fluxes from one set of branches to the other through mutual induction.

Let two currents $\underline{I}_1 = I_1e^{j\psi_1}$ and $\underline{I}_2 = I_2e^{j\psi_2}$ are known for two coupled coils. Work out the expressions for complex powers of the first and the second coils which are caused by a mutual induction in a case of their aiding connection

$$\underline{S}_{M1} = \underline{U}_{M1}\overset{*}{\underline{I}}_1 = j\omega M\underline{I}_1\overset{*}{\underline{I}}_1 = e^{j\frac{\pi}{2}}\omega MI_1e^{j\psi_1}I_2e^{-j\psi_2} =$$

$$= \omega MI_1I_2e^{j\left(\frac{\pi}{2}+\psi_1-\psi_2\right)} = \omega MI_1I_2\cos\left(\frac{\pi}{2}+\psi_1-\psi_2\right)+$$

$$+ j\omega MI_1I_2\sin\left(\frac{\pi}{2}+\psi_1-\psi_2\right) = \omega MI_1I_2\sin(\psi_1-\psi_2)+$$

$$+ j\omega MI_1I_2\cos(\psi_1-\psi_2) = P_{1M} + jQ_{1M};$$

$$\underline{S}_{M2} = \underline{U}_{M2}\overset{*}{\underline{I}}_2 = j\omega M\underline{I}_2\overset{*}{\underline{I}}_2 = e^{j\frac{\pi}{2}}\omega MI_2e^{j\psi_2}I_2e^{-j\psi_1} =$$

$$= \omega MI_1I_2e^{j\left(\frac{\pi}{2}+\psi_2-\psi_1\right)} = \omega MI_1I_2\cos\left(\frac{\pi}{2}+\psi_2-\psi_1\right)+$$

$$+ j\omega MI_1I_2\sin\left(\frac{\pi}{2}+\psi_2-\psi_1\right) = -\omega MI_1I_2\sin(\psi_1-\psi_2)+$$

$$+ j\omega MI_1I_2\cos(\psi_1-\psi_2) = P_{2M} + jQ_{2M}$$

From the last equations it follows

$$\begin{aligned} P_{1M} &= -P_{2M} = \omega M I_1 I_2 \sin(\psi_1 - \psi_2); \\ Q_{1M} &= Q_{2M} = \omega M I_1 I_2 \cos(\psi_1 - \psi_2). \end{aligned} \tag{4.130}$$

Thus, the total active power which is caused by a mutual induction, is equal to zero, that is $P_M = P_{1M} + P_{2M} = 0$. It means, that coupled coils do not influence on the general balance of an active power.

If $0 < \psi_1 - \psi_2 < \pi$, then $P_{1M} > 0$, $P_{2M} < 0$. On such a condition the energy is transmitted from a circuit into a magnetic field through the first coil and returned back into the circuit through the second coil. If $-\pi < \psi_1 - \psi_2 < 0$, then $P_{1M} < 0$, $P_{2M} > 0$, and the energy is transmitted from a circuit into a magnetic field through the second coil and returned back into the circuit through the first one.

The total reactive power which is caused by a mutual induction is

$$Q_M = 2\omega M I_1 I_2 \cos(\psi_1 - \psi_2)$$

The energy which is reserved by two inductive coils at the expense of a mutual induction

$$W_M = M i_1 i_2$$

The results are true and in such a case if the coupled circuits are not connected electrically.

Summary review questions

1. Define the following terms as they relate to sinusoidal function: argument, amplitude, cycle, frequency, period, instantaneous value, angular velocity, phase.

2. Distinguish between a period time function and sinusoidal time function.

3. Define mathematically the average value of a periodic function. What is the average value of one period of a sine function having amplitude *A*? What is the average value of the positive one-cycle of a sine wave having amplitude *A*?

4. How is the effective value of a periodic function defined? Why is such a definition consistent with a description of power-transfer capability of the associate variable?

5. When the periodic function is a sinusoid of amplitude *A*, what is the effective of the sinusoid? Over what interval of time is the effective value valid?

6. Do all periodic functions possess an effective value that is different than zero? Explain.

7. Do all periodic functions possess an average value that is different than zero? Explain and illustrate,

8. In a sinusoidal steady-state circuit the average power is expressed in terms of the effective values of the current and its associated voltage. Explain why average power is dependent upon effective values of voltage and current.

9. Define power factor and explain why it arises in sinusoidal ac circuits.

10. Distinguish between instantaneous power and average power in a sinusoidal ac circuit.

11. What is apparent power? How is it related to average power? What practical significance does it have?

12. Describe several ways of adding two sinusoids of the same frequency.

13. How is the phasor of a sinusoidal quantity defined? Mention specifically the information that is conveyed by the phasor about the corresponding sinusoidal function.

14. It is said that the phasor representation of sinusoids is a mathematical transform. Explain this statement and identify specifically the transform variable.

15. Why is knowledge of the effective value of a sinusoid more useful than its maximum value?

16. When a circuit is driven by a sinusoidal function, state the phasor relationship that exists between the voltage across a resistor .and the current that flows through it.

17. Describe and illustrate the phasor relationship between the voltage that appears across the terminals of a pure inductor and the current that flows through it in steady state when the inductor is excited by a sinusoidal source. What is the power factor of such a circuit?

18. What is inductive reactance, and how does it arise in a-c circuits?

19. What is a phasor diagram? Why is it useful in the study of sinusoidal steady-state analysis of circuits?

20. Draw the phasor diagram of the purely capacitive circuit. Are the electrical quantities used in your phasor diagram effective or peak values? Does it matter? Explain.

21. What is capacitive reactance, and how does it arise in a-c circuits?

22. What is meant by the complex impedance of an *RL* circuit? How does it arise in the analysis of ac circuits.

23. Complex impedance is often -described as an operator that serves to convert a voltage source to a current response. Explain and illustrate the meaning of this statement.

24. What is reactive power? Why is such a term not encountered when d-c sources are used in an electric circuit?

25. Distinguish among admittance, susceptance, and conductance. What is complex admittance?

26. Draw the phasor diagram of a series *RLC* circuit using the source voltage as the reference phasor and assuming that the capacitive reactance exceeds the inductive reactance. Show the corresponding time diagram of the source voltage and response current.

27. Can the phenomenon of resonance occur in engineering situations when a single energy-storing element is present? Explain.

28. In a series *RLC* circuit, how is the resonant frequency related to the natural frequency? What is the character of the impedance at resonance for this circuit? How is the current response related to the voltage source at the resonant frequency?

29. In the series *RLC* circuit, identify the character of the complex impedance when the circuit is driven at a frequency that exceeds the resonant frequency,

30. How is the bandwidth of a series *RLC* circuit defined? Upon which circuit parameters does the bandwidth depend? What role do the remaining circuit parameters play is this consideration?

31. How is the quality factor of a series *RLC* circuit defined? Of what practical significance is this number? Illustrate.

32. Is it possible for the voltage that appears across the *L* or *C* elements in a series *RLC* circuit to exceed the voltage applied to the circuit? Explain.

33. At the resonant frequency in a series *RLC* circuit, how is the voltage across the inductor coil related to the applied source voltage in phase and magnitude?

34. The instantaneous power per phase in a single-phase circuit contains a double-frequency sinusoidal component. Is a similar component present in the total instantaneous power of a three-phase system?

Problems

4.1. A sinusoidal source of $e(t) = 170 \sin 377t$ is applied to an *RL* circuit. It is found that the circuit absorbs 720 *W* when an effective current of 12 *A* flows.

(a) Find the power factor of the circuit.

(b) Compute the value of the impedance.

(c) Calculate the inductance of the circuit in henrys.

4.2. A voltage source of $e(t) = 141 \sin 377t$ is applied to two parallel branches. The time expression for the current in the first branch is

$$i_1(t) = 7.07 \sin\left(\omega t - \frac{\pi}{3}\right)$$

In the second branch it is

$$i_2(t) = 10 \sin\left(\omega t + \frac{\pi}{6}\right)$$

Compute the total power supplied by the source.

4.3. Refer to Prob. 4.2 and answer the following questions:

(a) Find the resultant current delivered by the source expressed in effective amperes and write the expression for the instantaneous value of the resultant current.

(b) Compute the apparent power of the complete circuit.

(c) Find the resultant power factor of the circuit.

(d) Draw the phasor diagram showing the voltage and all currents.

4.4. A voltage of $e(t) = 141 \sin\left(377t + \frac{\pi}{3}\right)$ is applied to a 20-Ω resistor. Find the effective value of the current in amperes, and compute the power supplied by the source.

4.5. A sinusoidal voltage $e(t) = 170 \sin\left(377t + \frac{\pi}{3}\right)$ is applied to a 0.1-*H* inductor.

(a) Find the effective value of the steady-state current in amperes.

(b) Write the expression for the instantaneous current.

(c) Draw a properly labeled phasor diagram. Use the rms value of the voltage phasor as reference.

4.6. A sinusoidal voltage $e(t) = 170 \sin\left(377t + \frac{\pi}{3}\right)$ is applied to a $100\mu F$ capacitor.

(a) Find the effective value of the steady-state current in amperes.

(b) Draw the phasor diagram. Use the rms value of the voltage phasor as reference.

4.7. A circuit is composed of a resistance of 9 Ω and a series-connected inductive reactance of 12 Ω. When a voltage *e(t)* is applied to the circuit the resulting steady-state current is found to be $i(t) = 28.3 \sin 377t$.

(a) What is the value of the complex impedance?.

(b) Determine the time expression for the applied voltage.

(c) Find the value of the inductance in henrys.

4.8. When a sinusoidal voltage of 120 *V* rms is applied to a series *RL* circuit, it is found that there occurs a power dissipation of 1200 *W* and a current flow given by $i(t) = 28.3 \sin(377t = \theta)$. Find the circuit resistance in ohms and the circuit inductance in henrys.

4.9. In an *RC* series circuit to which is applied a voltage of $170 \sin \omega t$ it is found that a steady-state current flows which leads the voltage by 30°. Find the effective voltage drops across the resistive and reactive elements.

4.10. A circuit is composed of a resistance of 6 Ω and a series capacitive reactance of 8 ohms. A voltage $e(t) = 141 \sin 377t$ is applied to the circuit.

(a) Find the complex impedance.

(b) Determine the effective and instantaneous values of the .current.

(c) Compute the power delivered to the circuit.

(d) Find the value of the capacitance in farads.

4.11. A sinusoidal voltage is applied to three parallel branches yielding branch currents as follows: $i_1(t) = 14.14 \sin(\omega t - 45^o)$, $i_2(t) = 28.28 \sin(\omega t - 60^o)$, $i(t) = 7.07 \sin(\omega t + 60^o)$.

(a) Find the complete time expression for the source current.

(b) Draw the phasor diagram in terms of effective values. Use the voltage as reference.

4.12. A sinusoidal voltage $U_m \sin \omega t$ is applied to three parallel branches. Two of the branch currents are given by

$$i_1(t) = 14.14 \sin\ (\omega t - 37^o)$$

$$i_2(t) = 28.28 \sin\ (\omega t - 143$$

The source current is found to be

$$i(t) = 63.8 \sin\ (\omega t + 12^o)$$

(a) Find the effective value of the current in the third branch.

(b) Write the complete time expression for the instantaneous value of the current in part (a).

(c) Draw the phasor diagram showing the source current and the three branch currents. Use voltage as the reference phasor.

4.13. A voltage wave $e(t) = 170 \sin 120t$ produces a net current of $i(t) = 14.14 \sin 120t + 7.07 \cos\ (120t + 30^o)$.

(a) Express the effective value of the current as a single phasor quantity.

(b) Draw the phasor diagram of part (a). Show the components of the current as well as the resultant.

(c) Find the power delivered by the voltage source of part (a).

4.14. A voltage of $e(t) = 150 \sin 1000t$ is applied across a series RLC circuit where $R = 40\Omega,\ L = 0.13\ H, and\ C = 10\ \mu F$.

(a) Compute the rms value of the steady-state current.

(b) Find the expression for the instantaneous voltage appearing across the capacitor terminals.

(c) Determine the expression for the instantaneous voltage appearing across the inductor terminals.

(d) Compare the rms value of the voltages appearing across L and C with that of the applied voltage and comment.

(e) Draw the complete phasor diagram for the solution of this problem showing all voltage components.

4.15. In the circuit of Prob. 4.14 determine:

(a) The power supplied by the source.

(b) The reactive power supplied by the source.

(c) The reactive power of the capacitor.

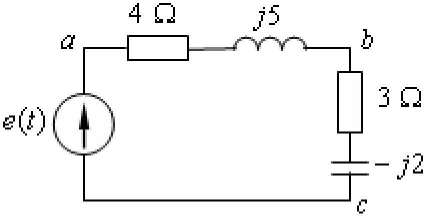


Fig. P4.16

4.16. In the circuit shown in Fig. P4.16 the applied forcing function is given by $e(t) = 141\sin\omega t$.

(a) Express the voltage drops across terminals *ab* and *bc* in terms of phasor notation.

(b) Draw a phasor diagram showing $\underline{U}_{ab} + \underline{U}_{bc}$ and using current as the reference phasor.

4.17. An rms voltage of 100*V* is applied to the series combination of $\underline{Z}_1$ and $\underline{Z}_2$ where $\underline{Z} = 20e^{j30^o}$ Ω. The effective voltage drops across $\underline{Z}_1$ is known to be $40e^{j30^o}$. Find the reactive component of $\underline{Z}_2$.

4.18. An effective voltage of 100 *V* is applied to the parallel combination of two complex impedances $\underline{Z}_1 = R_1 + jX_1$ and $\underline{Z}_2 = R_2 + jX_2$. Assuming that $R_1 = 3\Omega$ and $R_2 = 4\Omega$ and that the magnitudes of the two branch currents are the same, determine the values of X_1, X_2 and the resultant source current.

4.19. In the network configuration shown in Fig. P4.19 find the current which flows through the branch with Z_3 if $\underline{U}_1 = 100\ V$, $\underline{U}_2 = 100e^{-j60^o} V$. Use the nodal method.

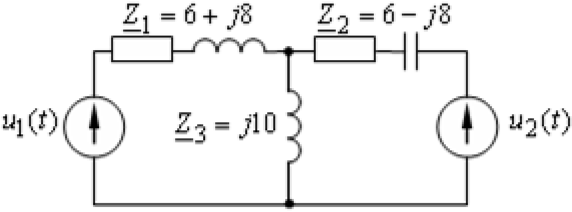


Fig. P4.19

4.20. A circuit has the configuration depicted in Fig. P4.20.

(a) Find the equivalent impedance appearing to the right of points *ab*.

(b) Determine the value of the reactance *X* which makes the source current in phase with the source voltage.

(c) Should the reactance *X* of part (b) be inductive or capacitive? Find the required value of *L* or *C*.

(d) Compute the effective value of the source current for the condition described in part (b).

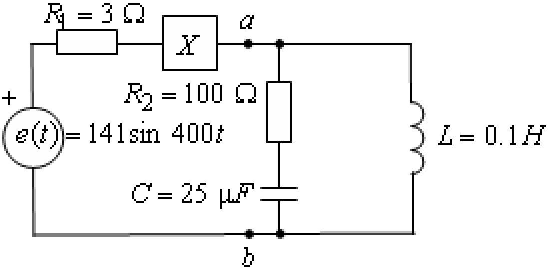


Fig. P4.20

4.21. Refer to the circuit of Fig. P4.21.

(a) Find the equivalent reactance for the parallel branch.

(b) Find the rms line current.

(c) Determine the rms voltage across the parallel branch. Comment on this result.

(d) What is the power dissipated?

(e) What is the power factor?

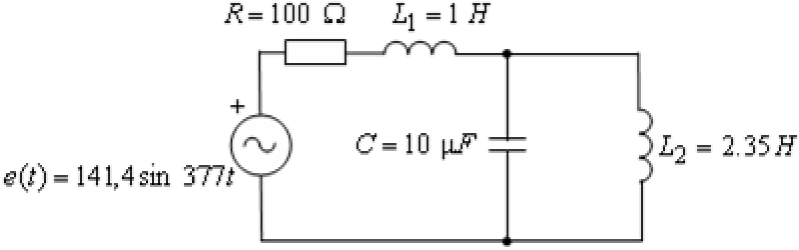


Fig. P4.21

4.22. Refer to the simple circuit depicted in Fig. P4.22 where the load $\underline{Z}$ is in series with the energy source $e(t) = 120\ V$, $f = 60\ Hz$.

(a) What is the power factor?

(b) Find the power dissipated.

(c) Find that value of capacitance which when placed across the load will make the overall power factor unity.

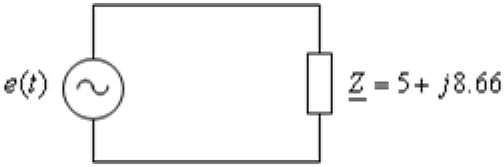


Fig. P4.22

4.23. In the circuit of Fig. P4.23 the following relationships hold:

$$u_{ab}(t) = 4\sqrt{2}\sin\ (\omega t + 135^o)$$

$$u_{bc}(t) = -4\sqrt{3}\sin\ (\omega t + 60^o)$$

$$u_{cd}(t) = 4\cos\ (\omega t - 150^o)$$

(a) Draw a clearly labeled phasor diagram for the voltages $u_{ab}(t),\ u_{bc}(t)$, and $u_{cd}(t)$.

(b) Find the phasor voltage $\underline{U}_{ad}$.

(c) Write the expression for $u_{ad}(t)$.

(d) If $i(t) = 2\sin(\omega t + 165^o)$ find the average power delivered by the current source.

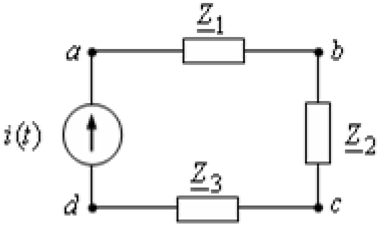


Fig. P4.23

4.24. A series *RLC* circuit has the following parameter values: $R = 10\ \Omega,\ L = 0.01\ H,\ C = 100\ \mu F$.

(a) Compute the resonant frequency in radians per second.

(b) Calculate the quality factor of the circuit.

(c) What is the value of the bandwidth?

(d) Compute the lower and upper frequency points of the bandwidth.

(e) If a signal of $e(t) = 1\sin 1000t$ is applied to this series *RLC* circuit, calculate the maximum value of the voltage appearing across the capacitor terminals.

4.25. A current source is applied to the parallel arrangement of *R*, *L*, and *C* where $R = 10\ \Omega$, $L = 1\ H$, $C = 1\ \mu F$.

(a) Compute, the resonant frequency. (b) Find the quality factor.

(c) Calculate the value of the bandwidth.

(d) Find the lower and upper frequency points of the bandwidth.

(e) If a signal of $i(t) = 1\sin 1000t$ is applied to this parallel *RLC* circuit, calculate the maximum value of voltage appearing across the capacitor terminals.

(f) What is the capacitor current in part (e)?

4.26. A voltage of $e(t) = 10\sin\omega t$ is applied to a series *RLC* circuit. At the resonant frequency of the circuit the maximum voltage across the capacitor is found to be 500 *V*. Moreover, the bandwidth is known to be 400 rad/sec and the impedance at resonance is 100 Ω.

(a) Find the resonant frequency.

(b) Compute the upper and lower limits of the bandwidth.

(c) Determine the value of L and C for this circuit.

4.27. A sinusoidal voltage $e(t) = 141\sin\omega t$ is applied to a series *RL* circuit.

In the steady state the effective value of the voltage measured across the terminals of the inductor is 60 volts.

What is the voltage appearing across the resistor terminals?

4.28. In the circuit shown in Fig. P4.28 the reactance of capacitor C_1 is 4 Ω, the reactance of C_2 is 8 Ω, the reactance of *L* is 8 Ω and active resistance *R* is 4Ω.

A sinusoidal voltage having an effective value of 120 *V* is applied to the circuit.

(a) Find the effective value of the current delivered by the source.

(b) Write the expression for the instantaneous value of the current found in part (a).

(c) Draw a carefully labeled phasor diagram showing the source voltage, the source current, and voltages $\underline{U}_{ab}$ and $\underline{U}_{bc}$.

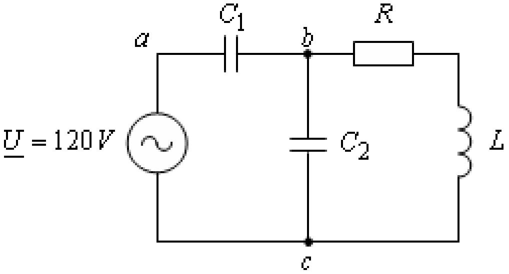


Fig. P4.28

4.29. Find the currents in the circuit Fig. P4.29 and plot a combine vector diagram of currents and voltages. The circuit parameters: $E = 100\,V$; $\omega L_1 = 2\ \Omega$; $\omega L_2 = 3\ \Omega$; $\omega M = 1\ \Omega$; $R = 4\Omega$.

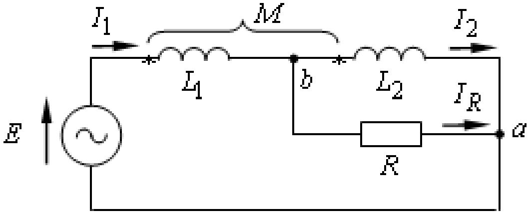


Fig. P4.29

4.30. Calculate the currents in parallel branches of Fig. P4.30 and plot a vector diagram of currents and voltages. The circuit parameters: $E = 100\,V$; $\omega L_1 = 3\ \Omega$; $\omega L_2 = 4\ \Omega$; $\omega M = 3\ \Omega$; $R_1 = R_2 = 4\Omega$.

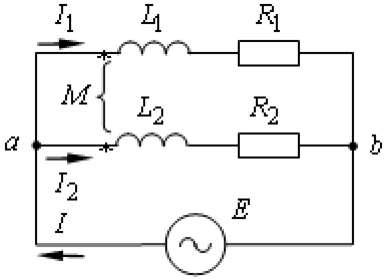


Fig. P4.30

Chapter 5

THREE-PHASE CIRCUITS

5.1 Three-phase System of Electromotive Forces

Suppose we have a system of three *ac* electromotive forces (e.m.f.s) of a certain frequency such that their amplitudes are equal but these e.m.f.s are displaced from one another by 120° in time. Such a set of three sinusoidal e.m.f.s make up a so-called *symmetrical three-phase system of electromotive forces*. The instantaneous values of such e.m.f.s are shown in Fig. 5.1 a , and their vector diagram is in Fig. 5.1 b .

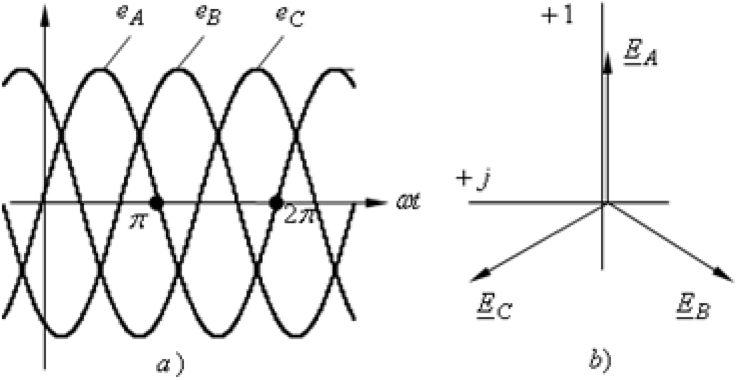


Fig. 5.1 Three-phase system of e.m.f.s.

The electromotive forces for a three-phase system are supplied by a three-phase generator (alternator). Such a generator has three identical (phase) coils rigidly attached to one another and rotating in a uniform magnetic field at a constant angular speed ω . The coils are displaced at 120° from one another, and the sine wave e.m.f.s induced in them, are also displaced at 120° in time phase. The beginnings of coils (windings) can be marked out with the first letters of Latin alphabet *A*, *B* and *C*, and the ends - with last letters *X, Y, Z*. In particular, we call this system a *three-phase balanced system* - in contrast to an unbalanced system, in which the magnitudes may be unequal and/or the phase displacements may not by 120°. For a balanced three-phase system, it follows from Eq. (5.1), that the phasor sum of these three electromotive forces is zero.

We may mathematically express this system of e.m.f.s as

$$e_A = E_m \sin \omega t = U_m \sin \omega t$$

$$e_B = E_m \sin(\omega t - 120^o) = U_m \sin(\omega t - 120^o) \tag{5.1}$$

$$e_C = E_m \sin(\omega t + 120^o) = U_m \sin(\omega t + 120^o)$$

For the identification, the three electromotive forces generated by the alternator are marked as follows. One of them is marked as $\underline{E}_A$. Then the one leading it is $\underline{E}_C$ and the one lags behind it, $\underline{E}_B$.

The order in time in which they go through a zero value and begin to increase in an arbitrary positive sense is termed the *phase sequence*. The order *ABC* is taken as normal or *positive* phase sequence. If the voltages have been caused to reverse their sequence, making it *ACB*, this will be a *negative* phase sequence. If the voltages are in time phase, their vectors coincide, and the system is described as one of *zero* phase sequence.

5.2 A Three-phase Circuit. Phase and Line Quantities

A three-phase circuit is a combination of three-phase supply, a three-phase load (or loads), and connecting wires. The term "phase" may be applied to that part of three-phase system which carries the same current. A source of alternating e.m.f. may be connected to an associated load in a variety of ways. Thus each phase winding of a generator can be connected to the load by two wires. The most common types of interconnections are *wye connection* and *delta connection*, applicable to both the supply and the load of a system. The number of connecting wires in a system is three or four.

In diagrams the convention for a generator is an arrangement of three phase windings, symmetrically disposed to give a phase displacement of 120°. The windings have beginnings (marked *A*, *B* and *C*) and ends. First we consider "wye-wye" connection with a neutral wire (also referred to as a three-phase four-wire system) (see Fig 5.2).

Three-phase circuits may be symmetrical and asymmetrical. The circuit, in which a symmetrical three-phase system of e.m.f.s and symmetrical loading operate, is called symmetrical or uniform.

For star connection the ends of three phases are connected together to form a node called a *neutral point N* or *O* (for the phases of generator) and *n* or *O'* (for the phases of the load), leaving the beginnings as terminals of three-phase star system. The wire connecting the neutral points of the alternator and the load is referred to as the *neutral wire*. The neutral wire carries a neutral current, $\underline{I}_N$ or $\underline{I}_o$, the positive direction for which is from *n* to *N***.**

The wires connecting the terminals *A*, *B* and *C* of the generator and the terminals *a*, *b* and *c* of the load are termed the *line wires*, or simply the *lines*. The currents in the line wires are referred to as *line currents*, $\underline{I}_A$, $\underline{I}_B$ and $\underline{I}_C$.

A positive direction for line currents is assumed to be from the supply to the load. When only the magnitudes of the line currents are involved, it is customary to use the symbol I_l, especially when all the line currents are equal in magnitude.

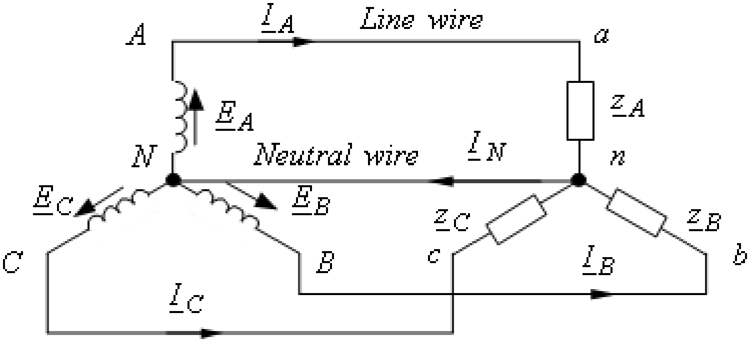


Fig. 5.2. Wye-wye connection with a neutral wire.

Fig. 5.2 shows a star-to-star system with a neutral wire (also referred to as a *three-phase four-wire system*). If there is no a neutral wire between neutral points it will be three-phase three-wire system.

The voltage between two wires is termed the *line voltage*. As with any voltage, it symbol has a two-letter subscript, for example, $\underline{U}_{AB}$, which is the line voltage between the terminals *A* and *B.* The symbol for the magnitude of the line voltage is U_l.

The voltage between the beginning and the end of a phase, or the voltage from line to earth is called a *phase voltage*, U_p. The currents in the line wires are referred to as the *line current.* They are marked with one letter as $\underline{I}_A$, $\underline{I}_B$ and $\underline{I}_C$. Where only the magnitudes of line currents are involved, it is customary to use the symbol I_l, especially when all the line currents are equal in magnitude. The currents in the phases of either the supply or the load are called the *phase currents*, I_p.

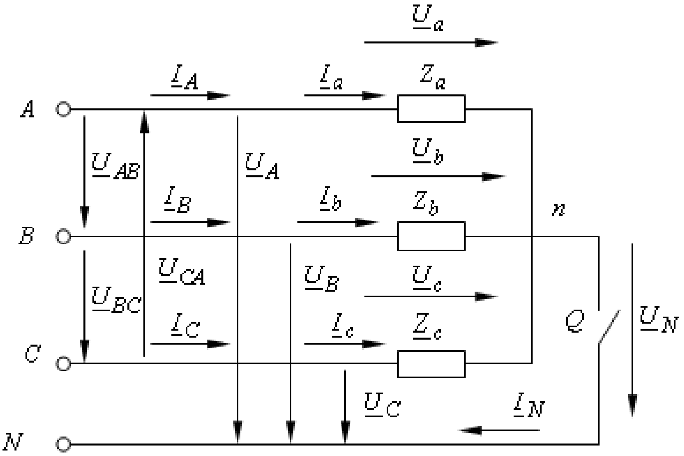


Fig. 5.3. A wye-connected load.

Hence, there are following names of voltages and currents for any electrical circuit (see Fig. 5.3): $\underline{U}_{AB}$, $\underline{U}_{BC}$, $\underline{U}_{CA}$ are the line voltages across the generator terminals; $\underline{U}_{ab}$, $\underline{U}_{bc}$, $\underline{U}_{ca}$ are the line voltages across the load terminals; $\underline{U}_A$, $\underline{U}_B$, $\underline{U}_C$ are the phase generator voltages; $\underline{U}_a$, $\underline{U}_b$, $\underline{U}_c$ are the phase voltages across the load terminals.

If one can neglect the impedances of connecting wires and internal impedances of generator windings, the line voltages across the generator terminals are equal to the line voltages across the load terminals, that is $\underline{U}_{AB} = \underline{U}_{ab}$, $\underline{U}_{BC} = \underline{U}_{bc}$, $\underline{U}_{CA} = \underline{U}_{ca}$.

For the same reason, generator phase voltages and phase voltages across the load terminals are numerically equal to e.m.f.s:

$\underline{U}_A = \underline{U}_a = \underline{E}_A$; $\underline{U}_B = \underline{U}_b = \underline{E}_B$; $\underline{U}_C = \underline{U}_c = \underline{E}_C$.

$\underline{I}_A$, $\underline{I}_B$, $\underline{I}_C$ are the line currents in line wires; $\underline{I}_a$, $\underline{I}_b$, $\underline{I}_c$ are the phase currents through the load phases. The line currents in line wires are equal to the load phase currents as generator windings and the load phases are connected in series: $\underline{I}_A = \underline{I}_a$, $\underline{I}_B = \underline{I}_b$, $\underline{I}_C = \underline{I}_c$. $\underline{I}_N$ is the current in a neutral wire, and $\underline{U}_N$ is the neutral voltage shift (or neutral point displacement voltage or the bias neutral voltage). It is the voltage between the neutral nodes of a generator and a load.

5.3 Relationship between Line and Phase Voltages and Currents

Between phase and line e.m.f.s, voltages and currents exist certain relationships for symmetrical circuits.

For a wye (star) connection the line voltages are related to the phase voltages such that

$$\underline{U}_{AB} = \underline{U}_A - \underline{U}_B$$
$$\underline{U}_{BC} = \underline{U}_B - \underline{U}_C \tag{5.2}$$
$$\underline{U}_{CA} = \underline{U}_C - \underline{U}_A$$

where $\underline{U}_{AB}$, $\underline{U}_{BC}$ and $\underline{U}_{CA}$ are the line voltages in a wye-connected three-phase generator; $\underline{U}_A$, $\underline{U}_B$ and $\underline{U}_C$ are the phase voltages in a wye-connected generator. When we have a symmetrical load in a wye connection, that is $Z_A = Z_B = Z_C$, the line voltages are $\sqrt{3}$ times more than the phase voltages. These relationships of the phase voltages and line voltages are illustrated in the phasor diagram of Fig. 5.4.

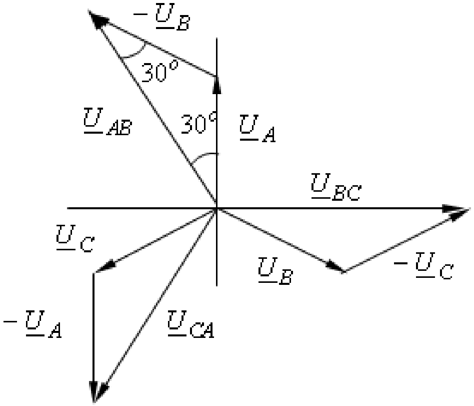


Fig. 5.4. The phasor diagram of line and phase voltages.

From the vector diagram, the line voltage may be visualized as the base of an equivalent triangle having acute angles of 30° (Fig.5.3)

$$U_l = U_{AB} = U_p \cdot 2\cos 30^o = \sqrt{3} \cdot U_p \tag{5.3}$$

The line currents in a wye supply are equal to the phase currents as line and phase wires are connected in series: $I_l = I_{ph}$.

Three-phase circuits are a modification of circuits carrying currents, and as such they can be solved by the same methods as single-phase sinusoidal circuits. Symbolic notation and vector diagrams are also applicable to three-phase circuits. Vector diagrams furnish a convenient check on the angles between current and voltage vectors, reveal mistakes in analytical calculations, and make all relationships more clear. At calculation of all the problems following further we connect a three-phase electrical circuit to the symmetrical three-phase generator. Now we will examine some of the three-phase circuits.

5.4 The Calculation of Wye-to-Wye with a Neutral Wire

For calculation of three-phase circuits we use definite methods. In a general case, any three-phase circuit may be considered as a single-phase in which several e.m.f.s operate. To calculate such a circuit we can use the symbolic method of calculation and some known laws: Ohm's law, Kirchhoff's laws, mesh-current method, node-analysis method, superposition theorem and so on.

If there is no impedance in a neutral wire in the circuit of Fig. 5.2, the potential at point *n* is the same as at point *N*, and two points are actually a single point Then three separate meshes are formed in the system, carrying the following currents:

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}_A}; \quad \underline{I}_B = \frac{\underline{E}_B}{\underline{Z}_B}; \quad \underline{I}_C = \frac{\underline{E}_C}{\underline{Z}_C}. \tag{5.4}$$

By Kirchhoff's current law the current in the neutral wire is the vector sum of the phase currents:

$$\underline{I}_N = \underline{I}_A + \underline{I}_B + \underline{I}_C \tag{5.5}$$

If the load is balanced ($\underline{Z}_A = \underline{Z}_B = \underline{Z}_C$), the current $\underline{I}_N$ is zero, and a neutral wire can be dispensed with, without affecting the operation of the system. If the load is unbalanced, then in a general case the current $\underline{I}_N$ is other than zero. If the neutral wire has an impedance Z_o, the system should be calculated by the nodal-pairs method.

Example 5.1. The phase voltage of the alternator in Fig. 5.2 is 120 V. The phase impedances: $\underline{Z}_A = R = 30\Omega$; $\underline{Z}_B = j\omega L = 80\Omega$; $\underline{Z}_C = - j/\omega C = 80\Omega$. Find a current in a neutral wire.

Solution: Write phase voltages in a complex form:

$$\underline{U}_A = \underline{U}_p \cdot e^{j0^o} = 120\ V;$$

$$\underline{U}_B = \underline{U}_p \cdot e^{-j120^o} = 120\, e^{-j120^o}\ V;$$

$$\underline{U}_C = \underline{U}_p \cdot e^{j120^o} = 120\, e^{j120^o}\ V.$$

Calculate phase currents for each phase separately:

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}_A} = \frac{120}{30} = 4\,A; \qquad \underline{I}_B = \frac{\underline{E}_B}{\underline{Z}_B} = \frac{120e^{-j120^o}}{80e^{j90^o}} = 1.5e^{-j210^o}\ A;$$

$$\underline{I}_C = \frac{\underline{E}_C}{\underline{Z}_C} = \frac{120e^{j120^o}}{80e^{-j90^0}} = 1.5\,e^{j210^o}\ A.$$

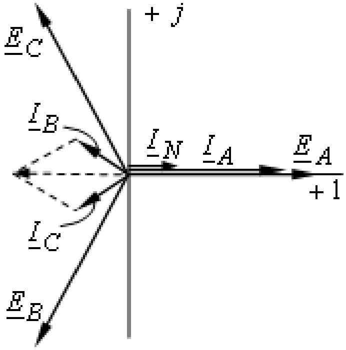


Fig. 5.5

The vector diagram is shown in Fig. 5.5.

Current $\underline{I}_A$ is in phase with $\underline{E}_A$. Current $\underline{I}_B$ is lagging with $\underline{E}_B$. Current $\underline{I}_C$ is in leading with $\underline{E}_C$. The algebraic sum $\underline{I}_A + \underline{I}_B + \underline{I}_C$ gives the current in a neutral wire $\underline{I}_N$ the value of which may be found as following:

$$\underline{I}_N = \underline{I}_A + \underline{I}_B + \underline{I}_C = 4 + 1.5e^{-j210^o} + 1.5e^{j210^o} =$$
$$= 4 - 1.48 + j0.235 - 1.48 - j0.235 = 1.04\ \ A.$$

5.5 The Calculation of Wye-to-Wye without a Neutral Wire

When the load is balanced (or uniform) and there is no a neutral wire between two neutral points, *N* and *n*, the voltage between these nodes is equal zero: $\underline{U}_N = 0$.

If the load is unbalanced the voltage appearing between the neutral points of the load and the supply can be found by the formula

$$\underline{U}_N = \frac{\underline{E}_A \underline{Y}_A + \underline{E}_B \underline{Y}_B + \underline{E}_C \underline{Y}_C}{\underline{Y}_A + \underline{Y}_B + \underline{Y}_C} \tag{5.6}$$

where $\underline{E}_A$, $\underline{E}_B$ and $\underline{E}_C$ are respective voltages at the generator end; $\underline{Y}_A, \underline{Y}_B$ and $\underline{Y}_C$ are the admittance of three phases. This voltage is called the neutral voltage shifting or the bias neutral voltage.

Now we can find the phase currents through the load

$$\underline{I}_a = \frac{\underline{U}_a}{\underline{Z}_a} = \frac{\underline{E}_A - \underline{U}_N}{\underline{Z}_a} = (\underline{E}_A - \underline{U}_N) \cdot \underline{Y}_a; \tag{5.7}$$

$$\underline{I}_b = \frac{\underline{U}_b}{\underline{Z}_b} = \frac{\underline{E}_B - \underline{U}_N}{\underline{Z}_a} = (\underline{E}_B - \underline{U}_N) \cdot \underline{Y}_b; \tag{5.8}$$

$$\underline{I}_c = \frac{\underline{U}_c}{\underline{Z}_c} = \frac{\underline{E}_C - \underline{U}_N}{\underline{Z}_c} = (\underline{E}_C - \underline{U}_N) \cdot \underline{Y}_c. \tag{5.9}$$

where $\underline{U}_a, \underline{U}_b, \underline{U}_c$ are phase voltages at the load, $\underline{Y}_a, \underline{Y}_b, \underline{Y}_c$ are phase admittances, that is the values inverse to complex impedances.

For the numerical data see Example 5.2. (Fig.5.6.)

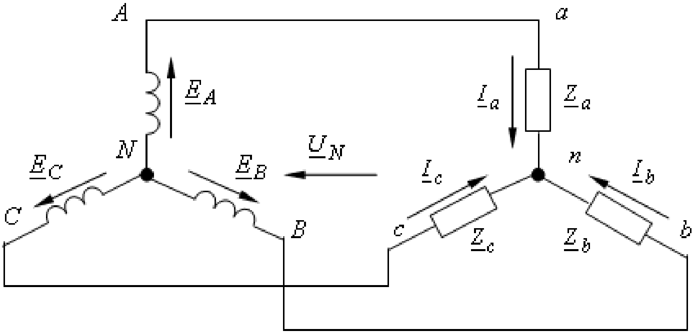


Fig.5.6

Example 5.2. Determine currents and voltages and draw a vector diagram in the circuit without a neutral wire, for $E_p = 50\ V$, $\underline{Z}_a = 20\ \Omega$, $\underline{Z}_b = 10\ \Omega$, $\underline{Z}_c = -j20\ \Omega$.

Write phase voltages in a complex form:

$$\underline{U}_A = \underline{E}_p \cdot e^{j0^o} = 50\ V\ ,\ \underline{U}_B = \underline{E}_p \cdot e^{-j120^o} = 50\,e^{-j120^o}\ V$$

$$\underline{U}_C = \underline{E}_p \cdot e^{j120^o} = 50\,e^{j120^o}\ V\ .$$

Calculate complex phase admittances:

$$\underline{Y}_a = 1/\underline{Z}_a = 1/20 = 0.05\ \Omega^{-1};\quad \underline{Y}_b = 1/\underline{Z}_b = 1/10 = 0.1\ \Omega^{-1};$$

$$\underline{Y}_c = 1/\underline{Z}_c = 1/20e^{-j90^o} = 0.05e^{j90^o} = j0.05\ \Omega^{-1}.$$

For the asymmetrical load the neutral voltage shifting appears between neutral nodes. We define it, using Eq. (5.4).

$$\underline{U}_N = \frac{50 \cdot 0.05 + 50e^{-j120^o} \cdot 0.1 + 50e^{j120^o} \cdot 0.05e^{j90^o}}{0.05 + 0.1 + j0.05} =$$

$$= \frac{-2.165 - j5.58}{0.15 + j0.05} = -20.981 - j23.66 = 31.62e^{-j132^o}\ \ V\ .$$

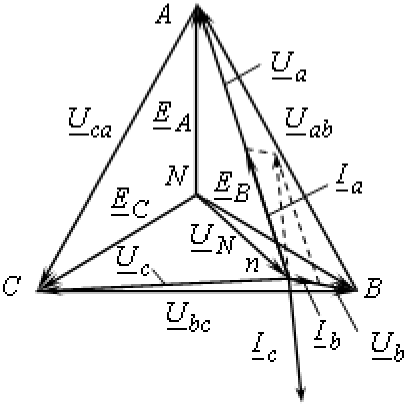


Fig. 5.7

Find the currents. using Eq. (5.7), (5.8), (5.9):

$$\underline{I}_a = (50 + 23{,}85 + j29.45)\cdot 0.05 = 3.55 + j1.183 = 3.74e^{j18^o}\ A;$$

$$\underline{I}_b = (-25 - j43.3 + 23{,}85 + j29.45)\cdot 0.1 = 1.002e^{-j102^o}\ A;$$

$$\underline{I}_c = (-25 + j43{,}3 + 23.85 + j29.45)\cdot j0.05 = 3.354e^{-j177^o}\ A.$$

The vector diagram for this problem one can see in Fig. 5.7.

5.6 The Calculation of Damage Conditions

There are two damage rates in three-phase circuit: a short circuit of any phase and an open circuit of a phase. Let consider these two conditions solving problems.

Example 5.3. Symmetrical load $\underline{Z}_a = \underline{Z}_b = \underline{Z}_c = (240 + j100)$ is connected up by way of line wires with line impedances $\underline{Z}_l = (30 + j40)\ \Omega$ to the three-phase generator with the phase e.m.f. $E_p = 220\ V$. The impedance of a neutral wire $\underline{Z}_N = 40 + j20$.

Calculate damage conditions for two cases:

- a *short circuit* of the phase *A*;
- an *open circuit* of a line wire between nodes A and a.

When a short circuit in phase *A*, (Fig.5.8), it is asymmetrical load, $\varphi_n = \varphi_a$, i.e. "*n*" and "*a*" are the same node and.

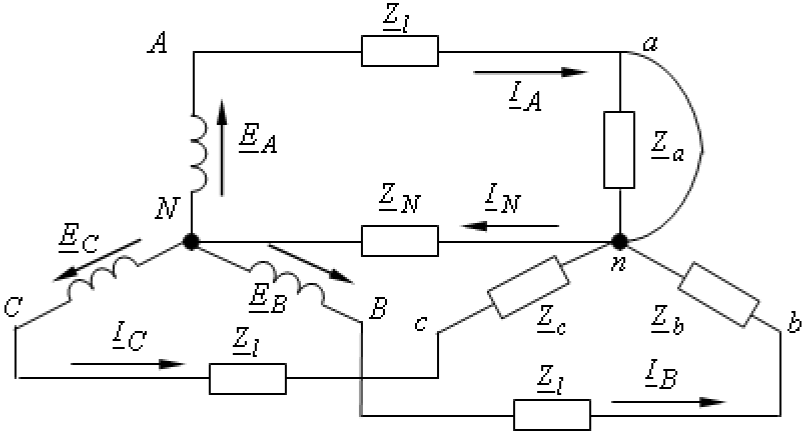


Fig.5.8. A short circuit of a load in the phase *A*

Define the neutral voltage shift appeared between neutral nodes:

$$\underline{U}_N = \frac{\underline{E}_A\underline{Y}_A + \underline{E}_B\underline{Y}_B + \underline{E}_C\underline{Y}_C}{\underline{Y}_A + \underline{Y}_B + \underline{Y}_C + \underline{Y}_N} \tag{5.10}$$

At first we calculate complex branch admittances:

$$\underline{Y}_A = \frac{1}{\underline{Z}_l} = \frac{1}{50e^{j53^o}} = 0.02e^{-j53^o} = (0.012 - j0.016)\ \Omega^{-1};$$

$$\underline{Y}_B = \frac{1}{\underline{Z}_b + \underline{Z}_l} = \frac{1}{304.14e^{j27^o}} = 0.0033e^{-j27} = (0.003 - j0.0015)\ \Omega^{-1};$$

$$\underline{Y}_C = \underline{Y}_B = 0.0033e^{-j27} = (0.003 - 0.0015)\ \Omega^{-1};$$

$$\underline{Y}_N = \frac{1}{\underline{Z}_N} = \frac{1}{44.72e^{j27^o}} = 0.022e^{-j27^o} = (0.02 - j0.01)\ \Omega^{-1}.$$

Determine the bias neutral voltage:

$$\underline{U}_N = \frac{127 \cdot 0.02e^{-j153^o} + 127e^{-j120^o} \cdot 0.0033e^{-j27^o} + 127e^{j120^o} \cdot 0.022e^{-j27^o}}{0.012 - j0.016 + 0.003 - j0.00115 + 0.003 - j0.0015 + 0.02 - j0.01} =$$

$$= \frac{3.761e^{-j57^o}}{0.048e^{-j37^o}} = 73.915 - j27.525 = 78.87e^{-j20^o}\ V.$$

Calculate the phase currents by Ohm's law:

$$\underline{I}_A = (\underline{E}_A - \underline{U}_N) \cdot \underline{Y}_a = \underline{U}_{an} \cdot \underline{Y}_a = 0.02e^{-j53^o} \cdot (127 - 78.87e^{-j20^o}) =$$

$$= 2.193 - j2.007 = 2.973e^{-j42^o}\ A;$$

$$\underline{I}_B = (\underline{E}_B - \underline{U}_N) \cdot \underline{Y}_b = \underline{U}_{bn}\underline{Y}_b = 0.0033e^{-j27^o}(127e^{-j120^o} - 59e^{-j5^o}) =$$

$$= -0.784 - j0.197 = 0.808e^{-j166^o}\ A;$$

$$\underline{I}_C = (\underline{E}_C - \underline{U}_N) \cdot \underline{Y}_c = \underline{U}_{cn} \cdot \underline{Y}_c = 0.0033e^{-j27^o}(127e^{j120^o} - 59e^{j5^o}) =$$

$$= -0.207 + j0.915 = 0.938e^{j103^o}\ A.$$

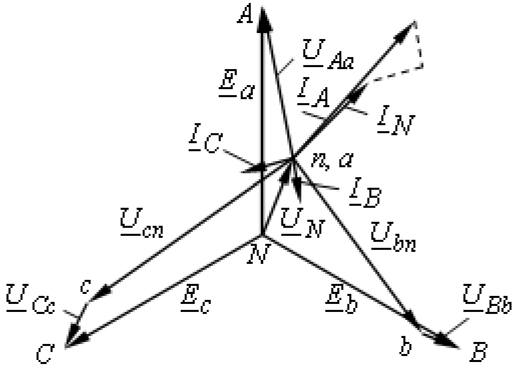


Fig.5.9. A vector diagram for a short circuit in the phase *A*.

The current in a neutral wire

$$\underline{I}_N = \frac{\underline{U}_N}{\underline{Z}_N} = \frac{78.87e^{-j20^o}}{44.7e^{j27^o}} = 1.764e^{-j47^o} \ \ A.$$

Check up the calculation, using Kirchhoff's current law:

$$\underline{I}_N = 2.193 - j2.007 - 0.784 - j0.197 - 0.207 + j0.915 =$$

$$= 1.203 - j1.2 = 1.764e^{-j47^o} \ A.$$

Determine the phase voltage drops across the load and the voltage drops in line wires: $\underline{U}_{an} = 0$

$$\underline{U}_{bn} = \underline{Z}_b \cdot \underline{I}_b = 260e^{j23^o} \cdot 0.808e^{-j166^o} = 210.1e^{-j143^o} \ \ V$$

$$\underline{U}_{cn} = \underline{Z}_c \cdot \underline{I}_c = 260e^{j23^o} \cdot 0.938e^{j103^o} = 243.858e^{j125^o} \ \ V.$$

$$\underline{U}_{Aa} = \underline{Z}_l \cdot \underline{I}_a = 50e^{j53^o} \cdot 2.973e^{-j42^o} = 148.655e^{j11^o} \ \ V$$

$$\underline{U}_{Bb} = \underline{Z}_l \cdot \underline{I}_b = 50e^{j53^o} \cdot 0.808e^{-j166^o} = 40.40e^{-j113^o} \ \ V$$

$$\underline{U}_{Cc} = \underline{Z}_l \cdot \underline{I}_c = 50e^{j53^o} \cdot 0.938e^{j103^o} = 46.9e^{j156^o} \ \ V$$

A vector diagram for this condition is shown in Fig. 5.9.

When we have *an open circuit* of the phase *A*, (Fig.5.10) the working condition also becomes asymmetrical. Hence a neutral voltage shift appears between the neutral junctions of a generator and a load.

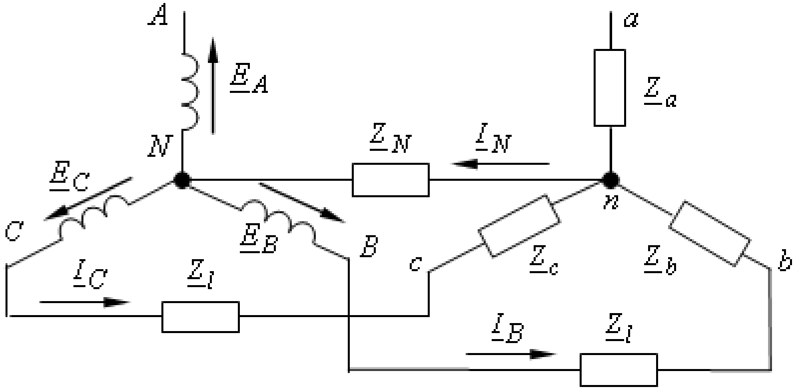


Fig.5.10. An open circuit in the line wire *Aa*.

As $\underline{U}_A = 0$, Eq.(5.5) is used in the following condition:

$$\underline{U}_N = \frac{\underline{E}_B \underline{Y}_B + \underline{E}_C \underline{Y}_C}{\underline{Y}_B + \underline{Y}_C + \underline{Y}_N} \tag{5.11}$$

$$\underline{U}_N = \frac{220e^{-j120^{o}} \cdot 0.0033e^{-j27^{o}} + 127e^{j120^{o}} \cdot 0.0033e^{-j27^{o}}}{0.003 - j0.0015 + 0.003 - j0.0015 + 0.02 - j0.01} =$$

$$= -24.997 - j0.284 = 24.998e^{-j179^{o}} \ V.$$

Calculate the phase currents and the current in the neutral wire:

$$\underline{I}_B = (\underline{E}_B - \underline{U}_N) \cdot \underline{Y}_b = \underline{U}_{bn} \cdot \underline{Y}_b = 0.0033e^{-j27^{o}} \cdot (127e^{-j120^{O}} -$$

$$- 24.998e^{j179^{o}}) = -0.537 - j0.43 = 0.69e^{-j141^{o}} \ A;$$

$$\underline{I}_C = (\underline{E}_C - \underline{U}_N) \cdot \underline{Y}_c = \underline{U}_{cn} \cdot \underline{Y}_c = 0.0033e^{-j27^{o}} \cdot (127e^{j120^{O}} -$$

$$- 24.998e^{j179^{o}}) = 0.04 + j0.684 = 0.69e^{j87^{o}} \ A$$

$$\underline{I}_N = \underline{U}_N \cdot \underline{Y}_N = 0.0022e^{-j27^{o}} \cdot 24.998e^{-j179^{O}} = 0.559e^{-j206^{o}} \ A.$$

Find the phase voltages and the voltage drops in line wires:

$$\underline{U}_{Bb} = \underline{Z}_l \cdot \underline{I}_b = 50e^{j53^o} \cdot 0.69e^{-j141^o} = 34.5\,e^{-j88^o}\ V$$

$$\underline{U}_{Cc} = \underline{Z}_l \cdot \underline{I}_c = 50e^{j53^o} \cdot 0.69e^{j87^o} = 34.5e^{j140^o}\ V$$

$$\underline{U}_{bn} = \underline{Z}_b \cdot \underline{I}_b = 0.69e^{-j141^o} \cdot 260e^{j23^o} = 179.4e^{-j118^o}\ V$$

$$\underline{U}_{cn} = \underline{Z}_c \cdot \underline{I}_c = 0.69e^{j87^o} \cdot 260e^{j23^o} = 179.4e^{j110^o}\ V$$

The vector diagram for this case is represented in Fig. 5.11.

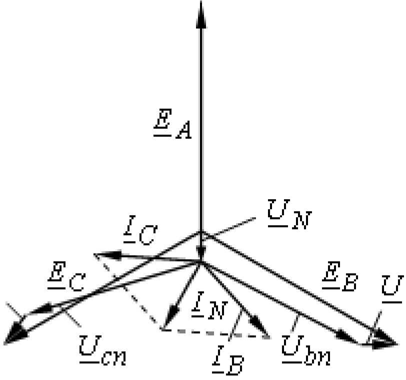


Fig. 5.11. A phasor diagram for the open circuit in the line wire *Aa*

5.7. Delta-Connected Load

For delta connection (Fig. 5.12), the end of the first phase of the load is connected to the beginning of the second one, the end of the latter is connected to the beginning of the third phase, and its end - to the beginning of the first, thus making symmetrical cyclic junctions.

The junctions make the terminals of the three-phase delta system. The vector sum of the e.m.f.s in a closed triangle is zero.

In a delta-connected load, phase currents I_{ab}, I_{bc}, I_{ca} flowing through phase impedances Z_{ab}, Z_{bc}, Z_{ca} are supplied with two-letter subscripts. The positive direction of a current flow is assumed to be clockwise.

The sequence of the letters in the subscript corresponds to the direction of current flow, the first letter standing for the sending end, and the second letter for the receiving end of a given current.

Let's mark the voltages and currents in the delta-connection:

U_{AB}, U_{BC}, U_{CA} are the line voltages in a three-phase delta-connected supply;

U_{ab}, U_{bc}, U_{ca} are the phase voltages in a delta-connected load;

I_A, I_B, I_C are the line currents in line wires connecting the supply and the load;

I_{ab}, I_{bc}, I_{ca} are the phase currents in a delta-connected load.

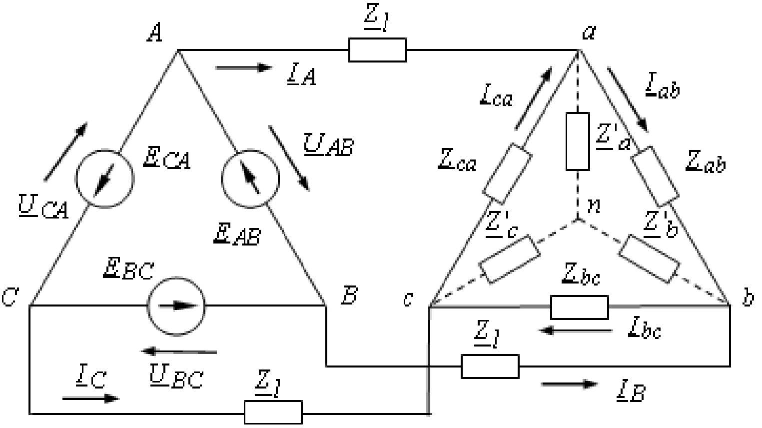


Fig.5.12. The delta connection.

The current I_{ab} is due to the voltage U_{ab}. Its magnitude and phase displacement with respect to U_{ab} are determined by the load impedance Z_{ab}. The current I_{bc} is due to U_{bc}. Its magnitude and phase displacement with respect to U_{bc} are governed by the impedance Z_{bc}. The current I_{ca} is due to U_{ca} and is determined by the load impedance Z_{ca}.

In a delta-connected supply the line voltages are equal to the phase voltages $U_l = U_p$. If there are no impedances in line wires, the line voltages in a delta-connected supply are equal to the phase voltages in a delta-connected load.

If the load is balanced, the line currents, however, $\sqrt{3}$ times the phase currents, as above for the voltages in a wye supply.

If the load is unbalances, the line currents can be found in terms of the phase currents by Kirchhoff's current law:

$$\underline{I}_A = \underline{I}_{ab} - \underline{I}_{ca},$$
$$\underline{I}_B = \underline{I}_{bc} - \underline{I}_{ab}, \qquad (5.12)$$
$$\underline{I}_C = \underline{I}_{ca} - \underline{I}_{bc}.$$

In practice it is often necessary to determine currents and voltages in three-phase delta-connected circuit when there are line impedances in line wires. For this case the line voltages across the load aren't equal to the generator line voltages.

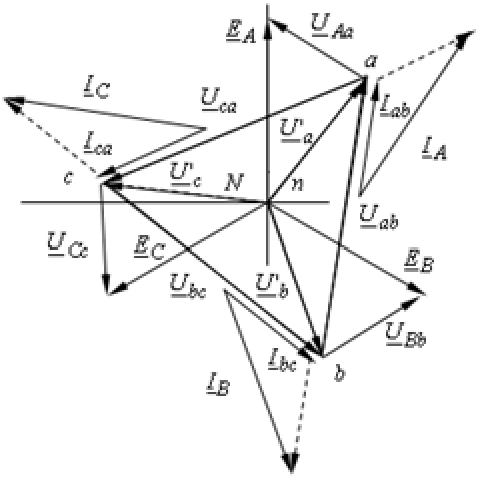


Fig. 5.13. A vector diagram for a symmetrical delta connection.

That is why we must at first change delta-connected load into wye-connected one (see Fig. 5.12).

One can use the following formulas for transformation:

$$\underline{Z}'_a = \frac{\underline{Z}_{ab} \cdot \underline{Z}_{ca}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}};$$
$$\underline{Z}'_b = \frac{\underline{Z}_{bc} \cdot \underline{Z}_{ab}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}}; \qquad (5.13)$$
$$\underline{Z}'_c = \frac{\underline{Z}_{ca} \cdot \underline{Z}_{bc}}{\underline{Z}_{ab} + \underline{Z}_{bc} + \underline{Z}_{ca}}.$$

If there is a symmetrical load and the phase impedances are equal ($\underline{Z}_{ab} = \underline{Z}_{ba} = \underline{Z}_{ca}$), the neutral voltage shift (or the bias neutral voltage) $\underline{U}_N$ is equal to *0*.

Then we can calculate currents:

$$\underline{I}_A = \frac{\underline{E}_A}{\underline{Z}_l + \underline{Z}'_a}; \quad \underline{I}_B = \frac{\underline{E}_B}{\underline{Z}_l + \underline{Z}'_b}; \quad \underline{I}_C = \frac{\underline{E}_C}{\underline{Z}_l + \underline{Z}'_c}. \qquad (5.14)$$

The phase currents are $\sqrt{3}$ times less the line currents and angle shift between them is 30^O (that is a phase current leads the corresponding line current).

$$\underline{I}_{ab} = \frac{\underline{I}_A}{\sqrt{3}} e^{j30^o}; \; \underline{I}_{bc} = \frac{\underline{I}_B}{\sqrt{3}} e^{j30^o}; \; \underline{I}_{ca} = \frac{\underline{I}_C}{\sqrt{3}} e^{j30^o} \qquad (5.15)$$

As the result of the calculation, we draw a vector diagram of currents and voltages (Fig. 5.13).

The description of the vector diagram:

$\underline{E}_A$, $\underline{E}_B$, $\underline{E}_C$ are the phase e.m.f.s of a generator;

$\underline{U}'_A$, $\underline{U}'_B$, $\underline{U}'_C$ are the phase voltages across a wye-connected load. They can be found by Ohm's law:

$$\underline{U}'_a = \underline{Z}'_a \cdot \underline{I}_A; \; \underline{U}'_b = \underline{Z}'_b \cdot \underline{I}_b; \; \underline{U}'_c = \underline{Z}'_c \cdot \underline{I}_c. \qquad (5.16)$$

$\underline{U}_{Aa}$, $\underline{U}_{Bb}$, $\underline{U}_{Cc}$ are the voltages across line impedances. We can determine them by the formulas:

$$\underline{U}_{Aa} = \underline{Z}_l \cdot \underline{I}_A; \; \underline{U}_{Bb} = \underline{Z}_l \cdot \underline{I}_B; \; \underline{U}_{Cc} = \underline{Z}_l \cdot \underline{I}_C. \qquad (5.17)$$

Calculate the phase voltages across a delta-connected load. They can be found as

$$\underline{U}_{ab} = \underline{U}'_a - \underline{U}'_b; \quad \underline{U}_{ab} = \underline{U}'_a - \underline{U}'_b. \qquad 5.18)$$

Example 5.4. The three-phase ssymmetrical load $\underline{Z}_{ab} = \underline{Z}_{bc} = \underline{Z}_{ca} = (500 - j450)\,\Omega$ is connected up by way of line wires with line impedances $\underline{Z}_l = (35 + j45)\,\Omega$ to the symmetrical generator with the phase voltage equaled to $U_p = 380\,V$. Calculate phase and line currents and voltages. Draw a vector diagram.

*Solution: A*t first we change a delta-connected load into a wye-connected one (see Fig. 5.12). As we have a balanced load, one can use the following equation:

$$\underline{Z}'_a = \underline{Z}'_b = \underline{Z}'_c = \frac{\underline{Z}_{ab}}{3} = 166{,}44 - j150 = 224.23e^{-j42^o}\,\Omega\,.$$

Determine phase voltage across the generator and write them in a complex form:

$$\underline{U}_A = 380\ V;\ \underline{U}_B = 380e^{-j120^o}\,V;\ \underline{U}_C = 380e^{j120^o}\,V\,.$$

Define phase impedances:

$$\underline{Z}_a = \underline{Z}_b = \underline{Z}_c = \underline{Z}_l + \underline{Z}_a = 35 + j45 + 166.44 - j150 = 227.36e^{-j28^o}\,\Omega\,.$$

Calculate line currents by Ohm's law:

$$\underline{I}_A = \frac{\underline{U}_A}{\underline{Z}_a} = \frac{380}{227.36e^{-j28^o}} = 1.67e^{j28^o}\,A\,.$$

Then

$$\underline{I}_B = \frac{\underline{U}_B}{\underline{Z}_b} = 1.67e^{-j92^o}\,A;\quad \underline{I}_C = \frac{\underline{U}_C}{\underline{Z}_c} = 1.67e^{j148^o}\,A.$$

Determine phase currents across the load by Eq. (5.15):

$$\underline{I}_{ab} = \frac{\underline{I}_A}{\sqrt{3}}e^{j30^o} = \frac{1.67e^{j28^o}}{\sqrt{3}}e^{j30^o} = 0.965e^{j58^o}\,A;$$

$$\underline{I}_{bc} = 0.965e^{-j62^o}\,A;\quad \underline{I}_{ca} = 0.965e^{j178^o}\ \text{A}.$$

Define line and phase voltages:

$$\underline{U}_{ab} = \underline{Z}_{ab}\cdot\underline{I}_{ab} = 672.68e^{-j42^o}\cdot 0.965e^{j57^o} = 649.14e^{j58^o}\,V\,;$$

$$\underline{U}_{bc} = \underline{Z}_{bc}\cdot\underline{I}_{bc} = 649.14e^{-j104^o}\,V\,;\ \underline{U}_{ca} = \underline{Z}_{ca}\cdot\underline{I}_{ca} = 649.14e^{j136^o}\ V;$$

$$\underline{U}_{Aa} = \underline{Z}_l\cdot\underline{I}_A = 57e^{j52^o}\cdot 1.67e^{j27^o} = 92.27e^{j80^o}\ V;$$

$$\underline{U}_{Bb} = \underline{Z}_l\cdot\underline{I}_B = 92.27e^{-j40^o}\ V;\ \underline{U}_{Cc} = \underline{Z}_l\cdot\underline{I}_C = 92.27e^{j200^o}\,V\,.$$

At last we draw a vector diagram for this case (see Fig. 5.14).

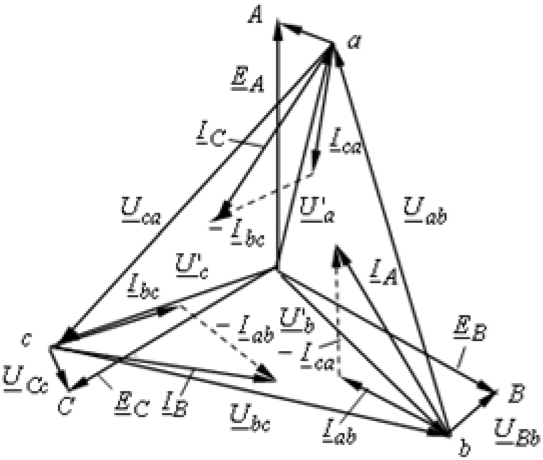


Fig. 5.14

5.8 Damage Conditions in a Delta-Connected Load

There are three damage rates in a delta-connected load:

an open circuit of any phase;

a short circuit of a phase;

an open circuit of a line wire.

The first damage condition we will consider for the case when the phase *bc* is broken. Then a delta-connected load changes into asymmetrical wye connection because we have got here an unbalanced load. Such a delta-connected circuit is depicted in Fig. 5.15.

So, it is the asymmetrical wye connection without a neutral wire. Hence, a neutral voltage shift appears between the two neutral nodes. We can find it by Eq. (5.6):

$$\underline{U}_N = \frac{\underline{E}_A\underline{Y}_A + \underline{E}_B\underline{Y}_B + \underline{E}_C\underline{Y}_C}{\underline{Y}_A + \underline{Y}_B + \underline{Y}_C}$$

where $\underline{Y}_{A,}$ $\underline{Y}_{B,}$ $\underline{Y}_C$ are complex phase admittances.

The phase currents may be determined by Eq. (5.9) as

$$\underline{I}_A = (\underline{E}_A - \underline{U}_N)\cdot\underline{Y}_A,\ \underline{I}_B = (\underline{E}_B - \underline{U}_N)\cdot\underline{Y}_B,\ \underline{I}_C = (\underline{E}_C - \underline{U}_N)\cdot\underline{Y}_C$$

The voltages $\underline{U}_{Aa}, \underline{U}_{Bb}, \underline{U}_{Cc}$ in line wires and the phase voltages $\underline{U}_{ab}, \underline{U}_{ca}$ are determined in a common way.

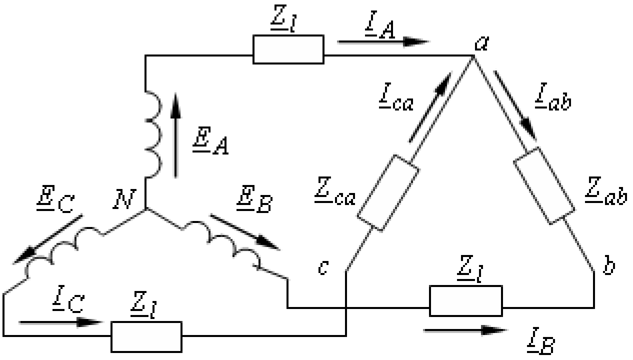


Fig. 5.15. An open circuit in the phase *bc*.

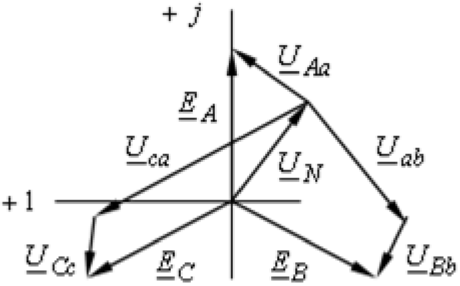


Fig. 5.16. A vector diagram for open circuit rate.

This diagram is drawn for the unbalanced case when $\underline{Z}_{ab} \neq \underline{Z}_{ca}$.

When we have a short circuit in the phase *bc*, a delta-connected circuit will change into asymmetrical wye connection without a neutral wire. Such a connection is depicted in Fig. 5.17. Hence, it appears the neutral voltage shift between the two neutral nodes *N* and *a*:

$$\underline{U}_N = \frac{\underline{E}_A\underline{Y}_A + \underline{E}_B\underline{Y}_B + \underline{E}_C\underline{Y}_C}{\underline{Y}_A + \underline{Y}_B + \underline{Y}_C},$$

where $\underline{Y}_A = \dfrac{1}{\underline{Z}_l + \dfrac{\underline{Z}_{ab} \cdot \underline{Z}_{ca}}{\underline{Z}_{ab} + \underline{Z}_{ca}}} = \dfrac{1}{\underline{Z}_l + \underline{Z}_{abc}}$; and $\underline{Y}_B = \underline{Y}_C = 1/\underline{Z}_l$ are phase complex admittances.

Knowing the neutral voltage shift, one can determine current.

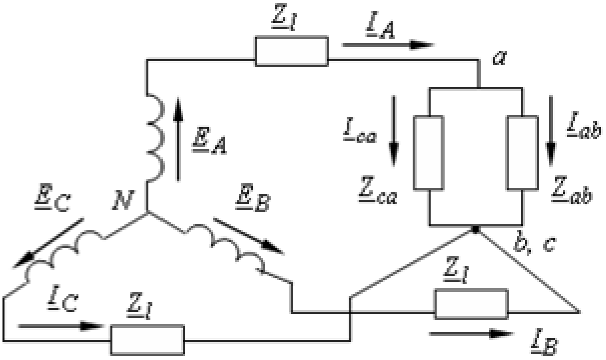


Fig. 5.17. The circuit with a short circuit in the phase *bc*.

Phase currents of the wye connection are found by Ohm's law:

$$\underline{I}_A = \frac{\underline{E}_A - \underline{U}_N}{\underline{Z}_l + \dfrac{\underline{Z}_{ab} \cdot \underline{Z}_{ca}}{\underline{Z}_{ab} + \underline{Z}_{ca}}} = (\underline{E}_A - \underline{U}_N) \cdot \underline{Y}_A$$

The currents in two parallel branches are determined by using so-called "resolving of the total current" (See Ch. 3, eq. 3.32):

$$\underline{I}_{ab} = \underline{I}_A \frac{\underline{Z}_{ca}}{\underline{Z}_{ca} + \underline{Z}_{ab}}; \qquad \underline{I}_{ca} = \underline{I}_A \frac{\underline{Z}_{ab}}{\underline{Z}_{ca} + \underline{Z}_{ab}}$$

When there is an open circuit in a line wire, a three-phase delta connection has become an usual one-phase circuit.

Let's consider the case when the line wire *Aa* is broken in the circuit with a balanced load. The circuit for this case is shown in Fig. 5.18, *a*. So, now we have the one-phase circuit with the input voltage $\underline{U}_{BC}$. For this case the currents across line impedances can be determined by the formula

$$\underline{I}_B = -\underline{I}_C = \frac{\underline{U}_{BC}}{\underline{Z}_{eq}}, \text{ where } \underline{Z}_{eq} = 2\underline{Z}_l + \frac{\underline{Z}_{bc} \cdot (\underline{Z}_{ab} + \underline{Z}_{ca})}{\underline{Z}_{bc} + \underline{Z}_{ab} + \underline{Z}_{ca}}.$$

Then we can calculate the currents through phase impedances:

$$\underline{I}_{bc} = \frac{\underline{U}_{bc}}{\underline{Z}_{bc}}; \quad \underline{I}_{ab} = \underline{I}_{ca} = \frac{\underline{U}_{cb}}{\underline{Z}_{ab} + \underline{Z}_{ca}}$$

where $\underline{U}_{bc} = -\underline{U}_{cb} = \underline{I}_B \cdot \dfrac{\underline{Z}_{bc} \cdot (\underline{Z}_{ab} + \underline{Z}_{ca})}{\underline{Z}_{bc} + \underline{Z}_{ab} + \underline{Z}_{ca}}$.

Now we can find voltages across series impedances $\underline{Z}_{ab}, \underline{Z}_{ca}$:

$$\underline{U}_{ca} = \underline{U}_{ab} = \frac{\underline{U}_{bc}}{2}.$$

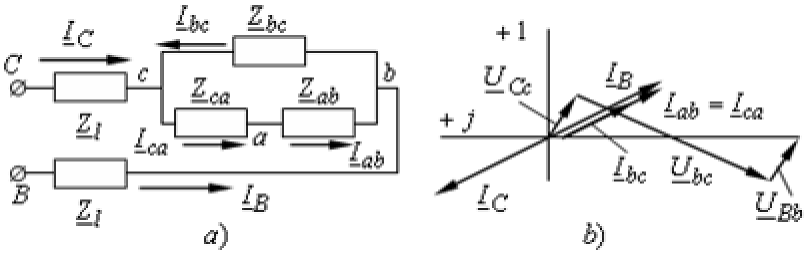


Fig. 5.18. An open circuit in the line wire *Aa*.

The vector diagram for this case is shown in Fig. 5.18, *b*.

5.9 Active, Reactive and Apparent Power in Three-phase Systems

The instantaneous power for a single-phase sinusoidal source varies itself sinusoidally at twice the frequency of the source. The expression for the single-phase sinusoidal source can be applied to each phase of the three-phase system.

The active power of a three-phase system is the sum of the active powers in each phase plus the active power dissipated across the resistance of a neutral wire (if it is not equal zero):

$$P = P_A + P_B + P_C + P_0 \tag{5.19}$$

where P_O is the active power in the resistance of a neutral wire.

The reactive power is the sum of the reactive powers in each phase plus the reactive power in the reactance of a neutral wire:

$$Q = Q_A + Q_B + Q_C + Q_0 \tag{5.20}$$

where Q_O is the reactive power in the reactance of a neutral wire.

The apparent (or total) power

$$S = \sqrt{P^2 + Q^2} \tag{5.21}$$

With a balanced load

$$
\begin{aligned}
&P = 3U_pI_p\cos\varphi_p = \sqrt{3}U_LI_L\cos\varphi_p,\\
&Q = 3U_pI_p\sin\varphi_p = \sqrt{3}U_LI_L\sin\varphi_p,\qquad (5.22)\\
&S = 3U_pI_p = \sqrt{3}U_LI_L
\end{aligned}
$$

where φ_p is the angle between phase voltage U_p and phase current I_p; U_L and I_L are line voltage and line current, respectively.

The equation for the power in a three-phase system is the same either for a wye or a delta connection when the power is expressed in terms of line quantities.

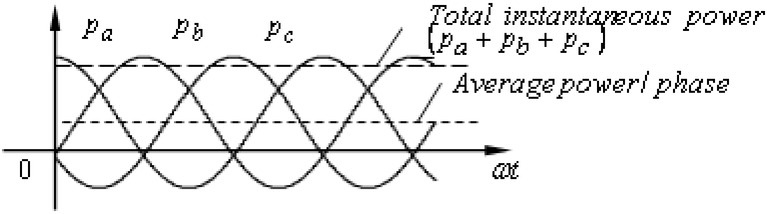


Fig. 5.19.

Before leaving this discussion on three-phase power, let us emphasize the fact that the total instantaneous power for the three-phase system is a constant as illustrated by Fig. 5.19. This stands in sharp contrast to the single-phase case where the single-phase power pulsates at twice the line frequency. Herein, then, lies another significant advantage of the three-phase system. Wherever large loads must be driven mechanically in commercial and industrial applications, the three-phase motor rather than the single-phase motor is used.

5.10 The Advantages of Three-phase System

The popularity of three-phase systems can be explained by the three principal advantages they offer, namely: (a) over long distances it is more economical to transmit alternating current power three-phase then with any other number of phases; (b) the components of three-phase systems, such as three-phase induction motors and three-phase transformers, are simple to manufacture and economical and reliable in service; (c) given certain conditions, including a balanced load on the phases, the instantaneous power of system remains unchanged over a period of the sinusoid.

5.11 The Generation of a Revolving Magnetic Field

A very important feature of polyphase, notably three-phase, systems is the ability to produce a ***revolving magnetic field***. It is defined as the field that has the vector of the resultant magnetic induction constant in magnitude and rotating at a constant angular velocity.

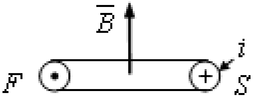


Fig.5.20.

Let us prove that the magnetic field due to a single coil carrying a sinusoidal current is a pulsating and not a revolving magnetic field.

A *pulsation field* is defined as such that has the magnetic induction vector varying (pulsating) along the axis of the current-carrying coil producing the field.

To begin with, we shall refer to Fig.5.20. It shows a coil energized by a sinusoidal current $i = I_m \sin \omega t$. The magnetic field is described by the vector of the magnetic induction B. The direction of B depends on the direction in which the coil is wound and the direction of current flow in it at a given instant of time. Let S be the start and F the finish of the coil winding. When the current flows into the terminal S and flows out of the terminal F (which is assumed to be the positive direction of current flow in the time interval from zero to π), the vector of the magnetic induction points upwards. During the next half-cycle, when the current is negative, the vector B points downwards. Thus, the locus of the tip of the vector B is the axis of the coil.

Let three identical coils be arranged so that their axes are displaced at 120° with respect to one another (Fig. 5.21, *a*).

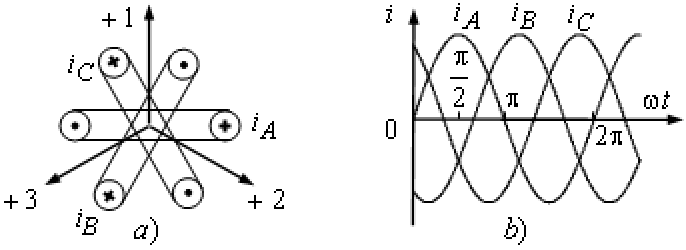


Fig. 5.21 Three identical coils

We connect the coils to a symmetrical three-phase system so that currents flow into the beginnings of the coils and vary as follows

$$i_a = I_m \sin \omega t;\ i_B = I_m \sin\left(\omega t - 120^o\right);\ i_C = I_m \sin\left(\omega t + 120^o\right).$$

Their graphs are shown in Fig.5.21, *b*. Each current produces a pulsating field directed along the axis of the respective coil.

We assume that the positive direction for the first coil is along the +1 axis, for the second - along +2 axis, and for the third - along the +3 axis. Each coil has its magnetic induction. The induction of the first coil is B_1, of the second coil -, and of the third coil - B_3. Fig. 5.22 shows the instantaneous values of B_1, B_2 and B_3 and the resultant induction for the instants of time $\omega t = 0, \pi/2, 3/2\pi.$ The first sketch (Fig. 5.22, *a*) applies to $\omega t = 0$, the second (Fig. 5.22, *b*) to $\omega t = \pi/2$, etc.

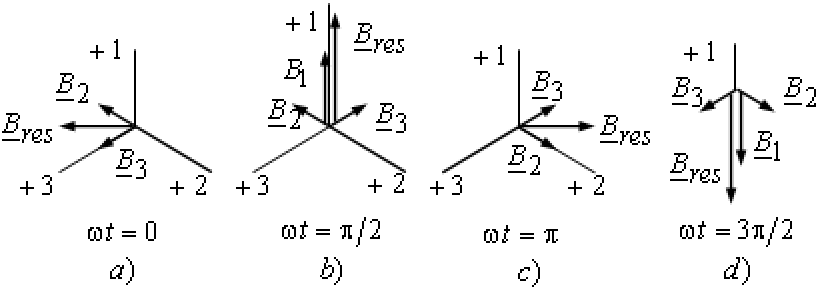


Fig. 5.22

As we advance in time, the vector of the resultant magnetic induction, remaining constant in magnitude $(1.5B_m)$, rotates at an angular velocity ω in the direction from the beginning of the first coil carrying the current $I_m \sin \omega t$ to the beginning of the second coil carrying the current $I_m \sin\left(\omega t - 120^o\right)$ which lags behind the first one by the angle 120^{O}. One may say that the vector of the resultant magnetic induction revolves towards the coil with a lagging current.

If the current $I_m \sin\left(\omega t - 120^o\right)$ be passed through the third coil, and the current $I_m \sin\left(\omega t + 120^o\right)$ through the second, the direction of field rotation will be reversed. If a magnetic circuit consists of a ferromagnetic material fully or in part, its magnetic flux is many times the one in the absence of any ferromagnetic material.

So, to build up the revolving magnetic field, it is usual to fit a solid or hollow ferromagnetic cylinder in each coil and to place the coils in the slots of an external ferromagnetic cylinder (see Fig. 5.23). The revolving magnetic field is utilized in electric motors.

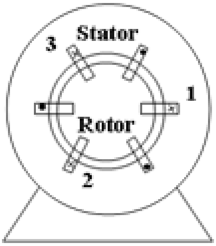


Fig. 5.23

5.12. Operating Principle of the Induction Motor

The three-phase ***induction motor*** is the most commonly used type. Normally an induction motor consists of an angular core (the stator) which carries the primary coils in slots on its inner periphery.

The primary coils are commonly arranged for a three-phase supply and serve to produce a revolving magnetic field. The stator encircles a cylindrical rotor carrying the secondary winding in slots on its outer periphery.

The rotor winding may be one of two types. These are *squirrel-cage* rotor winding and *slip-ring* (or *wound-rotor*) winding.

In a squirrel-cage machine, the rotor winding forms a complete closed circuit in itself.

The rotor winding of a slip-ring machine is completed when the slip rings are connected either directly together or through some resistance external to the machine. The rotor shaft is coupled to the shaft of the driven mechanism.

We assume that the rotor is stationary at some instant of time.

The revolving magnetic field due to the stator winding cuts across the stationary rotor winding at synchronous speed and induces an e.m.f. in it. The e.m.f. will give rise to a current which, by Lenz's law, sets up a magnetic field tending to reduce the magnetic field that causes the induced current.

Interaction between the rotor current and the revolving magnetic field causes the rotor to rotate in the same direction as the revolving magnetic field (which can be proved by the left-hand rule).

Under steady-state conditions, the speed of the rotor, is 95-98 per cent of the speed of revolving magnetic field of a stator

$\omega_r = (0.95 \div 0.98)\omega$, or is other than synchronous. Hence another name of this type of motor is *asynchronous motor*.

The speed of the rotor cannot be equal to synchronous speed. If it were equal to the latter, the revolving magnetic field would not be able to cut the secondary conductors and there would not be any current induced in the secondary winding and no interaction between the revolving field and the rotor, and the motor would not run.

5.13. The *a* Operator of a Three-phase Systems

Let the complex number e^{j120^o} equal to unity in magnitude be designated by *a* and termed the *operator of three-phase system*. Then

$$e^{j240^o} = (e^{j120^o})^2 = a^2.$$

The three vectors, *1*, *a*, and a^2, represent a symmetrical three-phase system (see Fig. 5.24).

$$1 + a + a^2 = 0 \tag{5.23}$$

Multiplying a vector by *a* turns it by 120°counter-clockwise without affecting its magnitude.

Multiplying a vector by a^2 turns it by 240°counter-clockwise or, which is the same, 120° clockwise.

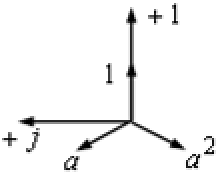


Fig. 5.24

Using the operator *a*, one can express $\underline{E}_B$ and $\underline{E}_C$ of a symmetrical three-phase system in terms of $\underline{E}_B$:

$$\left.\begin{aligned} \underline{E}_B &= a^2 \underline{E}_A; \\ \underline{E}_C &= a \underline{E}_A. \end{aligned}\right\}$$

5.14. Resolution of an Unsymmetrical System into Symmetrical Components

Any unsymmetrical system of three currents, voltages or fluxes of the same frequency which we designate $\underline{A}$, $\underline{B}$ $\underline{C}$ in general, can be resolved into three symmetrical systems of vectors with zero, positive and negative phase sequence.

These are called the *symmetrical components* of an unsymmetrical system.

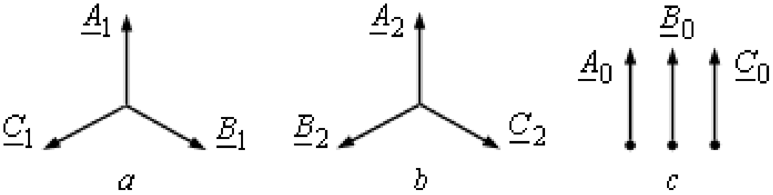


Fig. 5.25. A positive, negative and zero phase-sequence systems

A positive phase-sequence system (Fig. 5.25, *a*) is formed by three vectors $\underline{A}_1$, $\underline{B}_1$, $\underline{C}_1$, equal in magnitude and spaced 120° apart, so that $\underline{B}_1$ lags 120°behind $\underline{B}_1$, and $\underline{C}_1$ leads $\underline{A}_1$ by 120°.

Using the operator a of a three-phase system (see Sec. 5. 13), we may write:

$$\left.\begin{aligned} \underline{B}_1 &= a^2\underline{A}_1; \\ \underline{C}_1 &= a\underline{A}_1. \end{aligned}\right\} \tag{5.24}$$

A negative phase sequence A negative phase-sequence system (Fig. 5.25, *b*) is formed by three vectors $\underline{A}_2$, $\underline{B}_2$, $\underline{C}_2$ equal in magnitude and spaced 120° apart so that $\underline{B}_2$ leads $\underline{A}_2$ by 120°:

$$\left.\begin{aligned} \underline{B}_2 &= a\underline{A}_2; \\ \underline{C}_2 &= a^2\underline{A}_2. \end{aligned}\right\} \tag{5.25}$$

A zero phase-sequence system (Fig. 5.25, *c*) is formed by three vectors $\underline{A}_0$, $\underline{B}_0$, $\underline{C}_0$ coincident in phase:

$$\underline{A}_0 = \underline{B}_0 = \underline{C}_0. \tag{5.26}$$

The specified three vectors *A*, *B*, and *C* may be expressed in terms of the vectors of symmetrical systems as follows:

$$\left.\begin{aligned} \underline{A} &= \underline{A}_0 + \underline{A}_1 + \underline{A}_2; \\ \underline{B} &= \underline{B}_0 + \underline{B}_1 + \underline{B}_2; \\ \underline{C} &= \underline{C}_0 + \underline{C}_1 + \underline{C}_2. \end{aligned}\right\} \tag{5.27}$$

Noting Eqs. (5.24) and (5.25), we may rewrite Eq. (5.27) thus:

$$\underline{A} = \underline{A}_0 + \underline{A}_1 + \underline{A}_2; \tag{5.28}$$

$$\underline{B} = \underline{A}_0 + a^2\underline{A}_1 + a\underline{A}_2; \tag{5.29}$$

$$\underline{C} = \underline{A}_0 + a\underline{A}_1 + a^2\underline{A}_2. \tag{5.30}$$

In order to find $\underline{A}_0$, $\underline{A}_1$ and $\underline{A}_2$ in terms of the specified vectors *A*, *B*, and *C*, we add together Eqs. (5.28), (5.29) and (5.30), and note that $1 + a + a^2 = 0$. We get

$$\underline{A}_0 = \frac{1}{3}\left(\underline{A} + \underline{B} + \underline{C}\right). \tag{5.31}$$

In words, to find $\underline{A}_0$, one should add together the three specified vectors and take one-third of the sum.

To find $\underline{A}_1$ we add together Eq. (5.28) and (5.29) multiplied by *a*, and Eq. (5.30) multiplied by a^2 and get

$$\underline{A}_1 = \frac{1}{3}\left(\underline{A} + a\underline{B} + a^2\underline{C}\right) \tag{5.32}$$

Again stating this in words, $\underline{A}_1$ is equal to one-third of the sum of the vector $\underline{A}$, the vector $\underline{B}$ turned through 120° counter-clockwise, and the vector $\underline{C}$ turned through 120° clockwise.

$\underline{A}_2$ is given by the sum of Eq. (5.28), Eq. (5.29) multiplied by a^2, and Eq. (5.30) multiplied by *a*, or

$$\underline{A}_2 = \frac{1}{3}\left(\underline{A} + a^2\underline{B} + a\underline{C}\right). \tag{5.33}$$

5.15. The Method of Symmetrical Components

The method of symmetrical phase-sequence components is fundamentally a mathematical technique for practical solution of polyphase (including three-phase) circuit problems under dissymmetry conditions (which may be due to faults or unbalanced loads, etc.).

The method of symmetrical components as applied to problems of network faults involves the use of equivalent circuit diagrams in which all units of the system should be represented by their respective impedances. However, the phase impedance of a piece of equipment with zero phase-sequence (ZPS) – (Z_0) may be other than it is with positive phase-sequence (Z_P). Furthermore, the positive phase-sdsequence (PPS) impedance of some machines (such as induction motors) cannot be the same as their negative phase-sequence (NPS) impedance (Z_n).

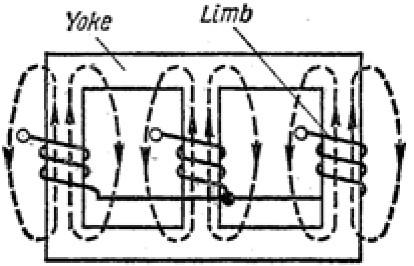


Fig. 5.26. The three-phase three-limb transformer.

Accordingly, calculations have to be performed for each phase sequence separately. Then, by the superposition principle, one can find the current or voltage for any portion of the system as the sum of ZPS, PPS and NPS currents or voltages.

Before going any further, consider why Z_P, Z_n and Z_0 differ from one another.

Let the phase impedances of a three-phase transmission line be Z_{p1}, Z_{n1} and Z_{01} for positive, negative and zero phase-sequences respectively. The PPS impedance of the line, Z_{p1}, is equal to its NPS impedance Z_{n1}, but is not equal to its ZPS impedance Z_{01}. This is because the phase inductance of a three-phase line is different for positive phase-sequence than it is for zero phase-sequence. The difference is due to two causes.

In the case of PPS and NPS the inductance of a transmission line is determined solely by the geometrical dimensions of the loops formed by the line conductors. In the case of ZPS the inductance of the same line depends also on the geometrical dimensions of the loops formed by the line conductors and the neutral wire.

In the case of PPS and NPS the e. m. f.s induced in the line conductors are the vector sums of e. m. f. s due to the line currents spaced at 120° in time phase. In the case of ZPS the e. m. f.s induced in the line conductors are due to ZPS currents whose vectors are coincident in time phase.

In the three-phase three-limb transformer of Fig. 5.26 the ZPS phase impedance is not equal to the PPS phase impedance, but the PPS impedance is equal to the NPS impedance. The explanation is this. The ZPS magnetic fluxes $\underline{\Phi}_0$ are all in time phase and link through the air (as shown by the dotted lines in Fig. 5.26) and not through the adjacent limbs. The PPS fluxes, are spaced at 120° in time phase and link through the adjacent limbs. So, with the same magnitude of ZPS and PPS currents, $\underline{\Phi}_0$ is considerably smaller than $\underline{\Phi}_1$.

The difference between Z_0, Z_P and Z_n is still greater in induction motors. The reason lies in the following. The PPS voltages produce a revolving magnetic field in the motor. The field carries along the rotor, so that it rotates at asynchronous speed ω_{rot}. The NPS voltages also produce a revolving magnetic field, but its sense of rotation is opposite to that of the field caused by the PPS voltages.

The ZPS voltages produce no revolving magnetic field at all. Instead, they establish pulsating fluxes around the stator winding, which link through the air gap between the stator and the rotor, exactly as the ZPS fluxes do in the three-phase three-limb transformer of Fig. 5.26.

The phase impedances of a motor for a given phase-sequence depend not only on the resistance and reactance of the stator phase windings, but also on the resistance and reactance of the rotor phase windings for the same phase sequence. In turn, the inductive reactance of a rotor phase is directly proportional to frequency. The PPS revolving field induces in the rotor currents with a frequency $(\omega - \omega_{rot})$ which is 0.02 to 0.05 of synchronous speed ω. On the other hand, the frequency of the rotor currents due to the ZPS revolving field is $\omega + \omega_{rot}$ = (1.98 to 1.95) ω. Since the rotor currents in either case (PPS and NPS) differ in frequency, the respective (PPS and NPS) phase impedances also differ.

As to the ZPS phase impedances of a motor, they differ from the. PPS and NPS components because the magnetic fluxes due to the ZPS currents do not link through the rotor, but link through the gap between the stator and the rotor.

The procedure by the method of symmetrical components consists in that the unsymmetrical system in question is resolved into three symmetrical systems of voltages (or currents, or e.m.f.s). Then, by the superposition principle, operation of the original system is presented as the result of the superposition of the three symmetrical sets of quantities.

In the PPS symmetrical system, all voltages, currents and e.m.f.s contain only PPS components, and all the pieces of electrical plant (transmission lines, rotary machines and three-phase transformers) are represented by their PPS impedances, Z_P, in equivalent circuit diagrams.

In the NPS symmetrical system, all voltages, currents and e.m.f.s contain only NPS components, and all the pieces of electrical plant are represented by their NPS impedances, Z_n in equivalent circuit diagrams.

In the ZPS symmetrical system, all voltages, currents and e.m.f.s contain only ZPS components, and all pieces of equipment are represented by their ZPS impedances, Z_0, in equivalent circuit diagrams.

In order to convert a given unsymmetrical system into three symmetrical ones, three unsymmetrical voltages $\underline{U}_A$, $\underline{U}_B$ and $\underline{U}_C$ are injected in the system at the point of dissymmetry.

On the basis of the compensation theorem, these voltages can replace the three unequal impedances at the fault point that have resulted in the dissymmetry of the entire system.

Next, the system of the three unsymmetrical voltages is resolved into three symmetrical systems (as explained in the preceding paragraph), and the main vectors $\underline{U}_0$, $\underline{U}_1$ and $\underline{U}_2$ are determined.

Similarly, the system of three unsymmetrical currents $\underline{I}_A$, $\underline{I}_B$ and $\underline{I}_C$ is resolved into three symmetrical systems of currents, and the main vectors $\underline{I}_0$, $\underline{I}_1$ and $\underline{I}_2$ are determined.

Thus, the method of symmetrical components involves six unknowns: $\underline{U}_0$, $\underline{U}_1$, $\underline{U}_2$, $\underline{I}_0$, $\underline{I}_1$, and $\underline{I}_2$ in terms of which any voltages and currents in a given circuit can be determined.

To find the six unknowns, we write six equations, one each for the three symmetrical systems, and the other three for the section of the circuit where conditions of dissymmetry have been observed. The form of the last three equations depends on the nature of dissymmetry in the circuit.

5.16. Application of the method of symmetric components for symmetrical circuits with asymmetrical system of generator e.m.f.s

Let's consider a three-phase electric circuit with the asymmetrical generator and the symmetric load which is presented in Fig. 5.27.

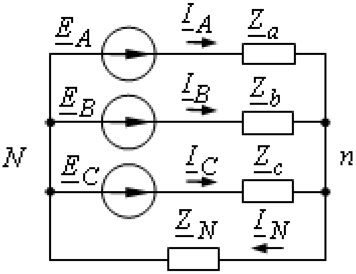


Fig. 5.27. The asymmetrical generator with a symmetric load

Phase impedances of a load have various meanings for various components, but they are equal for each symmetrical component.

The initial data are: the asymmetrical system of phase generator e.m.f.s $\underline{E}_A, \underline{E}_B, \underline{E}_C$, the phase impedances of the load for various sequences $\underline{Z}_1, \underline{Z}_2, \underline{Z}_0$, and the impedance of a neutral wire $\underline{Z}_N$.

Taking into account Eqs. (5.27) - (5.30) one can expand the asymmetrical system into the symmetrical components for phase generator e.m.f.s. The corresponding electric circuit obtained as a result of such a transformation, is presented in Fig. 5.28.

Find the symmetrical components of e.m.f.s using Eqs. (5.31) – (5.33):

$$\begin{cases} \underline{E}_1 = \frac{1}{3}(\underline{E}_A + a\underline{E}_B + a^2\underline{E}_C); \\ \underline{E}_2 = \frac{1}{3}(\underline{E}_A + a^2\underline{E}_B + a\underline{E}_C); \\ \underline{E}_0 = \frac{1}{3}(\underline{E}_A + \underline{E}_B + \underline{E}_C). \end{cases} \tag{5.34}$$

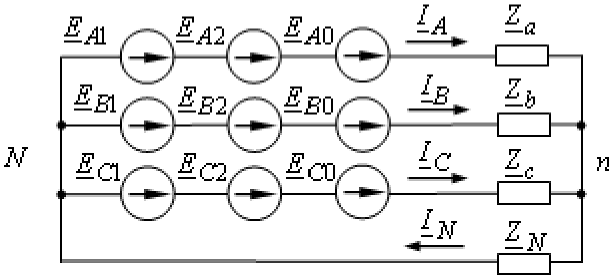


Fig.5.28. The transformed electric circuit.

Let's take advantage of the superposition method and transfer to three equivalent circuits of substitution for each of groups of symmetrical components (Fig. 5.29, *a, b, c).*

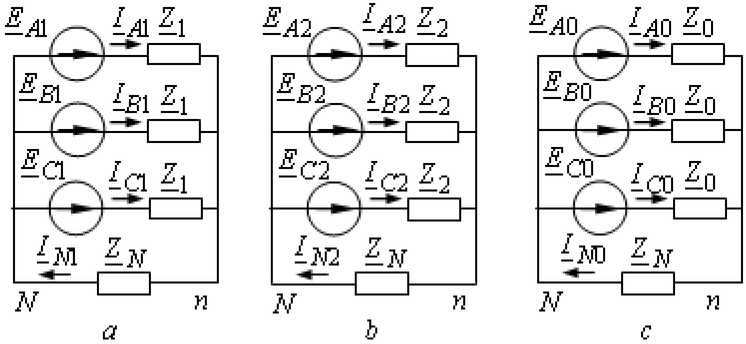


Fig. 5.29. Three equivalent circuits of substitution.

Let's fulfil the calculation of currents for each symmetrical component of e.m.f.

As for each symmetrical component the three-phase circuit is completely symmetrical, it is more convenient at the calculation to transfer to single-phase circuits of substitution and make the calculation for one phase termed as basic (usually it is phase *A*).

In symmetrical rate for positive and negative sequences the current in a neutral wire is equal zero and, hence, the voltage $\underline{U}_{nN1} = \underline{U}_{nN2} = 0$. Therefore the impedance in a neutral wire does not influence on the symmetrical phase currents of these sequences, and should not be considered in the circuits of substitution of these sequences. For current calculations of a basis phase it is enough to observe the following equivalent circuits (Fig. 5.30, *a*, *b)*. For zero sequence the potentials of the beginnings of linear wires are equal among themselves (as $\underline{E}_{0A} = \underline{E}_{0B} = \underline{E}_{0C} = \underline{E}_0$). Therefore they may be combined in one node. Taking into account the equality of complex load impedances $\underline{Z}_a = \underline{Z}_b = \underline{Z}_c = \underline{Z}_l$ we will obtain a substitution equivalent circuit which is presented in Fig. 5.30, *c*.

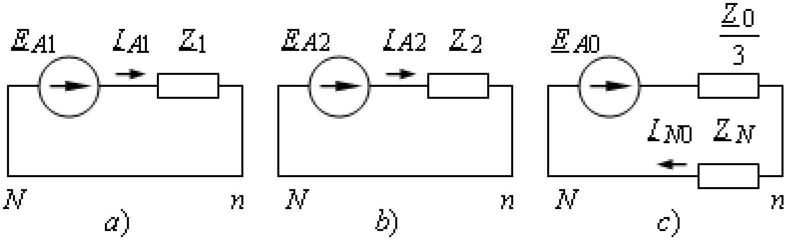


Fig.5.30. Equivalent circuits of substitution.

Using Ohm's law, we define currents for each equivalent circuit

$$\underline{I}_{A1} = \frac{\underline{E}_{A1}}{\underline{Z}_1}; \quad \underline{I}_{A2} = \frac{\underline{E}_{A2}}{\underline{Z}_2}; \quad \underline{I}_{N0} = \frac{3\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N}. \tag{5.35}$$

The currents of zero sequence in all phases coincide and become isolated through the neutral wire

$$\underline{I}_{N0} = \underline{I}_{A0} + \underline{I}_{B0} + \underline{I}_{C0} = 3\underline{I}_{A0}. \tag{5.36}$$

The component of zero sequence from here

$$\underline{I}_{A0} = \frac{\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N}. \tag{5.37}$$

On knowing symmetrical components of currents we can determine the actual linear currents in the initial three-phase circuit:

$$\underline{I}_A = \underline{I}_1 + \underline{I}_2 + \underline{I}_0 = \underline{I}_{A1} = \frac{\underline{E}_{A1}}{\underline{Z}_1} + \frac{\underline{E}_{A2}}{\underline{Z}_2} + \frac{\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N};$$

$$\underline{I}_B = a^2 \cdot \underline{I}_1 + a \cdot \underline{I}_2 + \underline{I}_0 = \frac{a^2 \cdot \underline{E}_{A1}}{\underline{Z}_1} + \frac{a \cdot \underline{E}_{A2}}{\underline{Z}_2} + \frac{\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N}; \qquad (5.38)$$

$$\underline{I}_C = a \cdot \underline{I}_1 + a^2 \cdot \underline{I}_2 + \underline{I}_0 = \frac{a \cdot \underline{E}_{A1}}{\underline{Z}_1} + \frac{a^2 \cdot \underline{E}_{A2}}{\underline{Z}_2} + \frac{\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N}.$$

The actual current in a neutral wire does not contain the components of positive and negative sequence. Therefore

$$\underline{I}_N = \underline{I}_{N0} = \frac{3\underline{E}_0}{\underline{Z}_0 + 3\underline{Z}_N} \qquad (5.39)$$

Complex impedances of phases of static three-phase receivers (lighting loadings, heaters etc.) do not depend on an aspect of sequence of a supply voltage, for such receivers $\underline{Z}_1 = \underline{Z}_2 = \underline{Z}_0$.

The calculation of currents of such receivers is simpler to fulfil by usual methods. For three-phase receivers for which the phase impedances for currents of various sequences essentially differ (first of all, for rotating electrical motors), the calculation of currents at any asymmetric voltage should be exclusively made by the method of symmetrical components.

5.17. Application of the method of symmetric components for asymmetrical circuits with symmetrical system of generator e.m.f.s

The majority of powerful electric circuits work in the symmetrical rates. Sharp asymmetry in the symmetrical three-phase circuits has emergency character and appears, as a rule, in any one cross-section.

By compensation theorem the non-symmetrical passive part of the circuit is exchanged by the system of three asymmetrical currents and three voltages. Then these currents and voltages are represented in the form of the sum of three while unknown symmetrical currents and three voltages. Thus, at the first stage the total unknown values are equal six.

Three rated equations are obtained on the basis of circuit designs on one phase, made for each of three sequences. The remaining three equations are written on the basis of concrete relationships between currents and voltages, formed in an asymmetry place.

There are two kinds of asymmetry: a cross asymmetry and the longitudinal one.

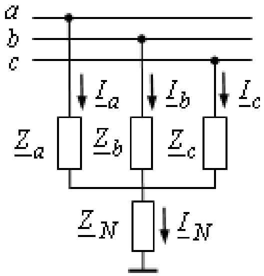


Fig. 5.31. The three-phase circuit for a case of cross asymmetry.

Cross asymmetry is caused by the difference of the resistances which have been switched on between phases and a neutral wire, or between separate phases. Such situation appears, when the asymmetric short circuit occurs or the asymmetric load is connected to a circuit. The section of the three-phase circuit for a general case of cross asymmetry is presented in Fig. 5.31. In general, cross asymmetry (Fig. 5.31) is characterised by following equations:

$$\begin{cases} \underline{U}_a = \underline{Z}_a \underline{I}_a + 3\underline{Z}_n \underline{I}_0; \\ \underline{U}_b = \underline{Z}_b \underline{I}_b + 3\underline{Z}_n \underline{I}_0; \\ \underline{U}_c = \underline{Z}_c \underline{I}_c + 3\underline{Z}_n \underline{I}_0. \end{cases} \tag{5.40}$$

At two-phase short circuit (for example, between phases *A* and *B*) the additional equations look like

$$\begin{cases} \underline{U}_a - \underline{U}_b = 0; \\ \underline{I}_a + \underline{I}_b = 0; \\ \underline{I}_c = 0, \end{cases} \tag{5.41}$$

and at two-phase earthing (for example, phases *A* and *B*)

$$\begin{cases} \underline{U}_a = 0; \\ \underline{U}_b = 0; \\ \underline{I}_c = 0. \end{cases} \tag{5.42}$$

At uniphase short circuit (for example, phase *A*) the additional equations are the following:

$$\begin{cases} \underline{U}_a = 0; \\ \underline{I}_b = 0; \\ \underline{I}_c = 0, \end{cases} \tag{5.43}$$

The calculation of short circuit currents in various points of electric power systems is the important engineering problem.

The longitudinal asymmetry in the three-phase circuit appears when unequal resistances are switched on in a cross-cut of phases. Such situation occurs, for example, when there is an open circuit in some linear wire. The circuit of a section of three-phase circuit for a general case of the longitudinal asymmetry is presented in Fig. 5.32. The longitudinal asymmetry for this case is characterised by following additional equations:

$$\begin{cases} \underline{U}_{Aa} = \underline{Z}_A \underline{I}_A; \\ \underline{U}_{Bb} = \underline{Z}_B \underline{I}_B; \\ \underline{U}_{Cc} = \underline{Z}_C \underline{I}_C. \end{cases} \tag{5.44}$$

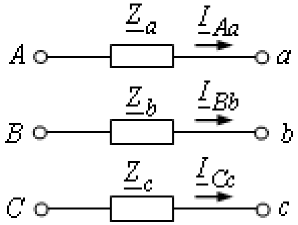


Fig.5.32. The three-phase circuit for a case of longitudinal asymmetry

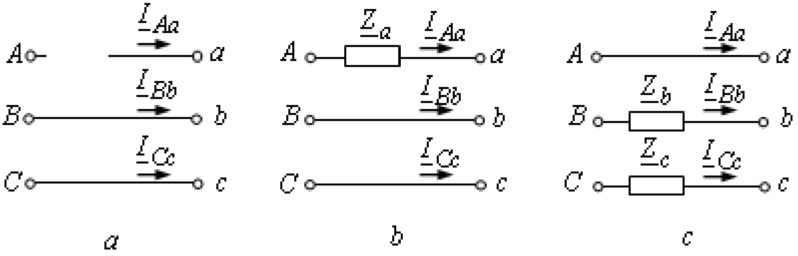


Fig. 5.33. Some special cases of the longitudinal asymmetry.

For some special cases of the longitudinal asymmetry, which circuits are presented in Fig. 5.33 the additional equations will be the following: – for the circuit presented in Fig. 5.33, *a*

$$\begin{cases} \underline{I}_A = 0; \\ \underline{U}_{Bb} = 0; \\ \underline{U}_{Cc} = 0; \end{cases} \tag{5.45}$$

– for the circuit presented in Fig. 5.33, *b*

$$\begin{aligned} &\underline{U}_{Aa} = \underline{Z}_A \underline{I}_A; \\ &\underline{U}_{Bb} = 0; \\ &\underline{U}_{Cc} = 0; \end{aligned} \tag{5.46}$$

– for the circuit presented in Fig. 5.33, *c*

$$\begin{cases} \underline{U}_{Aa} = 0; \\ \underline{U}_{Bb} = \underline{Z}_B \underline{I}_B; \\ \underline{U}_{Cc} = \underline{Z}_C \underline{I}_C. \end{cases} \tag{5.47}$$

Example 5.1. The three-phase engine with windings which are wye-connected, obtains the power supply through a three-wire line of transfers at line voltage 380 V. There is an emergency rate (an open circuit of the phase *A*) in the circuit.

At certain conditions the generator can go on working, obtaining a supply through two phases. The complex impedances of phases $\underline{Z}_1 = 3{,}6 + j3{,}6\ \ \Omega$, $\underline{Z}_2 = 0{,}15 + j0{,}5\ \ \Omega$. Determine the currents in feeders and the voltages $\underline{U}_{AA'}$, $\underline{U}_{A'n}$, $\underline{U}_{B'n}$, $\underline{U}_{C'n}$, $\underline{U}_{Nn}$.

Solution: The equivalent circuit of substitution is presented in Fig. 5.34. Knowing line voltages we determine phase electromotive forces: $E = 380/\sqrt{3} = 220\ V$. By compensation theorem we will mentally substitute the asymmetrical passive part of the circuit by e.m.f. sources and compose the circuits of positive, negative and zero sequences which one can see in Fig. 5.35.

A single-phase circuit design of zero sequence is broken, as there is no a fourth wire, and, hence, there are no the currents of zero sequence $\underline{I}_0 = 0$.

The main equations for circuits of positive and negative sequences:

$$\underline{Z}_1 \underline{I}_1 + \underline{U}_1 = \underline{E}_1 \tag{a}$$

$$\underline{Z}_2 \underline{I}_2 + \underline{U}_2 = 0 \tag{b}$$

and additional equations

$$\begin{cases} \underline{I}_A = 0; \\ \underline{U}_{Bb} = 0; \\ \underline{U}_{Cc} = 0; \end{cases} \qquad (c)$$

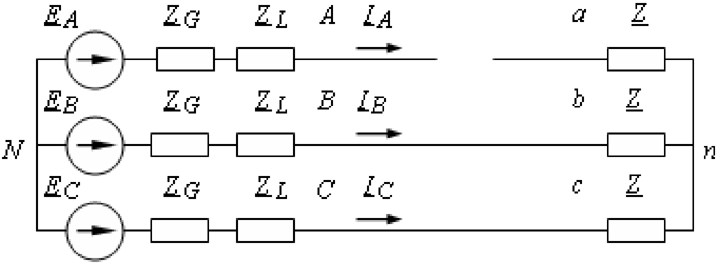


Рис. 5.34. The equivalent circuit of substitution.

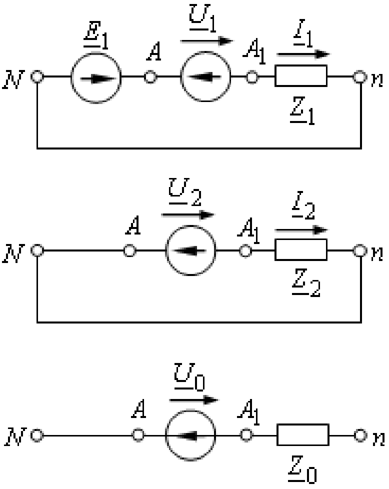


Рис. 5.35. The circuits of positive, negative and zero sequences.

Having expressed in these equations the currents and voltages through their symmetrical components, we will obtain:

$$\underline{I}_1 + \underline{I}_2 = 0 \quad or \quad \underline{I}_1 = -\underline{I}_2; \qquad (c)$$

$$a^2\underline{U}_1 + a\underline{U}_2 + \underline{U}_0 = 0; \qquad (d)$$

$$a\underline{U}_1 + a^2\underline{U}_2 + \underline{U}_0 = 0. \qquad (e)$$

On solving equations (*d*) and (*e*) we will find that $\underline{U}_2 = \underline{U}_1 = \underline{U}_0$. On substituting $\underline{U}_2 = \underline{U}_1$ and $\underline{I}_1 = -\underline{I}_2$ into Eq. (*b*), we will get

$$-\underline{Z}_2 \underline{I}_1 + \underline{U}_1 = 0 .$$

Now we subtract this equation from Eq. (*a*) and obtain the following result: $(\underline{Z}_1 + \underline{Z}_2)\underline{I}_1 = \underline{E}_1$.

Hence, we can calculate: – the currents in feeders:

$$\underline{I}_1 = -\frac{\underline{E}_1}{\underline{Z}_1 + \underline{Z}_2} = \frac{220}{(3.6 + j3.6) + (0.12 + j0.5)} = 39.6 e^{-j47^o} \text{ A};$$

$$\underline{I}_2 = -\underline{I}_1 = 39.6 e^{j133^o} \text{ A};$$

– the phase currents $\underline{I}_A = 0 \ A$;

$$\underline{I}_B = a^2 \underline{I}_1 + a\underline{I}_2 = (a^2 - a)\underline{I}_1 = -50 - j46.65 = 68.4 e^{-j137^o} \ A;$$

$$\underline{I}_C = a\underline{I}_1 + a^2 \underline{I}_2 = (a - a^2)\underline{I}_1 = 50 + j46.65 = 68{,}4 e^{j43^o} \ A.$$

And $\underline{U}_2 = -\underline{Z}_2 \underline{I}_2 = (0.15 + j0.5)39.6 e^{j133^o} = 20.7 e^{j25^o} V$;

$$\underline{U}_{AA'} = \underline{U}_1 + \underline{U}_2 + \underline{U}_0 = 3\underline{U}_2 = 62.1 e^{j25^o} V;$$

– the phase voltages: $\underline{U}_{A'n} = \underline{Z}_1 \underline{I}_1 + \underline{Z}_2 \underline{I}_2 = (\underline{Z}_1 - \underline{Z}_2)\underline{I}_1 =$

$$= (3.6 + j3.6 - 0.15 - j0.5)39.6 e^{-j47^o} = 183.5 e^{-j5^o} \ V;$$

$$\underline{U}_{B'n} = a^2 \underline{I}_1 \underline{Z}_1 + a\underline{I}_2 \underline{Z}_2 = (a^2 \underline{Z}_1 - a\underline{Z}_2)\underline{I}_1 = 203 e^{-j116^o} \ V;$$

$$\underline{U}_{C'n} = a\underline{I}_1 \underline{Z}_1 + a^2 \underline{I}_2 \underline{Z}_2 = (a\underline{Z}_1 - a^2 \underline{Z}_2)\underline{I}_1 = 218.8 e^{j114^o} \ V.$$

The voltage between the neutral points of the generator and the engine (the bias neutral voltage):

$$\underline{U}_{nN} = \underline{U}_0 = 20{,}7 \, e^{\,j25^o} \ V.$$

The procedure observed in this example may be used at the calculation of three-phase circuits for any case of the longitudinal asymmetry.

Summary review questions

1. Make the definition of a three-phase circuit and describe the process of deriving of three-phase EMF system.

2. What advantages have the power supply three-phase systems?.

3. How are winding connections of the three-phase generator "wye" and "delta" organized?

4. What is a three-phase network phase?

5. What voltages and currents are termed as linear?

6. What voltages and currents are termed as phase?

7. What relationships are between phase and line loading voltages if we have a wye-connection? What do you know about the relationships between phase and line currents in this case?

8. What relationships are between phase and line currents if we have a delta-connection? And what can you say about the relationship between phase and line loading voltages in this case?

9. Make definition of the symmetrical, asymmetric and uniform load. Illustrate with the examples.

10. What is the phase sequence?

11. What is called the symmetrical system of EMFs (voltages, currents)?

12. What is a neutral voltage shifting or a bias neutral voltage ? How is it possible to calculate theoretically?

13. How may phase voltages be found in the presence of a bias neutral voltage?

14. For what is a neutral (or zero) wire used in the three-phase circuit? How is it possible to find the value of a current in a neutral wire theoretically?

15. Under what condition doesn't the presence of a neutral wire influence on a mode of operation?

16. Why is the cross-section of a neutral wire less than cross-section of a linear wire?

17. What can you say about the short circuit rate in one of phases for wye-connection without a neutral wire at the symmetrical load?

18. What can you say about the short circuit rate in one of phases for wye-connection with a neutral wire at the symmetrical load? Explain the difference.

19. What can you say about an open circuit rate in one of phases for delta-connection at the symmetrical load?

20. What can you say about an open circuit rate in a line wire for delta-connection at the symmetrical load? Illustrate with the examples.

22. How can one define the apparent power of the three-phase circuit at the symmetrical load?

23. How can one define the apparent power of the three-phase circuit at the asymmetrical load?

How the chain total output is spotted at the asymmetric loading?

24. What requirement is fulfilled for active and reactive powers of the three-phase network and not fulfilled for the apparent one?

25. Which magnitudes must one use for power calculation that these expressions don't depend on the circuit design of the symmetrical load?

26. What requirements are necessary for making of a circular rotating magnetic field?

27. What principle of action has the asynchronous engine with the *squirrel-cage* rotor?

28 What principle of action has a synchronous engine?

29. In what cases are the components of zero sequence absent in the linear currents?

30. How is the resistance of a neutral wire considered at use of the method of symmetrical components?

31. Identify the salient features of a balanced three-phase voltage system of the Y and Δ types.

32. In balanced three-phase systems, explain what gives rise to the presence of the factor $\sqrt{3}$ in expressions for power and between line and phase quantities as well.

33. The instantaneous power per phase in a single-phase circuit contains a double-frequency sinusoidal component. Is a similar component present in the total instantaneous power of a three-phase system? Explain.

34. Three fixed and balanced load impedances are placed across a balanced three-phase voltage source. What is the relationship between the resulting line currents when the load impedances are first connected in a Δ arrangement and then in a Y arrangement?

Problems

5.1. In a symmetrical three-wire circuit there was a phase break. What will a voltmeter connected between neutral points of a source and the receiver show?

5.2. The windings of the three-phase electromotor are calculated for operation in the nominal rate at voltage $U_n = 380\ V$. With which line voltages can this electromotor work?

5.3. If the phase voltage of one phase of a Y-connected three-phase source is know to be $\underline{U}_{na} = 120e^{j30^o} V$, what are the expressions for the other two phase voltages? What is the expression for the line voltage $\underline{U}_{ab}$?

5.4. Three-phase voltage (220 V) is applied to a balanced delta-connected load in the manner illustrated in Fig. P5.4. The rms value of the phase current measured between points *a* and *b* is $\underline{I}_{ab} = 10e^{-j30^o}$.

(a) Find the line current. Draw the phasor diagram showing clearly the line voltages, phase currents, and line currents.

(b) Compute the total power received by the three-phase load.

(c) Find the value of the resistive portion of the phase impedance.

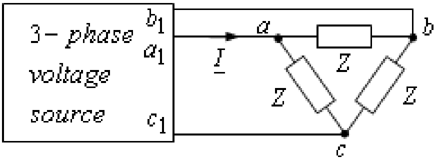


Fig. P5.4

5.5. The delta-connected load of Fig. P8.31 consists of phase impedances each equal to $15 + j20$.

(a) Find the phasor current in each line.

(b) What is the power consumed per phase?

(c) What is the phasor sum of the three line currents?

5.6. A three-phase, 208-V generator supplies a total of 1800 W at a line current of 10 A when three identical impedances are arranged in a wye connection across the line terminals of the generator. Compute the resistive and reactive components of each phase impedance.

5.7. A balanced three-phase wye-connected load has an impedance of $4e^{j50^o}$ Ω from line to neutral. Moreover, from line *a* to neutral *n* the voltage is $\underline{U}_{an} = 20e^{j30^o}$.

(a) What is the current in phases *b* and *c*?

(b) What is the voltage from line *b* to neutral?

(c) What is the phasor expression for the voltage from line *a* to line *c*, i.e., $\underline{U}_{ac}$?

5.8. In the electrical circuit (Fig. P5.8) $R_b = R_c = 2R_a$. How will the currents change after a safety fuse S_{f1} burns out?

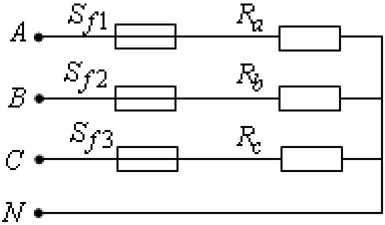


Fig. P5.8

5.9. The impedances ($\underline{Z}_a = \underline{Z}_b = \underline{Z}_c = 22e^{j30^o}$ Ω) are connected in “delta". The line voltage $U_l = 220\ V$. Define phase and line currents, the voltages across each phase and wattmeter readings P_1 and P_2 when *a*) there is a nominal work in the circuit;

b) an “open circuit” in line *C*;

c) an “open circuit” in phase *BC*.

5.10. In every phase of the load connecting in “delta” a current lags behind a voltage by 53°. All the impedances are equal 100 Ω. Calculate phase and line currents and the active power of the whole circuit if a line voltage U_L = 380 V. Draw a vector diagram.

5.11. Define voltages across the load impedances and currents in phases if three active resistances $R_a = 20\ \Omega$, $R_b = 40\ \Omega$, $R_c = 50\ \Omega$ in “wye” connection are joined to three-wires load with line voltage $U_l = 380\ V$.

5.12. Three phase impedances $\underline{Z} = 60 + j80\ \Omega$ are connected in “delta” with the line voltage $U_l = 220\ V$.Define phase and line currents, draw a vector diagram.

5.13. In every phase of the load connected in “delta” a current lags behind a voltage by 37^o and $Z_A = Z_B = Z_C = 45\ \Omega$. Calculate phase and line currents if the line voltage $U_L = 380\ V$. Draw a vector diagram.

5.14. The non-symmetrical load is connected to the asymmetrical source. The parameters of the circuit: $\underline{U}_{AB} = 80\ V$; $\underline{U}_{BC} = 60e^{-j90^o}\,V$; $\underline{Z}_A = 10e^{j30^o}\,\Omega$; $\underline{Z}_B = 5e^{-j60^o}\,\Omega$; $\underline{Z}_C = 10e^{j30^o}\,\Omega$. Define active, reactive and apparent powers of the load. Draw a vector diagram.

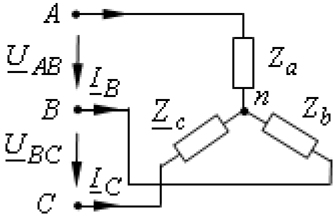


Fig. P5.14

5.15. The circuit devices in delta show the next readings: $U_{AB} = 100\,V$; $U_{BC} = 150\,V$; $U_{CA} = 120\,V$; $I_B = 15\ A$, $I_C = 12\ A$; $P_W = –600\ Wt$. Define phase impedances. Draw a vector diagram.

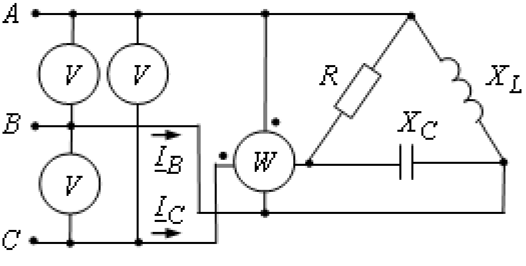


Fig. P5.1

Chapter 6

TRANSIENTS

In this chapter we will investigate the transient values of the element voltages and currents. In circuits containing energy storage elements, the inductor current and the capacitor voltage cannot change instantaneously; otherwise their stored energies would change instantaneously. Thus, there is a certain amount of "inertia" associated with these elements. Suppose, for example, that we find the solution for the element voltages and currents for the particular circuit and then, at some later time, open or close a switch located somewhere in the circuit; activating the switch changes the circuit, and now we must solve this new circuit.

The element voltages and currents will have changed due to the fact that we have a new circuit caused by the switching operation. The object of this chapter is to determine the behaviour of the element voltages and currents in the intermediate, or transient, time interval while they are adjusting to their new values.

The study of the dynamic behaviour of linear circuits and systems containing one or more energy-storing elements is of considerable importance to the engineer for two reasons. First, he often wants to know how long it takes for the circuit to respond to applied source functions. Recall that the energy in an energy-storing element such as an inductor or capacitor cannot change instantaneously. Accordingly, if an applied forcing function demands an increase in the amount of energy storage in a particular part of a circuit, this increase must occur gradually.

The change cannot take place in a discontinuous manner, because then infinite forces would be needed. The period of adjustment during which the stored energy changes from some initial level to a new, commanded, final level is called the settling time of the circuit (or system). In many engineering applications it is important to keep the response time within tolerable limits. Second, in situations where there are two or more energy-storing elements the engineer must be able to predict the occurrence of severe oscillations as the circuit (or system) changes from one energy state to another.

In electric circuits such oscillations can readily cause ruinous voltages or currents. In electromechanical systems, such as a servomechanism, these oscillations can cause excessively high torques which in turn may damage mechanical parts.

In fact, in some extreme cases, the interchange of energy during the transient state is such that the oscillations started by the application of the forcing function do not cease.

Accordingly, the new steady-state level is never reached. The system or circuit is then described as being in a state of sustained oscillations. Whether or not this is a desirable state of affairs depends on the objective which the engineer has in mind. If his invention is to build an oscillator, then his goal is achieved. However, if his intention is to build a follow-up control system which reaches its new steady-state position after the elapse of a reasonable amount of time, then clearly a sustained oscillation must be avoided at all costs.

It is the purpose of this chapter to furnish the background which will allow the student to determine the complete response of circuits and systems, when subjected to conventional forcing functions.

6.1 Transients Defined

Transients are the phenomena which occur between two permanent (or steady-state) conditions, as a rule periodic, differing from each other in, say, peak value, phase, wave-form or frequency of the e.m.f., the parameters or configuration of the circuit.

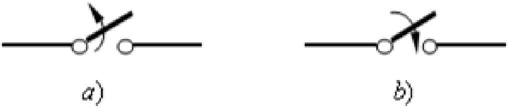


Fig. 6.1, *a* -a switch is to be opened, and *b* - a switch is to be closed.

Hence the changing process of any electrical circuit from one of the stationary conditions to another is called the transient. The stationary conditions is called such a rate when currents and voltages can exist infinitely long without changing of their character with given configuration of the circuit and its parameters. This rate depends on the configuration of the circuit and the value of energy sources.

Normally, transient phenomena are brought about in electric circuits by switching operations, that is by closing or opening switches. There may be, however, transients due to other causes, such as faults.

In diagrams the operation performed on a switch is shown by an arrow. Thus, the arrow in Fig. 6.1, *a* shows that the switch is to be opened, and the one in Fig. 6.1, *b* that it is to be closed.

Physically, transient phenomena associated with switching operations are changes leading from the energy state existing prior to a switching operation to an energy state which exists after a switching operation. Transients usually last for a few tenths, hundredths or even millionths of a second; it is seldom that they may last for several seconds. Their study is important, as it shows in advance what dangerous rises in voltage or current, many times their steady-state values may happen in the individual sections of a circuit. Transient analysis also throws light on how signals are distorted in wave-form or amplitude as they pass through different circuit elements.

6.2 Step Response of an *RL* Circuit

Any circuit composed of resistance and inductance represents a situation in which there occurs a dissipation of energy as well as storage of energy in a magnetic field. Since we now know how do deal with both voltage and current sources, the step response of the *RL* circuit is found for each source type. Attention is directed first to the step-voltage response.

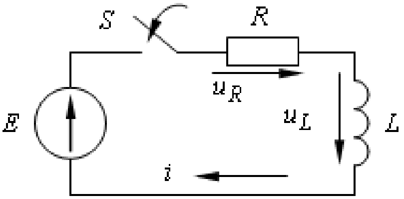


Fig. 6.2. An inductor in series with a resistor.

Appearing in Fig. 6.2 is the circuit arrangement of an initially deenergized inductor in series with a resistor. It is desired to find the complete solution for the current when the switch *S* is close. The governing differential equation results upon applying Kirchoff's voltage law to the circuit of this figure. Thus

$$E = Ri + L\frac{di}{dt} \tag{6.1}$$

We have got a linear *differential equation* with constant coefficients here. The current i is the solution being sought and E is the applied forcing function which causes the response i to exist.

From mathematics it is known that the general solution to a linear differential equation is the sum of a particular solution of an inhomogeneous equation and the complete solution of a homogeneous equation. The particular solution (or the particular integral) of Eq. (6.1) is

$$i_{ss} = I = \frac{E}{R},$$

where E is the direct current voltage. The homogeneous equation is obtained from the original one by equating its right-hand side to zero:

$$L\frac{di}{dt} + Ri = 0 \qquad (6.2)$$

Let's transfer from the differential equation to the algebraic one

$$R + Lp = 0\,. \qquad (6.3)$$

This algebraic expression is termed a *characteristic equation*, as its root determines a form of a free component.

Solving the equation, we will get

$$p = -\frac{R}{L}\,. \qquad (6.4)$$

The solution of the homogeneous equation (or the complementary solution or function) is an exponential function of the form

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t}, \qquad (6.5)$$

where A is a constant independent of time (time-invariant) and p is a root of the characteristic equation.

We assume that for all transient time $t = 0$ corresponds to the instant when the switch is thrown from one position to the other. A and p are constants independent of time (time-invariant). Without proof, for our case they are $A = -{}^{E}\!/_{R}$ and $p = -{}^{R}\!/_{L}$. Therefore, the solution to Eq. (6.1) may be re-written thus

$$i = \frac{E}{R} - \frac{E}{R}\cdot e^{-\frac{R}{L}\cdot t} \qquad (6.6)$$

where $\dfrac{E}{R}$ is the particular integral of Eq. (6.1) and the next value is the complementary solution to Eq. (6.5).

The particular integral represents the *forced* or *steady-state component* of the response (current or voltage), and the complementary solution is the *force-free,* or *transient component* of the response. The name "free" (or "force-free") is due to the fact that transient component of the response is independent of the excitation or drive.

To tell one component from the other and both from the total quantity, the convention is to supply the symbol of the steady-state component with the subscript "ss", the transient component with the subscript "t", and the total current with no subscript at all. Thus $i = i_{ss} + i_t$. Physically, the forced or steady-state component has the same frequency as the excitation or drive. If the drive is a sinusoid of frequency ω, the steady-state component will be also a sinusoid of the same frequency ω and of the same wave-form. In a circuit containing a *dc* voltage source (see Fig. 6.2), the forced component will be *dc* one. It should be remembered that inductance presents a short-circuit to *dc* circuits, and there is no forced voltage in the response of an inductor.

Of the three quantities, forced, free and total, the key ones are the total current and the total voltage. The total current is the one actually flowing through a given impedance for the duration of a transient. Similarly, the total voltage is the one actually existing across certain points in a network for the duration of a transient. These values can be measured or recorded by an oscilloscope. As to forced and free currents and voltages during a transient, they play an auxiliary part: they are those fictitious quantities whose sum gives the actual quantities.

6.3 Evaluation of initial conditions

An examination of Eq. (6.6) reveals that the presence of the *transient term* in the solution is demanded by the need to satisfy the boundary conditions immediately upon application of the forcing function, that is, the forcing function E wants to establish the forced solution E/R, but the presence of the inductance prevents this from occurring instantly. As a result the circuit reacts in such a way as to assure a smooth transition from the initial to the final energy state.

This is achieved through the generation of a component of the solution, which not only satisfies the boundary conditions at $t = 0^+$, but also permits the forced solution to exist after a suitable period of adjustment.

We can see that at $t = 0+$, i_t has the value $(-E/R)$ which it must have so that the total current at this time instant be zero as called for by the boundary condition. Equation (6.6) makes this obvious.

However, note that as time elapses the transient term decays to zero, leaving just the forced solution. Of course, as current i_t decays, time is allowed for transferring energy to the inductance.

When steady state is virtually reached, the energy stored in the inductor is

$$W = \frac{1}{2} L \left(\frac{E}{R} \right)^2 \tag{6.7}$$

It is because this energy cannot be transferred instantaneously that the need for the transient term arises.

If the changing of the inductive current carried out instantaneously (i.e. by great advance) it would lead to appearing of perpetually large e.m.f. of self-induction

$$e_L = -\frac{d\psi}{dt} = -L\frac{di_L}{dt} \to \infty .$$

The energy of magnetic field would also vary instantaneously and, hence, the circuit supply source should develop perpetually large power

$$p = \frac{dW_m}{dt} = Li_L \frac{di_L}{dt} \to \infty .$$

As the e.m.f. of self-induction and the power source in real circuits have finite values, the transition from one steady-state condition of the circuit with an inductive element to another one is possible only during some period of time.

If the voltage u_{C1} is applied to the terminals of a capacitor, some charge is retained on its facings, and the electrical field energy is reserved here:

$$q_1 = Cu_{C1}, \; W_{e1} = \frac{Ñu_{C1}^2}{2} . \tag{6.8}$$

If voltage is changed to the value u_{C2} the new installed condition will be characterized by the new values of a charge and an electrical field energy

$$q_2 = Cu_{C2},\ W_{e2} = \frac{\tilde{N}u_{C2}^2}{2}\,. \tag{6.9}$$

If this transition was carried out instantaneously, it would lead to appearing of the perpetually large charging (or discharge) current

$$i = \frac{dq}{dt} = C\frac{du_C}{dt} \to \infty$$

The electrical field energy would also change instantaneously, and, hence, the circuit supply source would develop perpetually large power

$$p = \frac{dW_e}{dt} = Cu_C\,\frac{du_C}{dt} \to \infty\,. \tag{6.10}$$

As the source current and the source power in real circuits have finite values, the transition from one steady-state condition of the circuit with a capacitive element to another one is possible only during some period of time.

To calculate transient means to define the character of time changing corresponding electrical magnitudes - currents or voltages.

The transient analysis in the linear circuits becomes simpler, if transient is considered as effect of superposition of two processes: the first is a new installed condition, considering that it comes just after switching, and the second is a transient process which ensures the circuit transition from one installed rate to another one.

The values of the transient current and voltage components depend on a level of disparity of the energy reserved in magnetic and electric fields to a new steady-state rate. In process of diminution of this disparity the transient currents gradually decrease to zero values.

One can represent a real circuit current during transient as a sum of two components: a new steady-state current and a transient current

$$i = i_{ss} + i_t\,.$$

The transient time is spotted by a small time interval, at the end of which currents and voltages so come nearer to the steady-state values, that the distinction appears to become practically imperceptible.

Consider the transient solution

$$i_t = -\frac{E}{R} e^{-\frac{R}{L}t} \tag{6.11}$$

which makes it clear that the decay of the transient term is solely dependent upon the circuit parameters R and L and entirely independent of the source function.

This is certainly not unexpected when it is recalled that the transient term is really a description of the manner in which the circuit reacts to external disturbances, whatever their origin or nature.

The only influence that the forcing function has on the transient term is in determining the magnitude of the coefficient of the exponential, but it in no way influences the rate of decay of the transient term. A plot of Eqs. (6.6) and (6.11) appears in Fig. 6.3.

Note that at $t = 0$ the steady-state and the transient solutions have equal and opposite values, thus yielding a zero value for the total current as called for by the boundary condition. Note also that as the transient term decays to zero, the total current reaches its forced value. In the interest of establishing a convenient measure of the duration of transient, let us look more closely at the exponential function of Eq. (6.11).

A glance at the power of the exponential, which must be a numeric, reveals that the quantity L/R must in turn bear units of time. Because of this fact and because the quantity is determined in terms of the circuit parameters, it is called the time constant of the circuit and is denoted by τ.

Thus time constant $\tau = {}^{L}\!/\!_{R}$. Its physical sense: it is the time during which a free component decreases in $e \approx 2{,}718$ times in comparison with its initial value $i_t(0+)$

$$i_t(\tau) = i_t(0+)e^{-1} = \frac{i_t(0+)}{e} = \frac{A}{e}. \tag{6.12}$$

This value can be found graphically. For this purpose it is necessary to spend a tangent to curve $i_t(t)$ in a point corresponding to the arbitrary time moment $t = t_1$ and prolong it to the point of intersection with a time axis where $t = t_2$.

Then time constant $\tau = t_2 - t_1$. Usually the time when $t = 0$ is chosen as the arbitrary (initial) time moment.

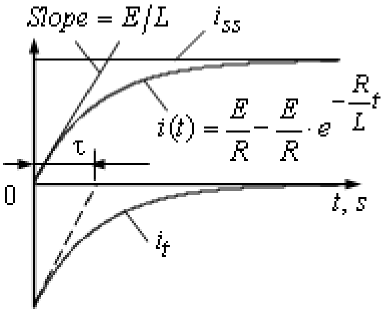


Fig. 6.3. The plot of inductive current solution.

The value inverse to the time constant termed as a *damping factor*

$$\alpha = \frac{1}{\tau}.$$

A transient current damps more slowly and, hence, the more a time constant (and the less a damping factor) the longer a new steady-state rate is installed. As we can see from the figure, in spite of the fact that transient process lasts perpetually long, in practice it is accepted to consider that the transient lasts only during the time $t = (3...5)\tau$.

6.4 The Rules of Switching

Under any transient or steady-state conditions two basic points hold: the current through any inductance and the voltage across a capacitance cannot change instantaneously.

The case of a current through inductance can be proved by reference to the network of Fig. 6.2.

Consider Eq. (6.1), made by Kirchhoff's voltage law:

$$L\frac{di}{dt} + Ri = E .$$

The current *i* and the voltage *E* can take on only finite values. Assume that the current *i* can change instantaneously. In other words, the current changes by a finite amount Δi during an interval of time $\Delta t \to 0$ tending to zero, such that $\Delta i / \Delta t = \infty$. If ∞ is substituted in Eq. (6.1), the left-hand side of the equation will no longer be equal to its right-hand side, and Kirchhoff's voltage law will not be satisfied.

In other words, the assumption that the current through an inductance can change instantaneously runs counter to Kirchhoff's voltage law. While the current through an inductance cannot change instantaneously, the voltage across it can. This does not contradict Kirchhoff's voltage law.

Consider now the simple circuit with a capacitance (see Fig. 6.4).

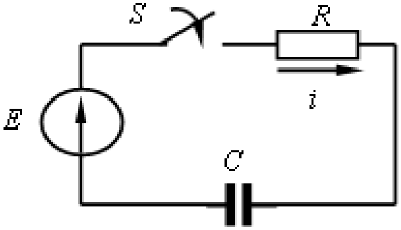


Fig.6.4. The circuit with a capacitance.

Write an equation by Kirchhoff's voltage law for this network

$$Ri + u_C = E \tag{6.13}$$

where E is the source voltage (which is a finite quantity) and u_C is the voltage across the capacitance. Since

$$i = C\frac{du_C}{dt},$$

it follows that

$$RC\frac{du_C}{dt} + u_C = E \tag{6.14}$$

If we assume that the voltage u_C can change instantaneously than $\frac{\Delta u_C}{\Delta t} \approx \frac{du_C}{dt} = \infty$, and the left-hand side of Eq. (6.14) will no longer be equal to its right-hand side.

Hence the assumption that the voltage across the capacitance can change instantaneously runs counter to Kirchhoff's voltage law.

However, a current equal to $C\frac{du_C}{dt}$ flowing through a capacitor can vary step-wise; this does not contradict Kirchhoff's voltage law.

There are two corollaries to the above two basic points, which may be called switching rules.

Let $i_L(0-)$ be the current through any inductance just prior to switching. It is equal to the current through the same inductance immediately after switching, $i_L(0+)$, or

$$i_L(0-) = i_L(0+) \tag{6.15}$$

Time $t = 0-$ is the instant immediately before switching; $t = 0+$ is the time just after switching. Eq. (6.15) expresses the first rule of switching: *The inductor current immediately prior to and immediately after a switching operation must be the same.*

Let the voltage across a capacitance just before switching be $u_C(0-)$, and that immediately after switching, $u_C(+)$. Since the voltage across a capacitance cannot change instantaneously, it follows that

$$u_c(0-) = u_c(0+)\,. \tag{6.16}$$

Eq (6.16) expresses the second rule of switching: *The capacitor voltage immediately prior to and immediately after a switching operation must be the same.*

6.5 Independent and Dependent Initial Conditions

In the literature, under *initial values* or *conditions* are meant the values of current and voltage in the circuit at time $t = 0$.

As have already been stated, currents through inductances and voltages across capacitances immediately before and after switching are respectively equal. Other quantities, such as voltages across inductances and resistances, currents through capacitances and resistances, can change instantaneously, and so their values immediately after switching may differ from those before switching.

Hence, a distinction should be made between initial conditions before and after switching. Initial conditions before switching are those which exist at time $t = 0-$. Initial conditions after switching are those which exist at time $t = 0+$.

For any circuit the currents and voltages that exist in or across any element at time $t = 0+$ (after switching) can be found by Kirchhoff's laws. In the equations written by these laws the currents through inductances and the voltages across capacitances known from the state of the circuit prior to switching are referred to as *independent initial conditions (i.i.c.)*

The remaining currents and voltages at $t = 0+$, found by Kirchhoff's laws are referred to as *dependent initial conditions*.

If by the time a transient is initiated, that is, before switching, all currents through and all voltages across passive elements have become equal to zero, the circuit is said to be in the state of zero initial conditions and the respective values are termed *zero initial conditions*. If, however, by the time of a transient the same values are other than zero, the circuit is said to be in the state of non-zero initial conditions, and the respective variables are termed *non-zero initial values*.

In the former case the currents through inductances and the voltages across capacitances will vary starting from zero values, in the latter, starting from non-zero values.

6.6 Short circuit in a *RL* circuit

Let's guess, that an inductive coil which has been connected through resistive element R_0 to a source of a direct current e.m.f., at the time instant $t = 0$ becomes to be short circuit (see Fig. 6.5).

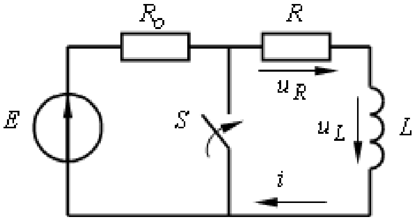


Fig. 6.5. RL circuit.

Before a switching at time instant $t = 0-$ the current flows through the circuit

$$i\ (0-) = \frac{E}{R_o + R}.$$

As a result of the switching in the organized contour which includes an inductive coil, thanks to a stored energy of a magnetic field, the current will not disappear instantaneously.

The e.m.f. of self-induction which has the same direction, as a current before switching, tends to support the current in the mesh at the expense of energy of a disappearing magnetic field.

As the magnetic field energy is gradually diffused, being transmuted into heat in a resistor, the current in a contour decreases to zero.

The steady-state value of a current in a short-circuited contour is equal to zero (the energy source does not influence on an inductive coil after a switching). The transient process taking place in this mesh is free or (transient) and the solution consists only of a free component, i.e. $i = i_t$. The free component of a current does not depend on the energy source, therefore the equation for it, a characteristic equation and its solution, a form of a transient component are the same

$$u_R + u_L = 0, \text{ or } Ri_t + L\frac{di_t}{dt} = 0.$$

The characteristic equation

$$R + pL = 0, \text{ from where } p = -\frac{R}{L}.$$

Then the transient component

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t}.$$

For integration constant definition we use initial conditions and the first law of switching

$$i(0-) = i(0+) = \frac{E}{R_o + R} = A$$

From here

$$i = \frac{E}{R_o + R}e^{-\frac{R}{L}t}.$$

The change of the current in a short-circuited contour and the voltage across the inductor are plotted in Fig. 6.6.

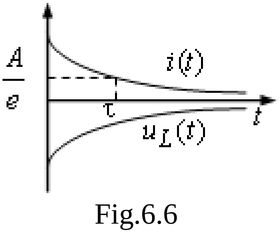


Fig.6.6

The transient process in a short-circuited contour theoretically lasts perpetually. Just as in the previous case, in a short-circuited contour it is possible to consider that the transient is practically over after the time $t = (3...5)\tau$. With that the energy is deposited in the form of heat in the resistor

$$\int_0^\infty Ri^2 dt = \int_0^\infty R\left(i(0-)e^{-\frac{R}{L}t}\right)^2 dt = Ri^2(0-)\int_0^\infty e^{-\frac{2R}{L}t} dt =$$

$$= Ri^2(0-)\left(-\frac{L}{2R}e^{-\frac{2R}{L}t}\right)\Bigg|_0^\infty = \frac{Li^2(0-)}{2},$$

i.e. all energy reserved in a magnetic field of the coil before a switching.

6.7 Step Response of an *RC* Circuit

Appearing in Fig. 6.7 is the circuit arrangement of a resistor in series with a capacitor. Let us find the complete expression for the current after the switch *S* is closed.

The governing differential equation is obtained upon applying Kirchhoff's voltage law to the closed circuit. Thus,

$$Ri + u_C = Ri + \frac{1}{C}\int i dt = E \qquad (6.17)$$

where E is the voltage source (which is a finite quantity) and u_C is the voltage across the capacitor.

We search the solution of the given differential equation as

$$u_C = u_{Css} + u_{Ct},$$

where u_{Css} is a steady-state component and u_{Ct} is a free (or transient) component of the capacitive voltage.

As the e.m.f. is a constant value, all steady-state electrical magnitudes (including voltage across the capacitor) will be also constant $u_{Css} = E$.

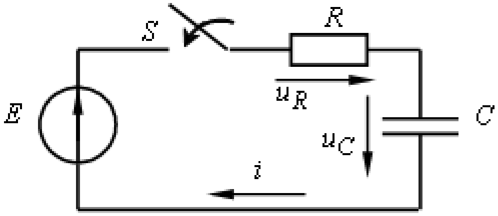


Fig. 6.7. RC series circuit.

As the current through the capacitor is $i = C\frac{du_C}{dt}$, the equation by the KVL may be rewritten as

$$RC\frac{du_C}{dt} + u_C = E$$

The solution of the given inhomogeneous differential equation is written as

$$u_C = u_{C_{ss}} + u_{C_t}$$

Before switching at $t = 0-$, the capacitor has not been charged. After closing a key by the second rule of switching the capacitive voltage must be the same immediately after switching

$$u_C(0-) = u_C(0+) = 0$$

Let's make a characteristic equation and, on solving it, we will find a form of the free component

$$pRC + 1 = 0$$

from where

$$p = -\frac{1}{RC},$$

Then we can determine transient component of the capacitive voltage as follows

$$u_{C_t} = Ae^{pt} = Ae^{-\frac{1}{RC}t}.$$

Then at the moment of switching we have got

$$u_C(0) = E + A.$$

From here $A = -E$.

Hence, on combining the solution for steady-state and free components, we will get a transient voltage across the capacitive element

$$u_C = E - Ee^{pt} = E - Ee^{-\frac{1}{RC}t} \tag{6.18}$$

The time constant $\tau = \frac{1}{|p|} = RC$.

Now we can find the expression for the current response as

$$i(t) = C\frac{du_C}{dt} = C\frac{d}{dt}\left(E - Ee^{-\frac{1}{RC}t}\right) =$$

$$= C\left(-\frac{1}{RC}\right)\left(-Ee^{-\frac{1}{RC}t}\right) = \frac{E}{R}e^{-\frac{1}{RC}t} \tag{6.19}$$

And the voltage across the resistor

$$u_R = Ri = R\frac{E}{R}e^{-\frac{1}{RC}t} = Ee^{-\frac{1}{RC}t} \tag{6.20}$$

A plot of these equations as functions of time is depicted in Fig. 6.8.

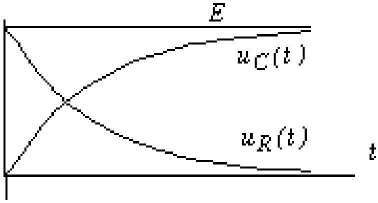


Fig.6.8

Example 6.1. In the circuit of Fig. 6.7 the initial voltage appearing across capacitor plates is 40 *V*. The resistance is equal to $0.2 \cdot 10^6\,\Omega$, the capacitance is equal to 10 *mF*. For the values of the parameters indicated find the expression for the current which flows when the switch S is closed.

Also, find the total energy dissipated in the resistor during the transient state and compare it with the energy initially stored in the capacitor.

Solution: Taking voltage drops in a clockwise direction as positive. Kirchhoff's voltage law applied to the circuit with *S* closed yields

$$Ri+\frac{1}{C}\int idt=0 \tag{6.21}$$

A comparison of Eq. (6.21) with Eq. (6.17) discloses that it is identical except for the fact that *E* is zero. Accordingly, the complete current response for this case follows from Eq. (6.19) upon setting *E* to zero. Thus

$$i=\frac{-u_C}{R}e^{-\frac{1}{RC}t}$$

The time constant is $\tau=RC=0.2\cdot 10^6\cdot 10\cdot 10^{-6}=2\ s$. During this time interval, the function $i(t)$ decreases by the factor e = 2.71.

The current solution may therefore be written as

$$i(t)=\frac{-40}{0.2\cdot 10^6}e^{-1/2t}=-2\cdot 10^4 e^{-1/2\,t}\ A$$

The amount of energy dissipated in the resistor is found from

$$W_{dis}=\int_0^\infty i^2(t)dt=\int_0^\infty \left(4\cdot 10^{-8}\right)\left(-0.2\cdot 10^6 e^{-t/2}\right)dt=$$

$$=-8\cdot 10^{-3}\left[e^{-t}\right]_0^\infty=8\cdot 10^{-3}\ J \tag{6.22}$$

The amount of energy initially stored in the electric field of the capacitor is given by

$$W_{el}=\frac{1}{2}CU^2=\frac{1}{2}10^{-5}\cdot 40^2=8\cdot 10^{-3}\ J\ . \tag{6.23}$$

A comparison of Eqs. (6.22) and (6.23) shows that all the capacitor energy is dissipated as heat after the switch is closed.

6.8 Short Circuit in *RC* Circuit

In dealing with a current source the resistance and capacitance parameters are considered to be in parallel as illustrated in Fig. 6.9. A step-current forcing function is applied to the parallel combination by opening the switch.

Then by Kirchhoff's current law we can write

$$J = i_R + i_C \tag{6.24}$$

Moreover, the same potential difference appears across R and C. Hence

$$R\, i_R = \frac{\int_0^t i_C \, dt}{C} \tag{6.25}$$

There can exist no initial charge on the capacitor because the switch S is initially assumed closed. Inserting Eq. (6.24) into Eq. (6.25) yields

$$R\ J = R\, i_C + \frac{1}{C}\int_0^t i_C \ dt \tag{6.26}$$

A study of the last equation can be made to reveal information about the steady-state value of i_C without resorting to the operator form of the governing differential equation and setting p to zero.

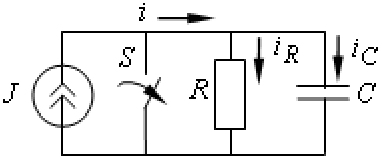


Fig. 6.9. The parallel connection of resistance and capacitance.

First note that the left side of the equation is a finite quantity, JR. Also note that, as time increases after opening the switch, the contribution from the integral term becomes larger and larger.

As time gets very large and approaches infinity, the only way the right side of the equation can remain finite so as to balance the left side is for the current to approach zero.

Our experience with differential equations having the form of Eq. (5.26) tells us that the capacitor carries a transient current given by

$$i_C = i_t = Ae^{-t/RC} \tag{6.27}$$

To evaluate *A*, we need the initial value of i_C. This too can be deduced from Eq. (6.26) upon inserting $t=0+$.

In this instance the contribution from the integral term is infinitesimal so that it follows that

$$i_C = J \tag{6.28}$$

Therefore the complete expression for the capacitor current is

$$i_C = J\,e^{-t/RC} \tag{6.29}$$

The time solution for the current through the resistor is found from Eq. (6.24) and (6.27). Thus

$$i_R = J - i_C = J(1 - e^{-t/RC}) \tag{6.30}$$

In the configuration of Fig. 6.9 the total source current initially flows entirely through the capacitor and then, as time elapses, it gradually transfers to the resistor. At steady-state all the current *J* flows through *R* and none through *C*.

6.9 Step Response of Second-order System (*RLC* Circuit)

The behaviour of a circuit or system which contains two independent energy-storing elements is completely described by a second-order differential equation.

Because the second-order system occurs frequently in engineering situations, considerable attention is given here to its analysis, (especially with a view to putting the results in a universal form). In this way the results can be applied to second-order systems generally, independently of their particular composition.

Thus the independent energy-storing elements may consist of inductance and capacitance or of mass and spring constant and so forth. The study of the second-order system is important also because the behaviour of many higher-order systems can frequently be described in terms of an equivalent second-order system.

Consequently the conclusions developed here can frequently be applied to such systems with satisfactory results.

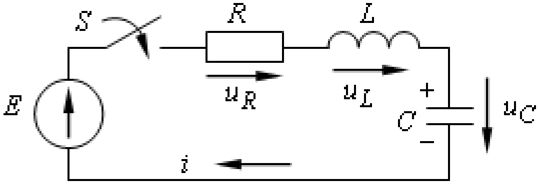


Fig.6.10. The series connection of the resistor, inductor and capacitor.

Let it be desired to find the complete current response in the circuit of Fig. 6.10 after the switch is closed. Assume that the capacitor has an initial charge of q_o. The governing differential equation for the circuit is found upon applying Kirchhoff's voltage law. Thus

$$E = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \tag{6.31}$$

Equation (6.31) is a second-order nonhomogeneous linear differential equation. Upon introducing the initial condition voltage we can then write

$$E = Ri + L\frac{di}{dt} + \frac{q_o}{C} + \frac{1}{C}\int idt \tag{6.32}$$

Rearranging and employing the differential operator, the expression becomes

$$E - \frac{q_o}{C} = \left(R + pL + \frac{1}{pC}\right)i \tag{6.33}$$

or

$$i = \left(\frac{1}{R + pL + 1/pC}\right)\cdot\left(E - \frac{q_o}{C}\right). \tag{6.34}$$

First let us discuss the steady-state solution.

One must remember; to find the steady-state solution for dc sources for any branch voltage of any energy storage element, we replace the inductor with a short circuit and the capacitor with an open circuit. Note that in the series *LC* problem only the steady-state capacitor voltage is nonzero and is determined by the remainder circuit.

Similarly, for the parallel *LC* problem, only the steady-state inductor current will be nonzero.

The steady-state (or forced) solution is found upon setting *p* equal to zero in the last equation which clearly leads to

$$i_{ss} = \text{forced solution} = 0$$

Since the capacitor represents an open circuit to a constant forcing function, this result is entirely expected.

The first step consists of finding the characteristic equation which here is readily determined by setting the denominator of Eq. (6.34) equal to zero. After rearranging terms, we get

$$Z(p) = p^2 + \frac{R}{L}p + \frac{1}{LC} = 0 \tag{6.35}$$

It is worth noting that this equation also results when the left side of Eq. (6.33) is set equal to zero, thus yielding the homogeneous differential equation in operator form.

Moreover, note that each of the three circuit parameters appears and that the highest order of ps is *2*, which is consistent with the presence of two independent energy-storing elements. If p_1 and p_2 denote the roots of this equation, the formal expression for the response can be written as

$$i(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{6.36}$$

Although Eq. (6.33) presents a formal solution of the problem at hand, we are interested in writing this solution in a more useful and significant form. In this connection, then, let us return to Eq. (6.35) and write the specific expressions for the roots. Thus

$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{6.37}$$

Depending upon the expression under the radical the transient response can be any one of the following:

(I) overdamped if $(R/2L)^2 > 1/LC$;

(2) critically damped if $(R/2L)^2 = l/LC$;

(3) underdamped if $(R/2L)^2 < 1/LC$.

Now let us investigate each of these three cases in more detail.

The first case**:** *Two unequal real roots*.

The characteristic equation has the form

$$p^2 + ap + b = 0 \tag{6.38}$$

The two roots are given by

$$p_{1,2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}\,, \tag{6.39}$$

which can be written, by factoring out $-a/2$, as

$$p_{1,2} = -\frac{a}{2}\left(1 \pm \sqrt{1 - \frac{4b}{a^2}}\right) \tag{6.40}$$

The two roots will be real and unequal if $4b/a^2 < 1$.

Then, from Eq. (6.40) it is clear that if a is positive, both of these roots will be negative.

For circuits containing positive resistors, inductors, and capacitors, *R*, *L* and *C* will be positive and also will a .

Therefore, for realistic circuits, both roots will be negative and we will designate them as $p_1 = -\alpha_1$; $p_2 = -\alpha_2$; where α_1 and α_2 are positive numbers.

Therefore transient solution will be of the form

$$i(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \tag{6.41}$$

where A_1 and A_2 are undetermined constants.

Thus, the transient solution consists of the sum of two decaying exponentials, as shown in Fig. 6.11, *a*. For this case the transient solution is said to be overdamped.

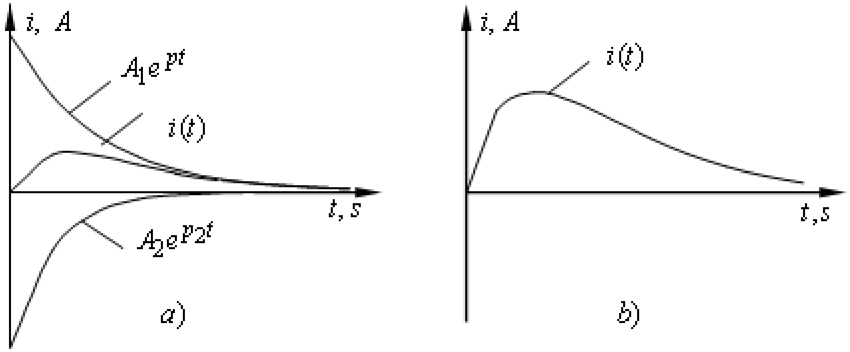


Fig.6.11. Transient solutions for two cases.

Thus, the transient solution consists of the sum of two decaying exponentials, as shown in Fig. 6.11, a. For this case the transient solution is said to be overdamped.

The second case: *Two equal real roots*. If $4b = a^2$. The radical in Eq. (6.40) will be zero and the roots will be equal: $p_1 = p_2 = -a/2 = -\alpha$.

From mathematics it is known that if among the roots of the characteristic equation there are two which are equal, the respective parts of the solution may be written thus

$$i_t = A_1 e^{pt} + A_2 t e^{pt} = (A_1 + tA_2) e^{-\alpha t} \qquad (6.42)$$

This solution cannot be reduced any further and that is contains exactly two undetermined constants. For this case, the transient solution is said to be critically damped, that is shown in Fig. 6.11, *b*.

The third case**:** *Complex Roots*. Because the underdamped case is the most interesting as well as the most frequently encountered case, attention is confined to it throughout the remainder of this section.

The term damping is an appropriate one to use in our description because, as previously demonstrated, it is characteristic of the resistance parameter to dissipate energy.

Consequently, its presence serves to prevent an uninterrupted interchange of energy between the two energy-storing elements. Such an uninterrupted interchange would constitute a sustained (or undamped) oscillation.

For the underdamped case, then, the expression for the roots of the characteristic equation becomes

$$p_{1,2} = -\frac{R}{2L} \pm j\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\,, \qquad (6.43)$$

This result is obtained by factoring out the minus sign under the radical and recalling that $j = \sqrt{-1}$.

The critical value of damping for fixed values of *L* and *C* corresponds to that value of *R* which makes the radical term go to zero. Hence

$$\left(\frac{R_c}{2L}\right)^2 = \frac{1}{LC} \qquad (6.44)$$

where *Rc* denotes the value of resistance which yields a critically damped transient response. In order to express the roots of the characteristic equation in a manner which makes the results applicable to all linear second-order systems as well as to provide a quick and convenient way of identifying the dynamic response, two figures of merit are introduced and defined.

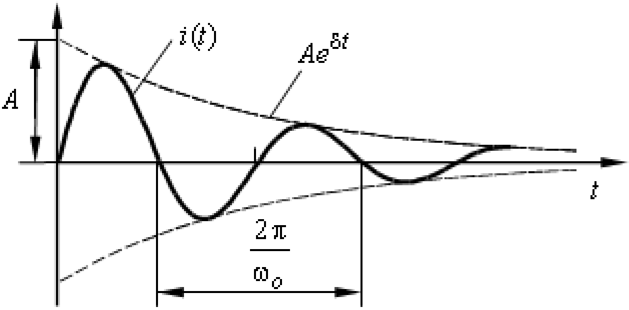


Fig. 6.12. Transient current for the case with complex roots.

If $\frac{4b}{a^2}>1$ in Eq. (6.45), we can rewrite this equation as

$$p_{1,2}=-\frac{a}{2}\left(1\pm j\sqrt{\frac{4b^2}{a}-1}\right) \tag{6.45}$$

So that we will denote the roots as

$$p_{1,2}=\delta\pm j\omega_o \tag{6.46}$$

Complex roots always occur in conjugate pairs. Thus, if one root is $p_1=-\delta+j\omega_o$, the other will be $p_2=-\delta-j\omega_o$. From mathematics it is known that the respective term, say the transient current i_t should be written thus

$$i_t=Ae^{-\delta t}\sin(\omega_o t+\gamma), \tag{6.47}$$

where ω_o is an angular frequency, and γ is called an initial phase (epoch angle). This is illustrated in Fig. 6.12 and is called a *damped sine-wave*. For our case (see Fig. 6.10) the initial phase $\gamma=0$. The envelope of the wave is decided by the curve $Ae^{-\delta t}$. The greater δ, the faster the transient decays. The magnitude A and an initial face of damped sinusoid γ depend on the circuit parameters, initial conditions and the energy source. On the other hand, ω_o and δ depend solely on the circuit parameters after switching. The complex frequency ω_o is termed the *angular natural frequency*, and δ is the *decay factor*, or *decrement*. The value $T=\frac{2\pi}{\omega_o}$ is called the period of a function.

The larger the ω_o of a circuit the smaller will be the settling time. Thus if two *RLC* circuits are compared and each has the same damping ratio but one has twice the natural frequency of the other, the transients in the circuit with the larger natural frequency will decay twice as fast.

6.10. General Outline of Transient Analysis as Applied to Linear Circuits

Transient analysis as applied to any linear circuit involves basically the following steps:

1. At first we represent an electric circuit before switching. Positive directions of currents through inductive elements and voltages across capacitive elements are assumed. Then one must find independent initial conditions (that is, the values of currents through inductive elements and voltages across capacitances immediately before switching). Independent initial conditions are found by some calculation method for direct current circuits, if energy sources are *dc*; or by the calculation method for sinusoidal alternating current, if energy sources are *ac*.
2. Now we represent an electric circuit after switching. Positive directions are assumed for branch currents and voltages. Positive directions of inductive currents and capacitive voltages must be the same, as in the circuit before switching. One must work out the integro-differential equations for transient, using Kirchhoff's laws.
3. Define the steady-state components of currents and voltages after switching when transient process has been over. The steady-state components are found by some calculation method for direct current circuits, if energy sources are *dc*; or by the calculation method for sinusoidal alternating current, if energy sources are *ac*.
4. The characteristic equation is written down and its roots are found. Now we can define the law of change of transient components by the form of the roots.
5. Required values are written down as the sum of steady-state and free components.
6. Then we can calculate the constants of integration which enter into transient components. If the transient component contains more than one constant of integration, it is required to find at first not only the

values of currents and voltages, but also their derivatives for the moment of time after switching. The main difficulty of a classical method just also consists in definition of dependent initial conditions and the constants of integration.

7. The unknown currents and voltages are expressed as functions of time in the form of algebraic sum of the steady-state and transient components (taking into account the calculated constants of integration).

6.11 Determination of Integration Constants in the Classical Method

The classical method shows that the current flowing under transient conditions is described by a differential equation whose solution is the sum of a particular integral (the steady-state component) and a complementary function (the transient component), The constants of integration of the complementary function (the transient current or voltage) are found by solving a system of algebraic equations from the known roots of the characteristic equation and from the known values of the transient currents and its derivatives for $t = 0+$.

Any transient current or voltage may be represented as the sum of exponential terms. The number of such terms is equal to the number of roots of the characteristic equation. For any network using Kirchhoff's laws and the switching rules one can find:

(1) the numerical value of $i_t(0+)$, or the transient current at $t = 0+$;

(2) the numerical value of the first and, if necessary, higher derivatives of the transient current for $t = 0+$.

Let's consider how the constants of integration A_1 and A_2 are found, assuming that $i_t(0+)$ and $i'_t(0+)$ and the roots p_1 and p_2 are known. When a network is described by a first-order equation, then $i_t = Ae^{pt}$. The constant of integration is determined from the transient current $i_t(0+)$: $A = i_t(0+)$.

When a network is described by a second-order equation which has real and unequal roots, then

$$i_t = A_1 e^{p_1 t} + A_2 e^{p_2 t} \tag{6.48}$$

Differentiating this equation with respect to time gives

$$i'_t = p_1 A_1 e^{p_1 t} + p_2 A_2 e^{p_2 t}\,. \tag{6.49}$$

Writing Eqs. (6.48) and (6.49) for $t = 0$ (and noting that for $t = 0$, $e^{p_1 t} = e^{p_2 t} = 1$) gives

$$\begin{cases} i_t(0+) = A1 + A2 \\ i_t'(0+) = p_1 A1 + p_2 A2 \end{cases} \tag{6.50}$$

Solving these two equations simultaneously gives us unknown values A_1 and A_2.

If the characteristic equation has conjugate complex roots, the transient current is given by Eq. (6.47), where the angular frequency ω_o and the decay factor δ are known from the solution of the characteristic equation.

So, the transient current is given by

$$i_t = Ae^{-\delta t} \sin(\omega_o t + \gamma)\,. \tag{6.51}$$

Differentiating Eq. (6.51) with respect to time gives

$$i'_t\,(0+) = -A\delta e^{-\delta t} \sin(\omega_o t + \gamma) + A\omega_o e^{-\delta t} \cos(\omega_o t + \gamma) \tag{6.52}$$

Writing Eqs. (6.51) and (6.52) for the time $t = 0$, we get

$$i_t(0+) = A \sin\gamma \tag{6.53}$$

$$i'_t\,(0+) = -A\delta \sin\gamma + A\omega_o \cos\gamma\,. \tag{6.54}$$

Thus, we have two equations to determine two unknown values: the constant in integration *A* and the initial phase γ.

Example 6.2. There were zero initial conditions in the network of Fig. 6.10 before switch closure. The parameters of the network are *E* = 60 *V*, *R*=10 *Ω*, *L*=0.1 *H*, *C*=10^{-4}*F*. Determine the law according to which the current vary with time.

Solution: (1) Assign positive directions to the current. The positive directions for the voltages will, as always, be assumed coincident with the current.

Before switch closure, the inductive current and the voltage across the capacitor were zero:

$$i(0-) = i(0+) = 0$$

$$u_C(0-) = u_C(0+) = 0$$

Hence, we have zero initial conditions here.

In accordance with the second switching rule, they remain zero immediately after closure of the switch, at *t = 0+*.

After switching the key Q is close. The equation by the second Kirchhoff's law:

$$u_R + u_L + u_C = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt = E \qquad (6.55)$$

where i *and* u_C are zero initial conditions. Then $u_L = E = 60\ V$.

(2) Consider steady-state condition. Remember: to find the steady-state solution for dc sources for any branch voltage of any energy storage element, we replace the inductor with a short circuit and the capacitor with an open circuit.

Note that in the series RLC problem only the steady-state capacitor voltage is nonzero and is determined by the remainder circuit.

Hence,

$$i_{ss} = 0,\ u_{Css} = E = 60\ V.$$

(3) To find the transient current, represent the current at t = 0+ as the sum of a steady-state and a transient component

$$i(0+) = i_{ss}(0+) + i_t(0+) = 0$$

The characteristic equation of the network

$$Z(p) = R + pL + \frac{1}{pC} = 0$$

or

$$p^2LC + pRC + 1 = 0$$

has two complex conjugate roots

$$p_{1,2} = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = \frac{-10^{-3} \pm j6.24 \cdot 10^{-3}}{2 \cdot 10^{-5}} = -50 \pm j312\ s^{-1}.$$

Since the characteristic equation has such roots, the transient component will have the form

$$i_t = Ae^{-50t}\sin(312t + \gamma)$$

where A and γ can be found from the transient solution and its first derivative for $t = 0+$.

From (1) above, $u_L = E = L\frac{di}{dt}$, from where $\frac{di}{dt} = \frac{E}{L} = 600\ {}^{A}\!/\!{}_{s}$.

The function it for t = 0+ is equal to $A\sin\gamma$.

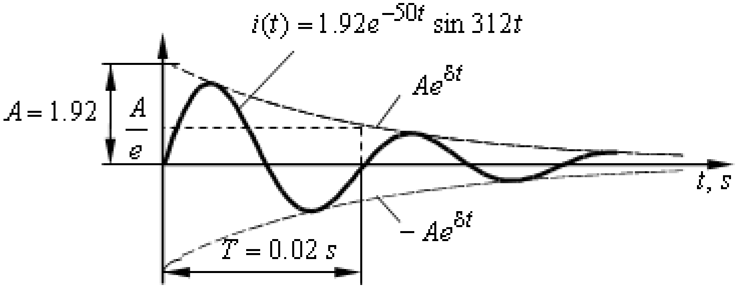


Fig. 6.13

The derivative of the function $Ae^{-\delta t}\sin(\omega_o t+\gamma)$ for t = 0+ is equal to $\frac{di}{dt}(0+) = -A\delta\sin\gamma + A\omega_o\cos$. To determine A and γ for the transient component of the current, write two equations:

$$\begin{cases} A\sin\gamma = 0 \\ -A\delta\sin\gamma + A\omega_o\cos\gamma = 600 \end{cases}$$

Solving them simultaneously gives $\gamma = 0^{o}$; A=1.92 A.
Therefore,

$$i(t) = i_t(t) = 1.92e^{-50t}\sin 312t\ .$$

This function is represented by curve in Fig. 6.13.

The period of the function is equal to $T = {}^{2\pi}\!/\!_{312} = 0.02\,s$, the time constant $\tau = \frac{1}{|50|} = 0.02\,s$. During this time interval, the function $1.92e^{-50t}$ decreases by the factor e = 2.71.

6.12. Complete Response of *RL* Circuit to Sinusoidal Input

So far in the chapter attention has been confined to just one type of input, the step forcing function. In this section we now determine the total response of a series *RL* circuit when a second type of input, the sinusoidal function, is applied to the circuit terminals. The circuit diagram is depicted in Fig. 6.14. The voltage source is assumed to be varying in a cosinusoidal fashion. When the switch is closed, the governing differential equation for the circuit becomes

$$e(t) = Em\cos\omega t = Ri + L\frac{di}{dt}\,. \tag{6.56}$$

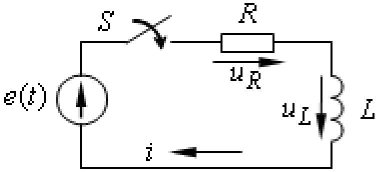


Fig. 6.14. The circuit diagram with sinusoidal input.

The solution is searched as the sum of two components

$$i = i_{ss} + i_t .$$

In operator form the response is

$$i = \left[\frac{1}{R + pL} \right] E_m \cos \omega t \tag{6.57}$$

The forced solution is found directly from the Eq. (6.57). Before doing so, recall that a considerable saving of effort results when the cosinusoidal function is replaced by the corresponding exponential function which contains it. Accordingly, we have

$$i_{ss} = \mathrm{Re}\left[\frac{E_m}{Z} e^{(j\omega t - \psi)} \right] = \frac{E_m}{Z} \cos(\omega t - \psi) =$$

$$= I_m \cos(\omega t - \psi) , \tag{6.58}$$

where $Z = \sqrt{R^2 + \omega^2 L^2}$; $\psi = \tan^{-1} \dfrac{\omega L}{R}$

It is instructive here to note that the forced solution has the same form as the forcing function but that it differs in two respects: amplitude and phase. The amplitude of the current is modified from that of the source voltage by the factor $1 \Big/ \sqrt{\left(R^2 + \omega^2 L^2\right)} = 1/Z$. The phase or argument is altered by the angle ψ.

To find the transient component of the solution, we first obtain the characteristic equation. This follows upon setting the denominator of Eq. (6.57) equal to zero. Thus

$$R + pL = 0 \tag{6.59}$$

from which $p = -R/L$.

This is a familiar result and leads to an expression for the transient solution that is the same as was found for the case with dc energy source. See Eqs. (6.3) and (6.4).

Thus

$$i_t = Ae^{pt} = Ae^{-\frac{R}{L}t} \qquad (6.60)$$

The complete solution is obtained by adding Eq. (6.58) to Eq. (6.60), which yields

$$i = i_{ss} + i_t = I_m \cos(\omega t - \psi) + Ae^{pt} \qquad (6.61)$$

The presence of the inductor means that $i(0+) = 0$. Inserting this result into Eq. (6.61) for t = 0+ identifies the value of A as

$$0 = I_m \cos\ (-\psi) + A$$

from here

$$A = -i_{ss}(0+) = -I_m \cos\ \psi$$

The final expression for the response thus becomes

$$i(t) = I_m \cos(\omega t - \psi) - I_m\, e^{pt} \cos\psi \qquad (6.62)$$

A graphical representation of Eq (6.62) appears m Fig 6.15.

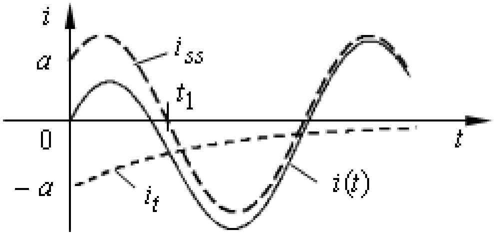


Fig. 6.15. The transient response of *RL* circuit to sinusoidal input.

The transient component of the solution behaves as expected. Initially it provides a magnitude of current which is equal and opposite to the instantaneous value of the steady-state current at the instant of switching. In Fig. 6.15 it is assumed that the switch is closed at the instant when the voltage has its positive maximum value. Note that the corresponding steady-state current is not zero at this instant.

In Fig. 6.15 it is shown to have a value 0*a*. Since the boundary condition demands that the current at this instant be zero, it is necessary for the transient term to have a value equal and opposite. This is depicted as –0*a*. As time elapses after switching, the transient term decays to zero at a rate determined by the circuit time constant, and the total current in Fig. 6.15 then becomes identical to the steady-state current.

The presence of a transient component in the total solution for the current is called for whenever the initial value of current in the *RL* circuit is at variance with the value of the steady-state current it the instant of switching.

As an illustration, if in the problem just discussed the initial current flowing through *L* were equal to 0*a* there would be no need for a transient term because the ensuing current would proceed directly into the steady state.

As a result of the correspondence of the currents before and after switching, then is no force attempting to bring about an abrupt change in the amount of energy stored in the inductor.

This behaviour can be described in another way. Let t_1 in Fig. 6.15 denote the time when the steady-state current is zero. If the switch in Fig. 6.14 is closed at this instant, no violation of the boundary condition occurs; consequently there is no need for a transient current In such a case the response proceeds directly into the steady state.

Incidentally, note that although the steady-state current is zero the corresponding value of the applied voltage is not. This is characteristic of the *RL* circuit for sinusoidal forcing functions.

Example 6.3. A sinusoidal forcing function $e(t) = 141 \sin 377t$ is applied to an initially de-energized series *RL* circuit in which $R = 100\ \Omega$ and L = 0.5 H.

(a) If the switch which applied the voltage to the *RL* circuit is closed at the instant when $e(t)$ is passing through zero with a positive slope, determine the initial value of the transient current.

(b) Write the complete expression for the transient solution.

(c) Write the expression for the complete solution of the current response.

(d) At what instantaneous value of the applied voltage will the closing of the switch result in no transient component of current?

Solution: (a) The general form of the steady-state solution is

$$i_{ss} = \frac{E_m}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \theta) \tag{6.63}$$

This follows directly from Eq. (6.58) except that the sine function is used in order to make it consistent with the assumed sinusoidal forcing function. Since $\omega = 377$ it follows that

$$\omega L = 377(0.5) = 188.5\ \Omega$$

and

$$\sqrt{R^2 + \omega^2 L^2} = \sqrt{100^2 + 188.5^2} = 213.5\Omega$$

Also

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1} 1.885 = 62^o$$

Therefore, the value of the steady-state current at t = 0+, which is the time immediately following application of the forcing function, is

$$i_{ss}(0+) = \frac{E_m}{\sqrt{R^2 + \omega^2 L^2}}\sin(-\theta) = \frac{141}{213.5}\sin(-62^o) = -0.584\ A$$

However, in accordance with the boundary condition the initial value of the current through the inductance must be zero. That is

$$i(0) = i_{ss}(0+) + i_t(0+) + 0$$

or 0 = -0.584 + i_t(0+). Hence the initial value of the transient current is i_t = 0.584 A.

(b) The rate of decay of the transient term is determined by the time constant of the circuit, which here is

$$\tau = \frac{L}{R} = \frac{0.5}{100} = \frac{1}{200}\ s$$

Therefore, the complete expression for the transient term becomes

$$i_t = 0.584e^{-200t}\ \ A$$

(c) Then the equation for the total solution results:

$$i(t) = 0.66\sin(377t - 62^o) + 0.584e^{-200t}\ \ A$$

(d) In order that there be no transient current in the solution, it is necessary to close the switch at that time instant for which it is zero. A glance at Eq. (6.63) shows that this occurs whenever

$$(\omega t - \psi) = 0,\ \pi,\ 2\pi,...$$

Choosing the first value, it follows that $\omega t = \psi = 62^{o}$.

The corresponding instantaneous value of the forcing function is then e = 141 sin 62^{o} =124.2 V. Therefore, if the switch is closed when the applied voltage has an instantaneous value of 124.2 volts, there is no need for a transient term and so none exists.

6.13. Response of *RLC* Circuit to Sinusoidal Inputs

The treatment of this case presents nothing new in the way of concepts about transients; therefore, the development of the solution is kept brief.

The circuit configuration is shown in Fig. 6.16.

With the switch closed, the differential equation for the circuit is

$$E_m \cos \omega t = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t idt$$

All initial conditions are assumed to be zero.

Proceeding as outlined in the previous section, we can write the expression for the forced solution directly as

$$i_{ss} = \frac{E_m}{Z} e^{j(\omega t-\theta)} = \frac{E_m}{Z}\cos(\omega t - \theta)$$

where now

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \text{ and } \theta = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

Again note that this forced solution has the same form as the source function but differs in its amplitude and argument.

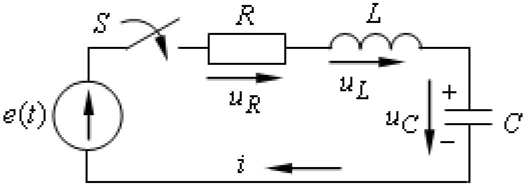


Fig. 6.16. The *RLC* circuit with sinusoidal source.

Write down the characteristic equation equalled to zero. Thus

$$R + pL + \frac{1}{pC} = 0 \ or \ p^2 LC + pRC + 1 = 0$$

This characteristic equation has two roots, p_1 and p_2. Accordingly, the transient component takes the form (if we have two real roots)

$$i_t = A_1 e^{p_1 t}\cos \omega t + A_2 e^{p_2 t}\sin \omega t$$

The expression for the total solution then becomes

$$i(t) = i_{ss} + i_t = \frac{E_m}{Z}\cos(\omega t - \psi) + A_1 e^{p_1 t}\cos\omega t + A_2 e^{p_2 t}\sin\omega t$$

Comparing the last equation, the current response of the *RLC* case, with the similar one, the current response for the *RL* case, shows that the responses differ essentially in the character of the transient terms. Because the *RL* configuration involves one energy-storing element and is described by a first-order differential equation there occurs just a simple exponential decay of the transient.

The steady-state solution merely rides on this exponential decay, as illustrated in Fig.6.15. For the underdamped *RLC* configuration, however, the transient solution itself involves an exponentially damped oscillation with definite frequency. Here the steady-state term can be looked upon as riding on the damped oscillation, as depicted in Fig. 6.17. It is basically in this respect that the two responses differ.

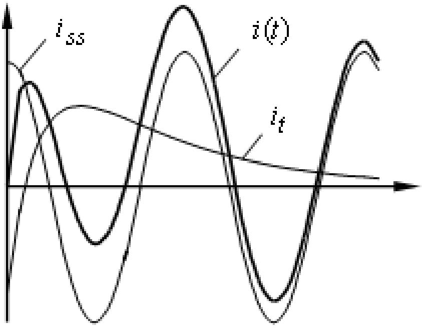


Fig.6.17. Transient solution for *RLC* circuit with sinusoidal source.

Example 6.4. In the figure of Fig. 6.18, the switch in the third branch is to be closed. Before closure the network is in a steady state; $e(t) = 127\sin(314t + 40^o)$ volts. The parameters of the circuit are $R_1 = R_3 = 50\,\Omega,\ R_2 = 10\,\Omega,\ L = 2\,H,\ C = 150\,mF$. Find the voltage across the capacitance after a switching.

Solution: Before closure of the switch

$$\underline{I}_{1m} = \frac{\underline{E}_m}{R_1 + R_2 + jX_L} = \frac{127e^{j40^o}}{60 + j628} = 0.202e^{-j44}\ A$$

The instantaneous steady-state current

$$i_1 = 0.202\sin(\omega t - 44^o)$$

At the close of the switch ($t = 0$)

$$i_1 = 0.202\sin(-44^o) = -0.14\ A$$

Write down independent initial conditions

$$i_1(0+) = -0.14\ A;\ u_C(0+) = 0$$

Now find the steady-state currents to determine the steady-state voltage across the capacitor after closure of the switch.

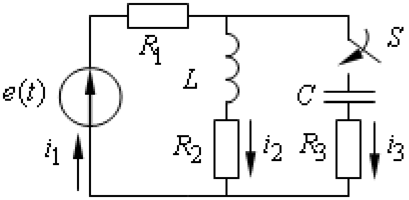


Fig. 6.18

The driving-point impedance of the network

$$\underline{Z}_{eq} = R_1 + \frac{(R_2 + j\omega L)(R_3 - \frac{j}{\omega C})}{R_2 + j\omega L + R_3 - \frac{j}{\omega C}} = 104.8e^{-j10^o}\ \Omega$$

Then

$$\underline{I}_{1m} = \frac{\underline{E}_{1m}}{\underline{Z}_{eq}} = \frac{127e^{j40^o}}{104.8e^{-j10^o}} = 1.213e^{j50^o}\ \ A$$

The instantaneous steady-state current after closure of the switch

$$i_{1ss} = 1.213\sin(\omega t + 50^o)\ A$$

$$i_{1ss}(0+) = 1.213\sin 50^o = 0.923\ A$$

The complex impedance of two parallel branches is

$$\underline{Z}_{23} = \frac{(R_2 + j\omega L)(R_3 - \frac{j}{\omega C})}{R_2 + j\omega L + R_3 - \frac{j}{\omega C}} = 56.3e^{-j18^o}\ \Omega$$

The complex voltage across the parallel portion of the network

$$\underline{U}_{23m} = \underline{I}_{1m} \cdot \underline{Z}_{23} = 1.213e^{j50^o} \cdot 56.3e^{-j18^o} = 68.2e^{j32^o}\ V$$

Then we can find the current through the capacitor

$$\underline{I}_{3m} = \frac{\underline{U}_{23}}{\underline{Z}_3} = \frac{68.2e^{j32^o}}{50 - j21.3} = 1.253e^{j54^o}\ A$$

The steady-state capacitive voltage after closure of the switch

$$\underline{U}_{Cm} = \underline{I}_{3m} \cdot \left(-\frac{j}{\omega C} \right) = 1.253e^{j54^o} \cdot 21.3e^{-j90^o} = 26.7e^{-j36^o}\ V$$

The instantaneous steady-state capacitive voltage and the instantaneous steady-state currents i_2 and i_3 after closure of the switch

$$u_{C_{ss}} = 26.7 \sin(\omega t - 36^o)\ V$$

$$u_{C_{ss}}(0) = 26.7 \sin(-36^o) = -15.57\ V$$

$$i_{2_{ss}} = 0.11 \sin(\omega t - 58^o)$$

$$i_{3_{ss}} = 1.253\ \sin(\omega t + 54^o)\ A$$

And their values at the moment of switching

$$u_{C_{ss}}(0) = 26.7 \sin(-36^o) = -15.57\ V$$

$$i_{2_{ss}}(0) = 0.11 \sin(-58^o) = -0.1$$

$$i_{3_{ss}}(0) = 1.253 \sin 54^o = 1.02$$

Now find i_{2t} (0+). By the first switching rule

$$i_2(0-) = i_2(0+) = -0.14 = i_{2_{ss}}(0+) + i_{2_t}(0+);$$

From here

$$i_{2_t}(0+) = i_2(0+) - i_{2_{ss}}(0+) = -0.14 + 0.1 = 0.04\ \text{A}$$

The transient voltage across the capacitor after closure of the switch is found by the second switching rule:

$$u_C(0+) = u_{C_{ss}}(0+) + u_{C_t}(0+);$$

$$u_{C_t}(0+) = u_C(0+) - u_{C_{ss}}(0+) = 0 - (-15.57) = 15.57\ \text{V}$$

To determine $i_{3t}(0+)$, write an equation for the loop formed by the first and third branches:

$$R_1 i_{1t}(0+) + R_3 i_{3t}(0+) + u_{Ct}(0+)$$

Substituting $-0.04 + i_{3t}(0+)$ for $i_{1t}(0+)$ and noting that $u_{C_t}(0+) = 15.57$ volts, we get

$$i_{3t}(0+) = \frac{-15.57 + 2}{50 + 50} = -0.1357 \quad \text{A}$$

From the equation $i_{3_t}(0+) = C\dfrac{du_{Ct}}{dt}$ we can determine

$$\left(\frac{du_{Ct}}{dt}\right)_{t=0+} = \frac{i_{3t}(0+)}{C} = \frac{-0.1357}{150 \cdot 10^{-6}} = -905 \text{ V/s}$$

The characteristic equation

$$p^2 LC(R_1 + R_2) + p\left[C\left(R_1R_2 + R_2R_3 + R_1R_3\right) + L_2\right] + R_1 + R_2 = 0$$

has two roots

$$p_1 = -42 + j15.2 \ s^{-1}; \ p_2 = -42 - j15.2 \ s^{-1}.$$

Therefore, the transient component should be given the form

$$u_{C_{ss}} = Ae^{-\delta t} \sin(\omega_o t + \gamma)$$

where $\delta = 42$ and $\omega_o = 15.2$. A and γ can be found from the transient component and its first derivative for $t = 0+$.

To determine A and γ for u_{C_t} write two equations

$$\begin{cases} A\sin\gamma = 15.57 \\ -\delta A\sin\gamma + \omega_o A\cos\gamma = -905 \end{cases}$$

Solving them simultaneously gives

A = 21.5 and γ = 137°;

Therefore

$$u_{C_{ss}}(0) = 26.7\sin(-36^o) = -15.57 \ V$$

The total voltage across the capacitance

$$u_C = u_{Css} + u_{C_t} = 26.7\sin(\omega t - 36^o) + 21.5e^{-42^o} \sin(15.2t + 137^o) \ V$$

6.14. The Laplace-Transform Method of Finding Circuit Solutions

In dealing with the subject matter of electrical engineering, it is frequently necessary to determine the solutions of linear differential equations. In this way information is obtained about the dynamic and forced responses of a circuit, device, or system to a driving function. Moreover, the development of methods of analysis, design, and synthesis as well as the interpretation of associated results are very often influenced by the particular mathematical formulations used to generate these solutions.

It is in the interest of furnishing the reader with the most consistent, systematic, and readily interpretable mathematical tool that attention is given in this chapter to the Laplace-transform method of solving linear differential equations. Certainly no more time is needed to learn the mechanics of the Laplace-transform method than the classical method. Moreover, the former has some very significant and important advantages to offer.

In the first place the Laplace-transform method of finding the solution to a defining differential equation involves a one-step systematic formulation which yields complete information about the sustained and transient solutions. This stands in sharp contrast to the three-part solution procedure of the classical method. In the latter, it is necessary first to find the particular solution associated with the forcing functions, then to find the complementary function which is the solution to the homogeneous differential equation, and finally to evaluate the constants of integration from the initial conditions.

Although two aspects of the procedure remain the same, namely finding the roots of the characteristic equation and evaluating the initial conditions, it often happens that where the governing differential equation is of an order higher than the second the amount of labour required to get the solution by the Laplace-transform method is less. However, by far the more important factor is the systematic formulation provided by the Laplace-transform method.

It permits the treatment of source functions and disturbances in the same fashion as initial conditions. This means that the same basic procedure is used to evaluate the forced solution as is used to evaluate the transient solution. This feature is particularly useful because it allows a unifying approach in the analysis and design of circuits, devices, and systems which cannot be otherwise achieved.

Thus, for example, in the study of electric circuits it is possible to treat the more general case of the transient response to deterministic inputs before the steady-state sinusoidal response. In addition, greater stress is placed on transform impedance as an operator which converts forcing function or initial condition to response.

Finally, the mathematical formulation of the Laplace transform permits a uniformity of analysis and of interpretation of results in situations which we find constantly recurring in all areas of electrical engineering and many other branches of engineering and science. Thus with this approach we will use the same techniques in arriving at answers to problems irrespective of whether they deal with circuits, amplifiers, feedback devices, control systems, or analogic computers.

In the treatment which follows, the emphasis is put on the mechanics of the Laplace-transform method. No attempt is made to present a rigorous exposition. This is left to books devoted primarily to that objective. Accordingly, a matter such as obtaining the final solution to a problem after applying the Laplace transform is accomplished through the use of tables rather than by means of integration in the complex plane, which is considered beyond the requirements for students of this book. When we accept the use of tables as a valid means of performing the inverse Laplace transformation, there is every reason to use the Laplace transform as a means of solving linear differential equations.

6.15. The Nature of a Mathematical Transform

It is the nature of a transform to simplify the analytical procedure leading to the solution of a problem irrespective of whether the problem is concerned with numbers, functions, or the calculus of differential equations. Specifically the Laplace transform is a mathematical tool for transforming functions. In linear analysis it also serves to convert operation in time, such as differentiation and integration, into simpler algebraic functions of an intermediate variable such as multiplication and division. The idea of using transforms to simplify the procedure leading to a solution should not be new to the student. It is assumed that he has used logarithms. Essentially the logarithm is a number transform which permits multiplication to be performed by means of the simpler operation of addition.

The first step in the procedure is to find the transform of each number. Finally, the solution in the original system of numbers is obtained by consulting the logarithm table. This last step is known as performing the inverse transformation through the use of tables. Thus multiplication, the desired operation, has actually been performed by addition, a simpler operation, in a transformed domain.

The Laplace transform achieves a similar result but on a different plane. Instead of simplifying the multiplication of numbers, it is concerned with functions and the simplification of operations of the calculus, such as differentiation and integration, into the easier operations of algebra such as multiplication and division. Note especially, however, that the method of procedure is identical.

Thus to solve an integrodifferential equation the same basic steps are involved. (1) The Laplace transform is used to convert the integrodifferential equation into an algebraic equation. The algebraic equation is expressed in terms of a new (or transformed) variable. (2) The simpler operation of solving the algebraic equation is performed in the transformed domain. (3) The solution in the original domain is obtained by consulting appropriate tables.

6.16. The Laplace Transform: Definition and Usefulness

A typical form of the differential equations encountered in the study of electrical engineering is illustrated by Eq. (6.64)

$$A_2 \frac{d^2 i(t)}{dt^2} + A_1 \frac{di(t)}{dt} + A_0 i(t) - A_{-1} \int i(t)dt = f(t) \tag{6.64}$$

where *f(t)* refers to the applied source function, the coefficients *A* denote constants, and *i(t)* is the solution or desired response. A test of the usefulness of the Laplace transform is that it must succeed in converting this integrodifferential equation to a form which is easier to handle in finding the solution. Since Eq. (6.64) involves derivatives as well as an integral, and because of the importance of preserving the identity of the response function in spite of these operations of differentiation and integration, it can reasonably be concluded that the Laplace transform must in some way involve the exponential function.

It will be recalled that the exponential function has the property that differentiation and integration always results in a new function which preserves the exponential character.

A time function (current, voltage, e.m.f.) is written f(t).The direct Laplace transform of a function f(t) is defined as

$$F(p) = \int_0^\infty f(t)e^{-pt}dt \tag{6.65}$$

where *p* is the intermediate or transformed variable

Note that *p* is part of the exponential function. Equation (6.65) states that to obtain the Laplace transform of a function *f(t)* it must be multiplied by e^{-pt} and then integrated from $t = 0$ to $t = \infty$. Furthermore, in order that *F(p)* be meaningful, it is necessary that the integral converge and that *f(t)* be defined for $t > 0$ and equal to zero for $t \leq 0$. In general the variable *p* is *a complex number* such that $p = a + jb$, where *a* is a real part, and *b* is the imaginary part.

Accordingly, for most of the functions encountered in electrical engineering, convergence is ensured by imposing the condition that the real part of *p* be positive, i.e., $Re\,[p] > 0$. Moreover, because *t* is used in this book exclusively to represent *time*, it follows that *p* must have the dimensions of inverse time, which is frequency.

It is for these reasons, then, that the transformed variable is described as a complex frequency. An examination of Eq. (6.65) then reveals that after integration many time functions can be expressed in terms of the complex frequency *p*, thus emphasizing that the Laplace transform is a mathematical tool which permits solving time-domain problems in the frequency domain.

To solve the integrodifferential equation by the Laplace transform, it is necessary to apply the Laplace integral to each term of the equation. This converts the original time-domain equation to one expressed entirely in terms of the intermediate variable *p*.

The function *f(t)* is the *inverse transform* of *F*. In Eq. (6.64) the unknown response *i(t)* is identified in the transform domain as *I(p)*, i.e., the correspondence between them is written thus

$$L[f(t)] = F(p)\,. \tag{6.66}$$

Also, a glance at Eq. (6.65) shows that $L[A_0 i(t)] = A_0 I(p)$. Thus one of the five terms of Eq. (6.64) is transformed. Let's desire to find the Laplace transform of a function *f(t)=A*, where A is a constant. Substituting *A* for *f(t)* in Eq. (6.66) and integrating gives

$$F(p) = \int_0^\infty Ae^{-pt} dt = A\left(-\frac{1}{p}\right)\int_0^\infty d\left(e^{-pt}\right) = -\frac{Ae^{-pt}}{p}\Bigg| = \frac{A}{p}. \quad (6.67)$$

Thus the Laplace transform of a constant may be written as

$$L[A] = \frac{A}{p}$$

or

$$A \mathrel{\dot{=}} \frac{A}{p} \quad (6.68)$$

where $\mathrel{\dot{=}}$ is the sign of correspondence. Accordingly the imagine of an exponential function $f(t) = e^{\alpha t}$,

$$F(p) = \int_0^\infty e^{\alpha t} e^{-pt} dt = \int_0^\infty e^{(\alpha - p)t} dt = \frac{e^{(\alpha-p)t}}{\alpha - p}\Bigg|_0^\infty = \frac{1}{\alpha - p}(0-1) = \frac{1}{p-\alpha}.$$

Thus

$$e^{\alpha t} \mathrel{\dot{=}} \frac{1}{p-\alpha}. \quad (6.69)$$

If $f(t) = Ae^{\alpha t}$,

$$Ae^{\pm\alpha t} \mathrel{\dot{=}} \frac{A}{p \mp \alpha}. \quad (6.70)$$

In deriving the transform Eq. (6.69), it is taken into account that the real part of the operator *p* is greater than α, that is *a* >α. It is only then that the integral converges. This transform (6.69) has a number of important corollaries. Thus putting α = jω gives

$$Ae^{j\omega t} \mathrel{\dot{=}} \frac{A}{p - j\omega}. \quad (6.71)$$

We next turn attention to the first derivative term. How is the Laplace transform of a derivative term found? The answer to this question is crucial in establishing whether or not the Laplace transform is useful in simplifying the solution of integrodifferential equations. To demonstrate that the Laplace transform does, in fact, achieve this objective, let us start with the statement that *F(p)* is the transform of *f(t)*.We inquire about the transform of a first derivative ${}^{df(t)}\!\!/_{dt}$, knowing that for $t = 0$, $f(t) = f(0)$. Converting the function ${}^{df(t)}\!\!/_{dt}$ by the Laplace transform method gives

$$\int_0^\infty \frac{df(t)}{dt} e^{-pt} = \int_0^\infty e^{-pt} d\left[f(t)\right].$$

Although we do not as yet know the specific form of *f(t)*, we are assuming that it is a continuous function for $t > 0$ and that its Laplace transform does exist. The right side of this equation is readily recognized as the integral of the product of two variables. From the calculus we know that

$$\int u\, dv = u\, v - \int v\, du\ .$$

The notation procedure in dealing with Laplace transforms is to use lowercase letters to denote the time functions and capital letters for the corresponding Laplace-transformed function.

Putting $e^{-pt} = u$ and $d\left[f(t)\right] = dv$, and integrating by parts gives

$$\int_0^\infty e^{-pt} d\left[f(t)\right] = e^{-pt} f(t)\Big\|_0^\infty - \int_0^\infty f(t)\, d\left[e^{-pt}\right].$$

However, $e^{pt} f(t)\Big|_0^\infty = 0 - f(0) = -f(0)$, and

$$-\int_0^\infty f(t)\, d\left[e^{-pt}\right] = p\int_0^\infty f(t) e^{-pt} dt = pF(p)\ .$$

Thus the Laplace transform of a derivative term is

$$\frac{df(t)}{dt}\,.\!=^{\cdot}\ pF(p) - f(0)\,. \tag{6.72}$$

Equation (6.72) is significant because it shows that the Laplace transform of a derivative in the original time domain carries over as the simpler operation of multiplication by the intermediate variable p in the transform domain. This result bears out the usefulness of the Laplace transform, for in transforming differentiation it preserves the transform of the original time function except for an algebraic operation. In addition, the term $f(0)$ describes in a direct and formal fashion the manner in which the initial condition is to be treated. Extension of the Laplace transform to derivatives of higher order is readily accomplished by repeated application of the general procedure implied in Eq. (6.72).

Thus the Laplace transform of the second derivative is

$$\frac{d^2 f(t)}{dt^2} .\!=^{\cdot}\; p^2F(p) - pf(0) - \left[\frac{df(t)}{dt}\right]_{t=0} . \qquad (6.73)$$

Therefore, the Laplace transform of the second derivative of the current i is

$$\frac{d^2 i(t)}{dt^2} .\!=^{\cdot}\; p^2 I(p) - p\,i(0) - pi'(0) .$$

Returning to Eq. (6.64) we see that we are now in a position to Laplace transform each term in the equation with the exception of the term involving the integral. By following a procedure similar to that used for differentiation it can be shown that the Laplace transform of an integral of time may be expressed as

$$\int_0^\infty \left[\int_0^t f(t)\,dt\right] e^{-pt}\,dt = -\frac{1}{p}\int_0^\infty \left[\int_0^t f(t)\,dt\right] d(e^{-pt}) =$$

$$-\frac{1}{p}\left[\int_0^t f(t)\,dt\right] e^{-pt}\Big|_0^\infty + \frac{1}{p}\int_0^\infty f(t)\,e^{-pt}\,dt = \frac{F(p)}{p} .$$

The substitution of the upper and the lower limits reduces the first term of the fight-hand side to zero. The substitution of the upper limit reduces it to zero by virtue of the limitation imposed of the function *f(t)* earlier, namely that the function *f(t)* should grow with increasing t more slowly than the exponential function, if at all. The substitution of the lower limit reduces zero because the term $\int_0^t f(t)\,dt$ reduces to zero.

Therefore, if $f(t) \mathrel{.\!=^{\cdot}} F(p)$, then

$$\int_0^t f(t)\ dt \mathrel{.\!=^{\cdot}} \frac{F(p)}{p}\,. \tag{6.74}$$

6.17. The Laplace Transform of the Voltage Across the Inductance

The Laplace Transform of a current *i* is *I*(*p*). Then the transform of the voltage across the inductance will be $u_L = L\dfrac{di}{dt}$.

To find the imagine of this function we will take the advantage of the Laplace transform (integrating in parts)

$$U_L(p) = \int_0^\infty L\frac{di}{dt}e^{-pt}dt = L\int_0^\infty e^{-pt}di = \begin{vmatrix} w = e^{-pt} & dw = -pe^{-pt}dt \\ dv = di & v = i \end{vmatrix} =$$

$$= iLe^{-pt}\Big|_0^\infty - \int_0^\infty iL(-p)e^{-pt}dt = 0 - Li(0) + pL\int_0^\infty ie^{-pt}dt = -Li(0) + pLI(p).$$

Therefore,

$$u_L = L\frac{di}{dt} \mathrel{.\!=^{\cdot}} pLI(p) - Li(0)\,. \tag{6.75}$$

6.18. The Laplace Transform of the Voltage Across a Capacitor

The voltage across a capacitor, u_C, if often given the form $u_C = \dfrac{1}{C}\displaystyle\int idt$. This form, however, does not specify the limits of integration with respect to time. The more exhaustive form of the capacitive voltage is $u_C = u_C(0) + \dfrac{1}{C}\displaystyle\int_0^t idt$. This expression takes into account the fact that by time *t* the voltage across a capacitor is decided not only by the current through the capacitor during the time interval from zero to the time *t*.

In accordance with Eq. (6.74) the transform of $\frac{1}{C}\int idt$ is $\frac{I(p)}{Cp}$, and the transform of the constant $u_c(0)$ is the constant itself divided by p.

To find the imagine of this function we will take advantage of Laplace transform (having presented the integral in the form of the sum of integrals and integrating in parts) as

$$U_C(p) = \int\limits_0^{\infty}\left(u_C(0) + \frac{1}{C}\int\limits_0^t idt \right) e^{-pt}dt = u_C(0)\int\limits_0^{\infty} e^{-pt}dt + \frac{1}{C}\int\limits_0^{\infty}\int\limits_0^t idt \cdot e^{-pt}dt =$$

$$= \frac{u_C(0)}{p} + \frac{1}{C}\int\limits_0^{\infty}\int\limits_0^t idt \cdot e^{-pt}dt = \left| \begin{array}{ll} w = \frac{1}{C}\int\limits_0^t idt & dw = \frac{1}{C}i \\ dv = e^{-pt}dt & v = -\frac{1}{p}e^{-pt} \end{array} \right| =$$

$$= \frac{u_C(0)}{p} - \frac{1}{pC}e^{-pt}\int\limits_0^t idt\,\Bigg|_0^{\infty} + \frac{1}{pC}\int\limits_0^{\infty} ie^{-pt}dt = \frac{u_C(0)}{p} + \frac{I(p)}{pC}.$$

Therefore, the Laplace transform of the capacitive voltage is

$$u_C = \frac{1}{C}\int\limits_0^t idt + u_C(0).=^{\cdot} \frac{I(p)}{pC} + \frac{u_C(0)}{p}\;. \qquad (6.76)$$

The Laplace transform and the respective inverse Laplace transform are referred to as a transform pair.

A formal solution to a linear integrodifferential equation is readily obtained in an intermediate form by Laplace transforming each term. This procedure leads to the represented results. Here all the circuit parameters as well as all initial conditions are known.

The only unknown term is $F(p)$. However, the specific form of $F(p)$ depends upon the nature of $f(t)$. Consequently, we next direct attention to the evaluation of $F(p)$ for those time functions which are commonly found in the study of electrical engineering.

Table 6.1

$F(p)$	$f(t)$
$\dfrac{1}{p \pm \alpha}$	$e^{\mp \alpha t}$
$\dfrac{1}{p - j\omega}$	$e^{-j\omega t}$
$\dfrac{1}{p\,(p+\alpha)}$	$\dfrac{1}{\alpha}\,(1 - e^{-\alpha t})$
$\dfrac{\alpha}{p\,(p+\alpha)}$	$1 - e^{-\alpha t}$
$\dfrac{1}{(p+\alpha)^2}$	$te^{-\alpha t}$
$\dfrac{p}{(p+\alpha)^2}$	$(1 - \alpha t)\; e^{-\alpha t}$
$\dfrac{1}{p\,(p+\alpha)^2}$	$\dfrac{t}{a} - \dfrac{1}{a^2} + \dfrac{e^{-\alpha t}}{a^2}$
$\dfrac{1}{(p+\alpha)(p+b)}$	$\dfrac{1}{a-b}(e^{-at} - e^{-bt})$
$\dfrac{p}{(p+\alpha)(p+b)}$	$\dfrac{1}{a-b}(a\,e^{-at} - b\,e^{-bt})$
$\dfrac{1}{p^2 + a^2}$	$\dfrac{1}{a}\sin\,(at)$
$\dfrac{p}{p^2 + a^2}$	$\cos\,(at)$
$\dfrac{1}{p^2}$	t
$\dfrac{1}{p^3}$	$\dfrac{1}{2}t^2$

Once the Laplace transform *F(p)* is found for a specific time function *f*(*t*), it is tabulated for future reference just as is done for logarithms or integrals.

Such a tabulation is particularly useful because, as can be shown, the Laplace transform of a time function, if it exists, is unique. The most useful of the transform pairs are listed above in Table 6.1.

6.19. Ohm's Law in Operational Form. Internal Electromotive Forces

Let's consider a part of the complex ramified electrical circuit. The electrical branch which consists of e.m.f. source and resistive, inductive and capacitor elements is connected to two nodes *a* and *b*. The current in the branch is directed from node *a* to node *b*.

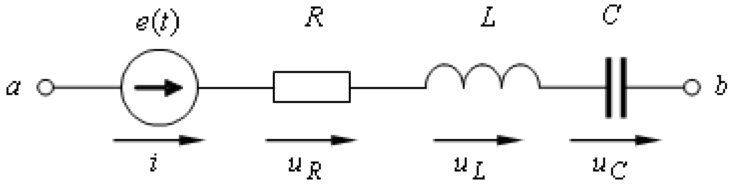


Fig. 6.19. A part of the complex ramified electrical circuit.

The switching in other part of the circuit leads to the transient. Independent initial conditions for a considered section are: the current through an inductive element $i(0)=i(0-)$ and the voltage across a capacitor $u_C(0)=u_C(0-)$.

The voltage between nodes *a* and *b* for the circuit after switching

$$u_{ab}=\varphi_a-\varphi_b=u_R+u_L+u_C-e(t) \tag{6.77}$$

where $u_R=Ri$; $u_L=L\dfrac{di}{dt}$; $u_C=\dfrac{1}{C}\int\limits_0^t idt+u_C(0)$. Substituting these values for u_R, u_L, u_C gives

$$u_{ab}=Ri+L\frac{di}{dt}+\frac{1}{C}\int idt-e(t). \tag{6.78}$$

Since the Laplace transform is a linear operator, it follows that the transform of a sum is equal to the sum of transforms. Therefore, we may substitute the respective transforms for each of the terms in Eq. (6.78), and we get

$$U_{ab}(p) = RI(p) + pLI(p) - Li(0) + \frac{1}{pC} I(p) + \frac{u_C(0)}{p} - E(p)\,. \qquad (6.79)$$

The effect of the above transformation is that instead of a differential equation (6.78) we have the algebraic equation (6.79) relating the current transform *I(p)* to the e.m.f. transform *E(p)* and the voltage transform $U_{ab}(p)$. From Eq. (6.78) it follows that

$$I(p) = \frac{U_{ab}(p) + E(p) + Li(0) - \dfrac{u_C(0)}{p}}{R + pL + \dfrac{1}{pC}} \qquad (6.80)$$

where $Z(p) = R + pL + \dfrac{1}{pC}$ is the *operation impedance* of the branch *ab*. It is similar in structure to the complex impedance of the same branch with *p* substituted for *jω*.

The term operation impedance is preferred in this description because resistance, inductance and capacitance are involved. Furthermore, the adjective operational is used because it is through the operation of differentiation that the factor p appears in equation as a multiplier of the both reactive parameters.

The concept of operational impedance is useful because by means of it one can write the response I(p) caused by a forcing function E(p) by merely applying Ohm's law in a general form.

Eq. (6.80) may be termed Ohm's law in operational form for the circuit branch containing a voltage source under non-zero independent initial conditions.

The term *Li*(0) is the internal e.m.f. due to the energy stored by the magnetic field of an inductive element, produced by the current i(0) through the inductance just before a switching.

The term $u_C(0)/p$ is the internal e.m.f. due to the energy stored by the electrostatic field of a capacitive element, produced by the voltage across the capacitor immediately before a switching.

In a special case, when a branch contains no e.m.f. and when we have zero initial conditions at the instant of switching, Eq. (6.80) takes on a simpler form

$$I(p) = \frac{U_{ab}(p)}{Z(p)}. \tag{6.81}$$

Eq. (6.81) is a mathematical expression for Ohm's law in operational form for the circuit branch containing no e.m.f., under zero independent initial conditions.

6.20. Kirchhoff's Laws in Operational Form

According to the *first Kirchhoff's law* an algebraic sum of instantaneous values of the currents converging in any junction of an electric circuit is equal to zero

$$\sum_{k=1}^{n} i_k = 0.$$

The current by means of Laplace transform can be presented in the operation form. As the Laplace transform is linear, the transform of the sum is equal to the sum of transforms.

$$\sum_{k=1}^{n} I_k(p) = 0. \tag{6.82}$$

According to the *second Kirchhoff's law* in any closed loop the sum of the instantaneous voltages across all the sections which enter into this contour, is equal to the algebraic sum of electromotive forces:

$$\sum_{k=1}^{n} R_k i_k + \sum_{k=1}^{n} L_k \frac{di_k}{dt} + \sum_{k=1}^{n}\left(\frac{1}{C_k}\int_0^t i_k + u_{Ck}(0)\right) = \sum_{k=1}^{n} e_k. \tag{6.83}$$

Having transferred to operational imagines, we have got the second Kirchhoff's law in the operational form as

$$\sum_{k=1}^{n} R_k I_k(p) + \sum_{k=1}^{n}\left(pL_k I_k(p) - L_k i_{Lk}(0)\right) + \sum_{k=1}^{n}\left(\frac{I_k(p)}{pC} + \frac{u_{Ck}(0)}{p}\right) = \sum_{k=1}^{n} E_k(p)$$

or

$$\sum_{k=1}^{n} Z_k(p) I_k(p) = \sum_{k=1}^{n} \left(E_k(p) + L_k i_{Lk}(0) - \frac{u_{Ck}(p)}{p} \right).$$

where $i_{Lk}(0)$, $u_{Ck}(0)$ are initial conditions of currents through inductive elements and voltages across the capacitive elements in corresponding branches. In the case of zero initial conditions the second Kirchhoff's law in the operational form looks like

$$\sum_{k=1}^{n} Z_k(p) I_k(p) = \sum_{k=1}^{n} E_k(p) \qquad (6.84)$$

Thus, Ohm's law, the first and second Kirchhoff's laws in the operational form are analogous on a notation to the same laws in the complex form. In the case of nonzero initial conditions (the general case), $E_k(p)$ also includes all internal electromotive forces.

6.21. The equivalent operational circuit

At a transient analysis by operational method it is convenient to write down the equations by Kirchhoff's laws at once in the operational form, and use the calculation methods which are based on Kirchhoff's equations, such as the mesh-current method, the node-analysis method, Thevenin's theorem, the superposition theorem, etc.

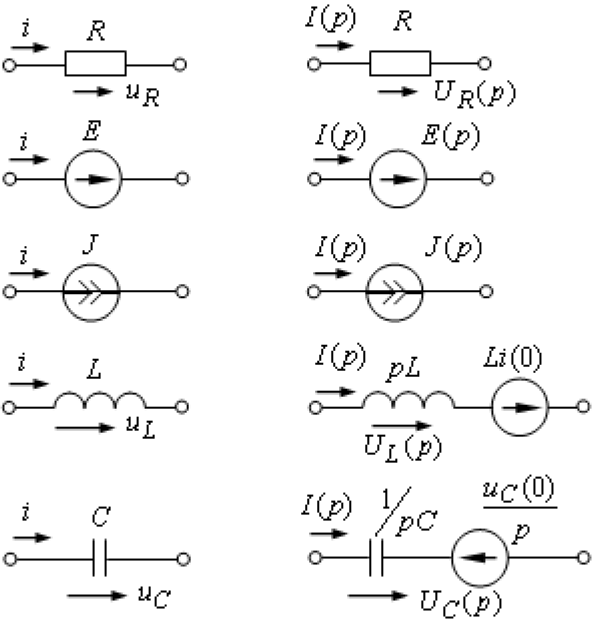
Each of these combined equations can be written having made equivalent operational circuit for the rated electrical circuit.

For this circuit each idealized passive element figured in an equivalent circuit for instantaneous values, should be exchanged by an equivalent operational figure, and voltages (or currents) of idealized voltage (or current) sources must be presented by operational images of corresponding functions.

The relationships between circuit elements for instantaneous values of currents and voltages and the elements of the operational circuit are presented at the end of this section..

The internal e.m.f.s caused by a reserve of energy in reactive elements at the moment of switching, are taken into account here.

In case of zero initial conditions the internal e.m.f.s are equal to zero. Then there are no internal electromotive forces in the equivalent operational circuit.



6.22. Inverse Laplace Transformation Through Partial Fraction Expansion

For determination of the original we will present the image having got in the form of a rational function, by the elementary addends for which originals are known. Let the image looks like a proper fraction (i.e. polynomial extent *m* in a numerator is less than the extent of a polynomial *n* in a denominator $m < n$)

$$F(p) = \frac{F_1(p)}{F_2(p)}, \tag{6.85}$$

and a numerator and a denominator have no common roots.

Now if this form of the *p*-plane solution were available in a table of Laplace-transform pairs, the time solution could be written forthwith. A little thought reveals, however, that such a compilation is impractical because to cover all cases the table would have to be endless. Therefore it is customary to make available only a limited number of the basic forms and then to treat all other cases by reducing them through partial-fraction expansion into the basic forms.

To accomplish this in the common case, it is necessary first to find the roots of this equation by any of the standard methods of algebra. For the sake of illustration assume these roots are found to be p_1, p_2,…and so on to p_n and that each is real. Then Eq. (6-85) may be rewritten in terms of a partial-fraction expansion as (6-86)

$$\frac{F_1(p)}{F_2(p)} = \frac{A_1}{p-p_1} + \frac{A_2}{p-p_2} + ... + \frac{A_n}{p-p_n} = \sum_{k=1}^{n} \frac{A_k}{p-p_k}. \qquad (6.86)$$

Now since each term on the right side of this expression is identifiable in Table 6-1, a complete description of the time solution merely requires an evaluation of the *n* coefficients.

Before proceeding further with the partial-fraction expansion as a means of facilitating the evaluation of the inverse Laplace transformation, let us formulate the transformed solution of Eq. (6-86) in completely general terms. Accordingly, we may write

$$F(p) = \frac{F_1(p)}{F_2(p)} = \frac{A_1}{p-p_1} + \frac{A_2}{p-p_2} + ... + \frac{A_n}{p-p_n} = \sum_{k=1}^{n} \frac{A_k}{p-p_k}. \qquad (6.87)$$

Since in the vast majority of physical situations the order of the denominator polynomial is very often greater than that of the numerator, i.e., n > m, the partial-fraction expansion is directly applicable. In those rare cases where *n* = *m* it is necessary to do longhand division in order that *F(p)* be in the form of a proper fraction. Because of its infrequent occurrence this case will receive no further attention. Hence in the material that follows *F(p)* is assumed to be expressible as a proper fraction so that partial-fraction expansion is immediately applicable. However, once this is done, the particular manner of evaluating the coefficients of the expansion is dependent upon the character of the roots of $F_2(p)$) in Eq. (6-87). Attention is now directed to all cases of interest which arise. Two cases are possible: all the roots are simple, or some or the roots are multiple. Let's observe these cases separately.

For definition of coefficient A_1 we will multiply both parts of Eq. (6.86) by $(p-p_1)$. We will get the following equation

$$\frac{F_1(p)}{F_2(p)}(p-p_1) = A_1 + (p-p_1)\sum_{k=2}^{n} \frac{A_k}{p-p_k}. \qquad (6.88)$$

Let's consider the given expression at p approaching p_1. The right-hand side of the equation gives A_1, and the left-hand side is an indeterminacy, because $(p-p_1)$ with $p \to p_1$ gives zero and the denominator $F_2(p_1)$ with $p = p_1$ also gives zero (p_1 is the root of the equation $F_2(p)=0$). Evaluate the indeterminate form by L'Hospital, rule. For this purpose the derivative of the numerator should be divided by the derivative of the denominator, and we will find the limit of the fraction

$$A_1 = \lim_{p\to p_1} \frac{F_1(p)}{F_2(p)}(p-p_k) = F_1(p_1) \lim_{p\to p_1} \frac{p-p_k}{F_2(p)} =$$

$$= F_1(p_1) \lim_{p\to p_1} \frac{1}{F_2'(p)} = \frac{F_1(p_1)}{F_2'(p_1)},$$

where $F_2'(p_1)$ is the derivative of $F_2(p_1)$ with respect to p; $F_1'(p_1)$ is the value of $F_1'(p)$ at $p = p_1$; and $F_1(p_1)$ is the value of $F_1(p)$ at $p = p_1$.

Therefore, at $p \to p_1$, Eq. (6.88) may be rewritten thus

$$A_1 = \frac{F_1(p_1)}{F_2'(p_1)}.$$

Analogically one can determine the remaining coefficients as

$$A_k = \frac{F_1(p_k)}{F_2'(p_k)}. \tag{6.89}$$

Hence,

$$F(p) = \frac{F_1(p)}{F_2(p)} = \sum_{k=1}^{n} \frac{A_k}{p-p_k} = \sum_{k=1}^{n} \frac{F_1(p_1)}{F_2'(p_1)} \cdot \frac{1}{p-p_k}. \tag{6.90}$$

As it has been shown earlier that

$$\frac{A_k}{p-p_k} \mathrel{\dot{=}} A_k e^{p_k t},$$

one can find the corresponding time function. The inverse transform of the left-hand side is *f(t)*; that of the right-hand side is the sum of the component time functions.

Noting that $F_1(p_k)/F_2'(p_k)$ on the right-hand side of Eq. (6.90) are constants, and that the functions of p on the right-hand side are solely the factors $1/(p-p_k)$ corresponding to a time function of the form $e^{p_k t}$ (see Eq. 6.69), we may write

$$f(t)=\sum_{k=1}^{n}\frac{F_1(p_k)}{F_2'(p_k)}e^{p_k t}\ . \tag{6.91}$$

The inverse transformation with the aid of Eq. (6.91) is based on the expansion of the transform into partial fractions of the form $\left[F_1(p_k)/F_2'(p_k)\right]\left[1/(p-p_k)\right]$, whose corresponding time functions are exponential functions $\left[F_1(p_k)/F_2'(p_k)\right]e^{p_k t}$.

The expression in the right-hand side of Eq. (6.90) has as many terms as there are roots of the equation *F(p)* = 0. The factors $F_1(p_k)/F_2'(p_k)$ may be likened to the constants of integration of a differential equation in the classical method of transient analysis.

If one of the roots of the equation *F(p)* = 0 is *p* = 0, than the corresponding term on the right-hand side of Eq. (6.91) will be

$$f(t)=\frac{F_1(0)}{F_2'(0)}+\sum_{k=2}^{n}\frac{F_1(p_k)}{F_2'(p_k)}e^{p_k t}\ . \tag{6.92}$$

The term $F_1(0)/F_2'(0)$ is the component of the unknown current or voltage due to a direct-current voltage. If there are no direct-current voltages in the circuit, this value is equal to zero. If equation $F_2(p)=0$ has complex conjugate roots there is no necessity to determinate the items in the right-hand side of the expansion theorem for each of complex conjugate roots separately. It is caused by that the functions with the real coefficients from complex conjugate values of an independent variable are complex conjugate themselves. The sum of complex conjugate magnitudes is equal to the doubled value of the real part of any of these numbers. Therefore, if roots p_1 and p_2 are complex conjugate it is enough to calculate the value of this function only for the root p_1 and find the real part of this term.

$$\frac{F_1(p_1)e^{p_1 t}}{F_2'(p_1)}+\frac{F_1(p_2)e^{p_2 t}}{F_2'(p_2)}=2\,\mathrm{Re}\left(\frac{F_1(p_1)e^{p_1 t}}{F_2'(p_1)}\right). \tag{6.93}$$

In a general case, when one of the roots, for example p_1, in the equation $F_2(p)=0$ to the power *n* has a multiplicity *m*, the rational fraction can be also apportioned into vulgar fractions

$$\frac{F_1(p)}{F_2(p)}=\frac{F_1(p)}{(p-p_1)^m F_3(p)}=$$

$$=\frac{A_{11}}{p-p_1}+\frac{A_{12}}{(p-p_1)^2}+\ldots+\frac{A_{1m}}{(p-p_1)^m}+\sum_{k=2}^{n}\frac{A_k}{p-p_k}$$

where

$$A_{1q}=\frac{1}{(q-1)!}\left(\frac{d^{q-1}}{dp^{q-1}}\left(\frac{(p-p_1)^m F_1(p)}{F_2(p)}\right)\right)_{p=p_1}=$$

$$=\frac{1}{(q-1)!}\left(\frac{d^{q-1}}{dp^{q-1}}\left(\frac{(F_1(p)}{F_3(p)}\right)\right)_{p=p_1}.$$

The apportionment into vulgar fractions can be made by the method of undetermined coefficients which is related to the solution of the system of equations, that it is not convenient in practice.

Required time function looks like

$$f(t)=e^{p_1 t}\sum_{q=1}^{m}\frac{1}{(q-1)!\cdot(m-q)!}\left(\frac{d^{q-1}}{dp^{q-1}}\left(\frac{F_1(p)}{F_3(p)}\right)\right)_{p=p_1}t^{m-q}+$$

$$+\sum_{k=2}^{n}\frac{F_1(p_k)}{F_2'(p_k)}e^{p_k t}\,. \tag{6.94}$$

6.23. Common steps in operational analysis

Transient analysis by an operational method is made in the following sequence in connection with the Heaviside expansion theorem:

1. The analysis of a circuit before switching and the definition of independent initial conditions. It is fulfilled in the same way as in a classical method of transient analysis. Note that the expansion formula is applicable under any initial conditions and to practically any drive wave-forms.

2. Non-zero initial conditions are dealt with by introducing "internal" electromotive forces in *N(p)*. Then we formulate an equivalent operational circuit immediately after switching. This formulation of an equivalent operational circuit is made by replacement of energy sources and passive elements by their operational images.

3. Making up the system of equations of electrical equilibrium in a circuit in an operational form. This system can be generated by any of methods which we use for the calculation of steady-state rates in the linear circuits immediately for the equivalent operational circuit.

4. The solution of this system of equations concerning the images of required currents or voltages. It can be made by any known method.

5. The definition of originals of required currents or voltages. The process of finding the time solution by going from the p domain to the time domain through the use of Eq. (6.91) or appropriate tables can be described as performing the inverse Laplace transformation.

In the next sections we consider the matter of manipulating complicated expressions of transformed solutions into forms which are available in our listing of Laplace-transform pairs as presented in table of inverse Laplace transformations.

If the image of required function represents the ratio of two polynomials rather p, it is possible to take advantage of expansion theorem for making up of an inverse Laplace transform.

6.24. Step Response of a Direct Current RL Circuit

As an example we will consider the transient in the direct current electrical circuit which is presented in Fig. 6.20.

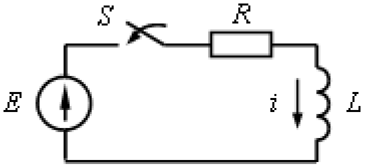


Fig. 6.20. Series *RL* circuit.

Appearing in Fig. 6.20 is the circuit arrangement of an initially deenergized inductor in series with a resistor. It is desired to find the complete solution for the current when the switch is closed.

The governing differential equation results on applying Kirchhoff's voltage law and giving due regard to the conventions of direction, we may write for clockwise summation round the loop. Hence

$$E = Ri + L\frac{di}{dt} \tag{6.95}$$

The current i is the solution being sought and E is the applied source function which causes the response i to exist. This equation is a linear differential equation of first order. It contains a single energy-storing element as evidenced by the first derivative term.

Because the switch in Fig, 6.20 is initially open, it follows that

$$i(0) = 0.$$

Since we have a zero initial condition here the internal e.m.f. due to the energy stored by the magnetic field of an inductive element is also equal to zero. Then we have the following operational circuit which is presented in Fig. 6.21. The desired solution in the time domain is identified as $i(t)$. The corresponding unknown function in the p domain is called $I(p)$.

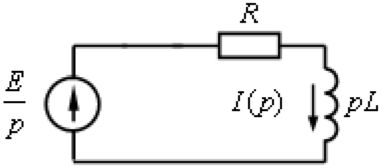


Fig. 6.21. Operational *RL* circuit.

Substituting the respective transforms for each term of Eq. (6.95) gives

$$\frac{E}{p} = I(p)\ (R + pL).$$

Therefore, the transformed solution for the response becomes

$$I(p) = \frac{E/p}{R + pL} = \frac{E/L}{p(p + R/L)}. \tag{6.96}$$

To obtain the time solution which corresponds to Eq. (6.96) we must perform the inverse Laplace transformation, which requires putting *I(p)* into a form whose terms are readily identifiable in Table 6.1. This result is achieved by using a partial-fraction expansion. Accordingly, we can rewrite this equation as

$$I(p) = \frac{E/L}{p\,(p + R/L)} = \frac{E}{L} \cdot \frac{1}{p\,(p + \alpha)}\,,$$

where $\alpha = R/L$.

From the table of Laplace transform pairs we find that

$$\frac{1}{p\;(p + \alpha)} \;\mathrel{.{=}\cdot}\; \frac{1}{\alpha}\,(1 - e^{-\alpha t})$$

Then by the application of this transform we have the following result

$$i(t) = \frac{E}{L}\left[\frac{1}{R/L}\left(1 - e^{-(R/L)t}\right)\right] = \frac{E}{R} - \frac{E}{R}\,e^{-(R/L)t} \tag{6.97}$$

where the term E/R is a steady-state component of the current, and the term ${}^{E}\!\!/_{R}\,e^{-(R/L)t}$ is a transient (or free) component of the required current. Thus, by means of Table 6.1 this operation is carried out by proceeding from the *p*-function column to the corresponding time-function column. It is interesting to note in Eq. (6.96) that the presence of *p* in the denominator is associated with the step forcing function, and this in turn is responsible for generating the steady-state solution. On the other hand, the operational impedance ($R + pL$) is associated with the generation of the transient term, as also revealed by Eq. (6.97).

In many instances in electrical engineering, particularly in control systems, it is useful to formulate the ratio of the output response (in this case, current) to the input forcing function (voltage) of the Laplace-transformed function for zero initial conditions. This ratio is called the *transfer function* of the circuit. In the case of the *RL* circuit the transfer function can be written as

$$\frac{I(p)}{E(p)} = \frac{1}{Z(p)} = \frac{1}{R + pL}\,. \tag{6.98}$$

Note that it is unnecessary to specify the specific form of the forcing function. As the right side of Eq. (6.98) indicates, the transfer function indirectly conveys information about the transient response of the circuit. The transfer function involves just the circuit parameters.

6.25. Step Response of a Direct Current RC Circuit

Appearing in Fig. 6.22 is the circuit arrangement of a resistor in series with a capacitor.

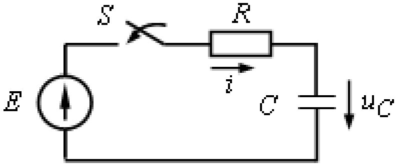


Fig. 6.22. Series RC circuit.

Let us now find the complete expression for the current after the switch *S* is closed.

Taking voltage drops in a clockwise direction as positive the governing differential equation is obtained upon applying Kirchhoff's voltage law to the closed circuit. Thus

$$E = Ri + \frac{1}{C}\int i dt. \tag{6.99}$$

Laplace transforming each term yields

$$\frac{E}{p} = I(p)\ (R + \frac{1}{Cp}). \tag{6.100}$$

Because the switch in Fig, 6.22 is initially open, it follows that $u_C(0) = 0$. Since we have a zero initial condition here the internal e.m.f. due to the energy stored by the electrostatic field of a capacitive element, produced by the voltage across the capacitor is also equal to zero. Then we have the following operational circuit described in Fig. 6.23.

Therefore, the transformed solution for the response becomes

$$I(p) = \frac{E/p}{R + \dfrac{1}{Cp}} = \frac{ECp}{p(RCp+1)} = \frac{E}{R}\cdot\frac{1}{\left(p + \dfrac{1}{RC}\right)} = \frac{E}{R}\cdot\frac{1}{(p+\alpha)} \tag{6.101}$$

where $\alpha = 1/RC$.

From the table of Laplace transform pairs we find that

$$\frac{1}{(p+\alpha)} \mathrel{.{=}^{\cdot}} e^{-\alpha t}.$$

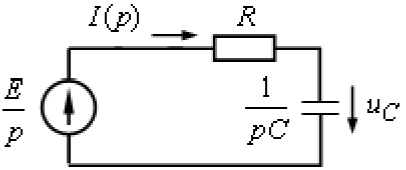


Fig. 6.23. Operational RC circuit

By the application of this transform we have the following result

$$i(t)=\frac{E}{R}\,e^{-\frac{1}{RC}t}. \tag{6.102}$$

So, by means of inverse Laplace-transformed functions (see Table 6.1) this operation is carried out by proceeding from the *p*-function form to the corresponding time-function form. An inspection of. the exponential term of the last equation reveals that the time constant of this circuit is given by $\tau = RC$. The complete current solution in the case of the simple *RC* circuit consists merely of the transient term because the capacitor presents an open circuit to the battery source at steady state. Hence the forced solution is zero in this instance.

6.26. Step Response of Second-Order system (*RLC* Circuit)

Let it be desired to find the complete current response in the circuit of Fig. 6.24 after the switch *S* is closed. For simplification of the problem we consider that the capacitor has no initial charge before switching.

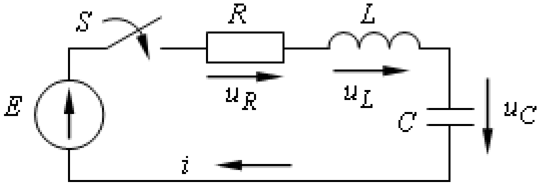


Fig. 6.24

Because the switch in Fig, 6.24 is initially open, it follows that $u_C(0) = 0$. Since we have a zero initial condition here the internal e.m.f. due to the energy stored by the electrostatic field of a capacitive element, produced by the voltage across the capacitor is also equal to zero. Then we have the following operational circuit

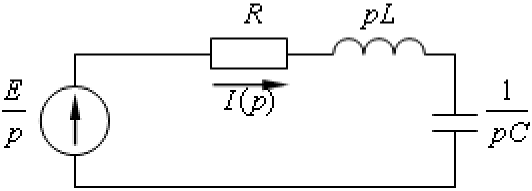


Fig. 6.25. Operational RLC circuit.

The governing differential equation for the circuit is found upon applying Kirchhoff's voltage law. Thus

$$E = Ri + L\frac{di}{dt} + \frac{1}{c}\int idt. \qquad (6.103)$$

Equation (6.103) is a second-order nonhomogeneous differential equation because it involves a derivative as well as an integral term. Laplace transforming each term of the equation yields

$$\frac{E}{p} = I(p)\cdot\left(R + pL + \frac{1}{pC}\right). \qquad (6.104)$$

The function in brackets on the right side is the operational impedance of the series *RLC* circuit. The expression for the current solution in the *p* domain then readily becomes

$$I(p) = \frac{CE}{p^2LC + pRC + 1} = \frac{E/L}{\left(p^2 + p\frac{R}{L} + \frac{1}{LC}\right)}. \qquad (6.105)$$

A glance at Eq. (6.105) shows that the denominator expression does not include an *p* factor standing alone as was the case for the *RL* circuit. This means that a constant term in the steady-state solution does not exist. A glance at the circuit configuration verifies this conclusion because in the steady state the capacitor presents an open circuit to the battery so that the particular solution must be zero. It is significant to note that the quadratic expression in the denominator of Eq. (6.105) results from an algebraic manipulation of the operational impedance.

Accordingly, it involves each of the three circuit parameters. Moreover, the expression is quadratic because of the presence of two energy-storing elements. From the experience we have gained so far in evaluating the inverse Laplace transform, we know that the first step in the procedure is to identify the specific root factors of the poles of the transformed solution *I(p)*. This then permits a partial-fraction expansion to be made, each root factor being readily identifiable in Table 6.I.

For each of the cases treated so far there was no need to find the root factors of the denominator of *I(p)* because they automatically appeared in the desired form, as reference to Eqs. (6.96) and (6.102) indicates. This simplicity was a consequence of dealing with first-order circuits. Before we can proceed further in the solution procedure for *i(t)* in the situation now under consideration, we must first find the specific roots of the quadratic expression. That is, we must determine those values of *p* which satisfy the equation

$$R + pL + \frac{1}{pC} = 0\,. \tag{6.106}$$

This expression is called the *characteristic equation* of the second-order system. Keep in mind that it results from setting the operational impedance equal to zero and then performing an algebraic manipulation which serves to isolate the p^2 term. The roots of this equation, which are also the poles of *I(p)* in this instance, depend solely upon the circuit parameters. In turn these roots determine entirely the nature of the transient response. To understand this better, consider that p_1 and p_2 are the roots of the characteristic equation. The formal expression for the corresponding time solution then becomes

$$i(t) = A_1\, e^{p_1 t} + A_2\, e^{p_2 t}\,. \tag{6.107}$$

Thus the characteristic manner in which the transient terms decay is solely dependent on the roots p_1 and p_2. The total solution in this case involves only a transient term because, as already pointed out, the forced solution is zero. As an example we will consider the transient in the direct current electrical circuit which it is presented in Figure 6.26.

Example 6.5. In the circuit of Fig.6.26, for the values of the parameters E = 60 V, $R_1 = 10\,\Omega$, $R_2 = 20\,\Omega$, $R_3 = 30\,\Omega$, $L = 0.1H$, $C = 0.002\,F$ find the expression for the currents which flow through branches when the switch *S* is closed.

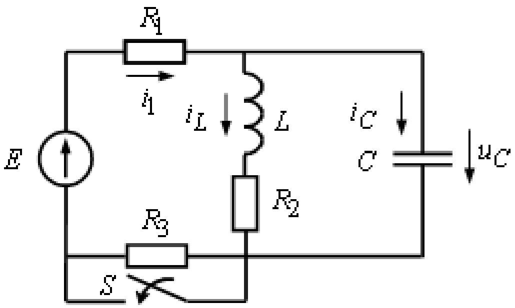


Fig. 6.26

Solution: Let's assume, that before switching the installed mode took place, that is currents through all branches and voltage across the circuit elements were invariable in time. Let's define independent initial conditions. The current through an inductive element

$$i_L(0) = i_L(0-) = \frac{E}{R_1 + R_2 + R_3} = \frac{60}{10 + 20 + 30} = 1\ A .$$

The voltage across the capacitive element

$$u_C(0) = u_C(0-) = R_2 i_L(0-) = 20\frac{60}{10 + 20 + 30} = 20\ V$$

Then the equivalent operational circuit after switching one can see in Fig. 6.27. Let's work out the equations by mesh-current method:

$$\begin{cases} I_{11}(p)\left(R_1 + R_2 + pL\right) - I_{22}\left(R_2 + pl\right) = \dfrac{E}{p} + Li_L(0) \\ -I_{11}\left(R_2 + pl\right) + I_{22}(p)\left(R_2 + pL + \dfrac{1}{pC}\right) = -Li_L(0) - \dfrac{u_C(0)}{p} \end{cases}$$

We solve the system of equations by a matrix method. Determine the main determinant of the system:

$$\Delta = \begin{vmatrix} R_1 + R_2 + pL & -R_2 - pL \\ -R_2 - pL & R_2 + pL + \dfrac{1}{pC} \end{vmatrix} =$$

$$= \left(R_1 + R_2 + pL\right)\left(R_2 + pL + \frac{1}{pC}\right) - \left(-R_2 - pL\right)^2 =$$

$$= R_1 R_2 + R_2^2 + pLR_2 + pLR_1 + p^2 L^2 + \frac{R_1}{pC} + \frac{R_2}{pC} + \frac{L}{C} .$$

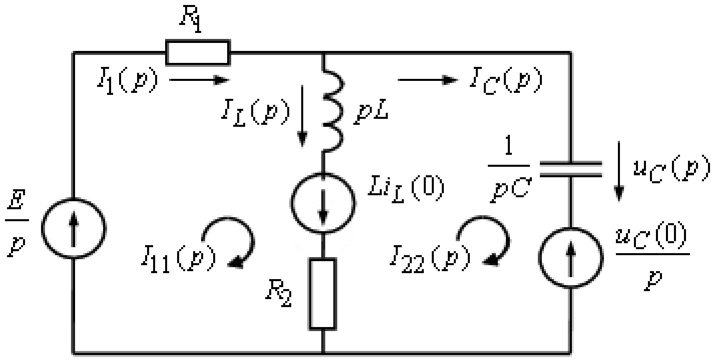


Fig. 6.27. The equivalent operational circuit after switching.

The first additional determinant

$$\Delta_{11} = \begin{vmatrix} \dfrac{E}{p} + Li_L(0) & -R_2 - pL \\ -Li_L(0) - \dfrac{u_C(0)}{p} & R_2 + pL + \dfrac{1}{pC} \end{vmatrix} =$$

$$= \left(\frac{E}{p} + Li_L(0)\right)\left(R_2 + pL + \frac{1}{pC}\right) - \left(-R_2 - pL\right)\left(-Li_L(0) - \frac{u_C(0)}{p}\right) =$$

$$= \frac{E + EpC(R_2 + pL) + pLi_L(0) - u_C(0)pC(R_2 + pL)}{p^2C}.$$

The second additional determinant

$$\Delta_{22} = \begin{vmatrix} R_1 + R_2 + pL & \dfrac{E}{p} + Li_L(0) \\ -R_2 - pL & -Li_L(0) - \dfrac{u_C(p)}{p} \end{vmatrix} =$$

$$= \left(R_1 + R_2 + pL\right)\left(-Li_L(0) - \frac{u_C(p)}{p}\right) - \left(-R_2 - pL\right)\left(\frac{E}{p} + Li_L(0)\right) =$$

$$= \frac{E(R_2 + pL) - R_1 pLi_L(0) - u_C(p)(R_1 + R_2 + pL)}{p}.$$

Now we can find the equations for currents:

$$I_{11}(p) = I_1(p) = \frac{E + EpC(R_2 + pL) + pLi_L(0) - u_C(0)pC(R_2 + pL)}{p\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} =$$

$$= \frac{E + EpC(R_2 + pL)}{pZ(p)} + \frac{Li_L(0)}{Z(p)} - \frac{Cu_C(0)(R_2 + pL)}{Z(p)}.$$

$$I_{22}(p) = I_2(p) = \frac{\left(E(R_2 + pL) - R_1pLi_L(0) - u_C(p)(R_1 + R_2 + pL)\right)C}{\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} =$$

$$= \frac{EC(R_2 + pL)}{Z(p)} - \frac{pLCR_1i_L(0)}{Z(p)} - \frac{Cu_C(p)(R_1 + R_2 + pL)}{Z(p)},$$

where $Z(p) = 0$ is the characteristic equation.

In the given expressions the first terms define transient currents when the circuit is at the action of electromotive force. Remaining terms determine the transient currents of appearing in branches at the expense of nonzero initial conditions of an inductive current and a capacitive voltage. If it were zero initial conditions, these terms would miss.

As we can see from this example, the operational image of a current represents a rational fraction where both a numerator and a denominator are polynomials of the term *p*.

Now we pass from the image of a current to the original with the use of expansion theorem in case of the real different roots.

$$I_1(p) = \frac{E + EpC(R_2 + pL) + pLi_L(0) - u_C(0)pC(R_2 + pL)}{p\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} = \frac{F_1(p)}{pZ(p)};$$

$$I_2(p) = \frac{\left(E(R_2 + pL) - R_1pLi_L(0) - u_C(p)(R_1 + R_2 + pL)\right)C}{\left[p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2)\right]} = \frac{F_2(p)}{Z(p)}.$$

Let's consider characteristic equation and find its roots:

$$p^2LCR_1 + p(CR_1R_2 + L) + (R_1 + R_2) = 0$$

$$p_{1,2} = \frac{-(CR_1R_2 + L) \pm \sqrt{(CR_1R_2 + L)^2 - 4LCR_1(R_1 + R_2)}}{2LCR_1} =$$

$$= \frac{-(0.002 \cdot 10 \cdot 20 + 0.1) \pm \sqrt{(0.002 \cdot 10 \cdot 20 + 0.1)^2 - 4 \cdot 0.1 \cdot 0.002 \cdot 10(10 + 20)}}{2 \cdot 0.1 \cdot 0.002 \cdot 10} =$$

=150;   -100 s^{-1}.

We have got the two real, negative, different in magnitude roots. Let's observe the receiving of the time solution for the current which flows through the first branch. As the denominator of the image contains a factor p we search the current original by the formula

$$i_1(t) = \frac{F_1(0)}{Z(0)} + \sum_{k=1}^{2} \frac{F_1(p_k)}{p_k Z'(p_k)} e^{p_k t} .$$

$$F_1(0) = E = 60 .$$

$$F_1(-150) = 60 + 60 \cdot (-150) \cdot 0.002 \cdot (20 + (-150) \cdot 0.1) + (-150) \cdot 0.1 \cdot 1 -$$
$$- 20 \cdot (-150) \cdot 0.002 \cdot (20 + (-150) \cdot 0.1) = -15 .$$

$$F_1(-100) = 60 + 60 \cdot (-100) \cdot 0.002 \cdot (20 + (-100) \cdot 0.1) + (-100) \cdot 0.1 \cdot 1 -$$
$$- 20 \cdot (-100) \cdot 0.002 \cdot (20 + (-100) \cdot 0.1) = -30 ;$$

$$Z(0) = R_1 + R_2 = 10 + 20 = 30 ;$$

$$Z'(p) = 2 pLCR_1 + (CR_1 R_2 + L) ;$$

$$Z'(-150) = 2 \cdot (-150) \cdot 0.1 \cdot 0.002 \cdot 10 + (0.002 \cdot 10 \cdot 20 + 0.1) = -0.1 ;$$

$$Z'(-100) = 2 \cdot (-100) \cdot 0.1 \cdot 0.002 \cdot 10 + (0.002 \cdot 10 \cdot 20 + 0.1) = 0.1 ;$$

$$i_1(t) = \frac{F_1(0)}{Z(0)} + \frac{F_1(p_1)}{p_1 Z'(p_1)} e^{p_1 t} + \frac{F_1(p_2)}{p_2 Z'(p_2)} e^{p_2 t} =$$

$$= \frac{60}{20} + \frac{(-15)}{(-150)(-0.1)} \cdot e^{-150t} + \frac{(-30)}{(-100)(0.1)} \cdot e^{-100t} =$$

$$= 2 - 1 \cdot e^{-150t} + 3 \cdot e^{-100t} \text{ A.}$$

Let's find the receiving of the time solution for the current which flows through the second branch. As the denominator of the image does not contain a factor p we search the current original by the formula

$$i_2(t) = \sum_{k=1}^{2} \frac{F_2(p_k)}{Z'(p_k)} e^{p_k t} ;$$

$$F_2(-150) = (60 \cdot (20 + (-150) \cdot 0.1) - 10 \cdot (-150) \cdot 0.1 \cdot 1) \cdot 0.002 -$$
$$- 20 \cdot (10 + 20 + (-150) \cdot 0.1) \cdot 0.002 = 0.3;$$

$$F_2(-100) = (60 \cdot (20 + (-100) \cdot 0.1) - 10 \cdot (-100) \cdot 0.1 \cdot 1) \cdot 0.002 -$$
$$- 20 \cdot (10 + 20 + (-100) \cdot 0.1) \cdot 0.002 = 0.6 ;$$

$$i_2(t)=\frac{F_2(p_1)}{Z^{'}(p_1)}e^{p_1t}+\frac{F_2(p_2)}{Z^{'}(p_2)}e^{p_2t}=\frac{0.3}{(-0.1)}\cdot e^{-150t}+\frac{0.6}{(0.1)}\cdot e^{-100t}=$$

$$=-3\cdot e^{-150t}+6\cdot e^{-100t}\ A.$$

Let's consider the case when we have two conjugate roots.

The roots of the characteristic equation and a form of a free component depend on the parameters of electric circuit. Suppose, that one of parameters has changed in the considered circuit (for example, inductance has increased four times and became equal $L=0.4\,H$. Other parameters have not changed. As independent initial conditions do not depend on the inductance in a considered circuit, they also remained invariable.

In this case the roots of the characteristic equation look like

$$p_{1,2}=\frac{-(0.002\cdot 10\cdot 20+0.4)\pm\sqrt{(0.002\cdot 10\cdot 20+0.4)^2-4\cdot 0.4\cdot 0.002\cdot 10(10+20)}}{2\cdot 0.4\cdot 0.002\cdot 10}=$$

$$=-50+j35.4;\quad -5-j35.4\ \text{s}^{-1}.$$

Let's observe the receiving of the time solution for the currents which flow through the branches.

Find the time function for the current in the first branch:

$$i_1(t)=\frac{F_1(0)}{Z(0)}+2\,\mathrm{Re}\left(\frac{F_1(p_1)}{p_1Z^{'}(p_1)}e^{p_1t}\right);$$

$$F_1(0)=E=60;$$

$$F_1(-50+j35.4)=60+60\cdot(-50+j35.4)\cdot 0.002\cdot(20+(-50+j35.4)\cdot 0.4)+$$
$$+(-50+j35.4)\cdot 0.4\cdot 1-20\cdot(-50+j35.4)\cdot 0.002\times$$
$$\times(20+(-50+j35.4)\cdot 0.4)=-0.10+j42.48;$$

$$Z(0)=R_1+R_2=10+20=30;$$

$$Z^{'}(p)=2pLCR_1+(CR_1R_2+L);$$

$$Z^{'}(-50+j35.4)=2\cdot(-50+j35.4)\cdot 0.4\cdot 0.002\cdot 10+(0.002\cdot 10\cdot 20+0.4)=$$
$$=-j0.566;$$

$$i_1(t)=\frac{60}{30}+2\,\mathrm{Re}\left(\frac{-0.10+j42.48}{(-50+j35.4)(-j0.566)}e^{-(50+j35.4)t}\right)=$$

$$=2+2\,\mathrm{Re}\left(1.224e^{-j35^o}e^{-(50+j35.4)t}\right)=2+2\cdot 1.224e^{-50t}\cos(35.4t-35^o)=$$

$$=2+2.448e^{-50t}\sin(35.4t+55^o)\,A\ .$$

Find the time function for the current in the second branch:

$$i_2(t) = 2\,\mathrm{Re}\left(\frac{F_2(p_k)}{Z'(p_k)} e^{p_k t}\right);$$

$$F_2(-50 + j35.4) = [60 \cdot (20 + (-50 + j35.4) \cdot 0.4) - 10 \cdot (-50 + j35.4) \cdot 0.4 \cdot 1) - \\ - 20 \cdot (10 + 20 + (-50 + j35.4) \cdot 0.4)] \cdot 0.002 = -j0.850\,;$$

$$i_2(t) = 2\,\mathrm{Re}\left(\frac{-j0.850}{-j0.566} e^{(-50+j35.4)t}\right) = 2\,\mathrm{Re}\left(1.5 e^{j0^o} e^{(-50+j35.4)t}\right) =$$

$$= 3e^{-50t}\cos(35.4t) = 3e^{-50t}\sin(35.4t + 90^o)\ A\,.$$

6.27. Complete Response of *RLC* Circuit to Sinusoidal Input

In this chapter up to this point attention has been confined to just one type of deterministic input, the step forcing function.

In this section we now determine the total response of a series *RLC* circuit when a second type of deterministic input, the sinusoidal function, is applied to the circuit terminals. The circuit diagram is depicted in Fig. 6.28.

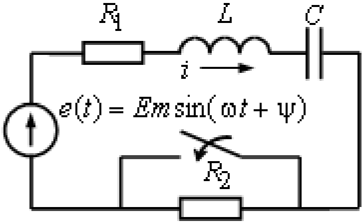
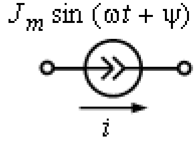


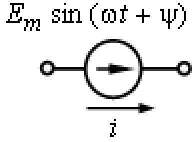
Fig. 6.28. Series *RLC* circuit with sinusoidal source.

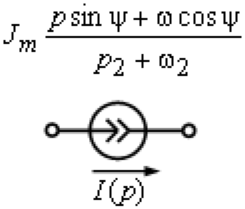
Since our concern is with finding the total solution, it follows that a steady-state solution as well as a transient solution must be determined. A point worthy of note here is that we shall be finding the steady-state solution to a sinusoidal forcing function in a linear *RLC* series circuit.

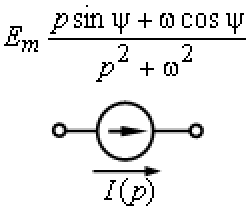
We can do this in a rather simple and direct fashion is attributable to the use of Laplace transforms as a means of solving linear nonhomogeneous differential equations. This is just another of the advantages the Laplace transform has to offer as a solution procedure.

The operational images of sinusoidal energy sources









The voltage source in the circuit of Fig. 6.28 is assumed to be varying in a sinusoidal fashion. When the switch is closed the governing differential equation for the circuit becomes

$$Em\sin(\omega t + \psi) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt \tag{6.108}$$

After formulation of an equivalent operational circuit the further calculation is conducted also as in a case of direct current energy source.

The presence of sinusoidal energy sources leads to appearing of the factor $p^2 + \omega^2$ in a denominator of the operational image of a current (or voltage), and accordingly the equation $F_2(p) = 0$ will have two complex conjugate imaginary roots $p_{1,2} = \pm j\omega$.

The steady-state meaning of the solution will be included into the addends in the form of $2\,\mathrm{Re}\frac{F_1(j\omega)}{F_2'(j\omega)}e^{j\omega t}$.

The calculation of the steady-state component in such a form in most cases is more complicated than its immediate calculation by a symbolical method. Therefore, for sinusoidal electrical circuits the steady-state component is recommended to be determined by a symbolical method. The operational method should be used for the calculation only a transient component. Equivalent operational circuits contain only internal e.m.f.s $L_k i_{L_k}(0)$ and $u_{C_k}(0)/p$.

Example 6.6. Let's consider the circuit (see Fig. 6.28) with the parameters $e(t) = 50\sin(200t + 30^o)\ V$, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $L = 0.1\ H$, $C = 0.0001\ F$. Find the circuit current after switching.

Let's calculate independent initial conditions by a symbolical method (the calculation is made for maximum values of currents and voltages).

$$X_L = \omega L = 200 \cdot 0.1 = 20\ \Omega;$$

$$X_C = \frac{1}{\omega C} = \frac{1}{200 \cdot 0.0001} = 50\ \Omega$$

Before switching the maximum value of the circuit current is

$$\underline{I}_m = \frac{\underline{E}_m}{R_1 + R_2 + j(X_L - X_C)} = \frac{50e^{j30^o}}{10 + 20 + j(20 - 50)} = 1.179e^{j75^o}\ A.$$

The instantaneous value of the circuit current is

$$i(t) = 1.179\sin(200t + 75^o)\ A.$$

Its value at the moment of switching is

$$i(0) = 1.179\sin 75^o = 1.138\ A.$$

The maximum complex capacitive voltage is

$$\underline{U}_{Cm} = \underline{I}_m \cdot X_C e^{-j90^o} = 1.179e^{j75^o} \cdot 50e^{-j90^o} = 59.93e^{-15^o}\ V;$$

The instantaneous value of the capacitive voltage is

$$u_C(t) = 59.93\sin(200t - 15^o)\ V;$$

Its value at the moment of switching is

$$u_C(0) = 59.93\sin(-15^o) = -15.51\ V.$$

Let's make an equivalent operational circuit

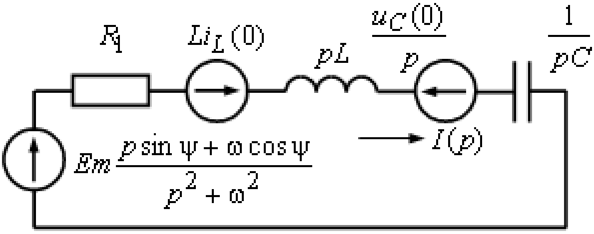


Fig. 6.29. Equivalent operational circuit.

When we write transform equations, non-zero independent initial conditionals are taken care of by introducing "internal" electromotive sources due to the initial currents through the inductances and the initial voltages across the capacitances.

The transform of the current after switching is

$$I(p)=\frac{F_1(p)}{F_2(p)}=\frac{E_m\dfrac{p\sin\psi+\omega\cos\psi}{p^2+\omega^2}+Li(0)-\dfrac{u_C(0)}{p}}{R_1+pL+\dfrac{1}{pC}}=$$

$$=\frac{(E_m p(p\sin\psi+\omega\cos\psi)+pLi(0)(p^2+\omega^2)-u_C(0)(p^2+\omega^2))C}{(p^2+\omega^2)(p^2LC+pCR_1+1)}.$$

The denominator of *I(p)* is directly available. Therefore, a partial-fraction expansion can be written forthwith.

The sinusoidal function is responsible for the presence of the purely imaginary complex conjugate poles. At first determine the roots of the equation $F_2(p)=0$:

$$F_2(p)=(p^2+\omega^2)(p^2LC+pCR_1+1)=0;$$

$$(p^2+200^2)(p^2\cdot 0.1\cdot 0.0001+p\cdot 0.0001\cdot 10+1)=0;$$

$$p_{1,2}=\pm j200\ s^{-1};\ p_{3,4}=-50\pm j312.2\ s^{-1};$$

Let's transfer from the transform of the current to the current original. At first we determine derivates of the denominator (at different roots):

$$F_2'(p)=2p(p^2LC+pCR_1+1)+(p^2+\omega^2)(2pLC+CR_1)$$

$$F_2'(p_1)=2\cdot j200((j200)^2\cdot 0.1\cdot 0.0001+j200\cdot 0.0001\cdot 10+1)=$$

$$=-80+j240;$$

$$F_2'(p_3)=((-50+j312.2)^2+200^2)\times$$

$$\times(2\cdot(-50+j312.2)\cdot 0.1\cdot 10^{-4}+10^{-4}\cdot 10=194.9-j343.0;$$

Let us find the values of the nominator (at different roots):

$$F_1(p_1) = 50 \cdot j200 \cdot (j200 \cdot \sin 30^o + 200 \cdot \cos 30^o) \cdot 10^{-4} +$$

$$+\, j200 \cdot 0.1 \cdot 1.138 \cdot ((j200)^2 + 200^2) \cdot 10^{-4} -$$

$$-\,(-15.51)((j200)^2 + 200^2) \cdot 10^{-4} = -100 + j173.2\,;$$

$$F_1(p_3) = 50 \cdot (-50 + j312.2)((-50 + j312.2)\sin 30^o + 200 \cdot \cos 30^o)10^{-4} +$$

$$+\,(-50 + j312.2) \cdot 0.1 \cdot 1.138 \cdot ((-50 + j312.2)^2 + 200^2) \cdot 10^{-4} -$$

$$-\,(-15.51)((-50 + j312.2)^2 + 200^2) \cdot 10^{-4} = -223.8 - j33.6\,;$$

Substitute found values into the initial formula to determine current original:

$$i(t) = 2\,\mathrm{Re}\left(\sum_1^2 \frac{F_2(p_k)}{Z'(p_k)} e^{p_k t} \right) =$$

$$= 2\,\mathrm{Re}\left(\frac{-100 + j173.2}{-80 + j240} e^{j200t} + \frac{-223.8 - j33.6}{194.9 - 343.0} e^{(-50 + j312.2)t} \right) =$$

$$= 2\,\mathrm{Re}\left(0.791 e^{j11^o} e^{j200t} + 0.574 e^{-j111.1^o} e^{(-50 + j312.2)t} \right) =$$

$$= 1.582\cos(200t + 11.6^o) + 1.148 e^{-50t}\cos(312.2t - 111.1^o) =$$

$$= 1.582\sin(200t + 101.6^o) + 1.148 e^{-50t}\sin(312.2t - 21.1^o)\ A.$$

Let's solve the problem by the second method. At first we find the steady-state values of the circuit current and the voltage across the capacitor by a symbolic method as

$$\underline{I}_{ss_m} = \frac{\underline{E}_m}{R_1 + j(X_L - X_C)} = \frac{50 e^{j30^o}}{10 + j(20 - 50)} = 1.581 e^{j101.6^o}\ A;$$

The instantaneous value of the steady-state component is

$$i_{ss}(t) = 1.581\sin(200t + 101.6^o)\ A.$$

Find this component at the moment of switching when $t = 0$:

$$i_{ss}(0) = 1.581\sin 101.6^o = 1.549\ A;$$

By analogy, determine the steady-state component of the capacitor voltage:

$$\underline{U}_{Css_m} = \underline{I}_{ss_m} \cdot X_C e^{-j90^o} = 1.581 e^{j101.6^o} \cdot 50 e^{-j90^o} = 79.05 e^{11.6^o}\ V;$$

$$u_{Css}(t) = 79.05 \sin(200t + 11.6^o)\ V;$$

$$u_{Css}(0) = 79.05 \sin(11.6^o) = 15.90\ V.$$

Let's determine the free components of the current and the voltage for the capacitor at the time moment.

$$i_t(0) = i(0) - i_{ss}(0) = 1.138 - 1.549 = -0.411 A;$$

$$u_{Ct}(0) = u_C(0) - u_{C\,ss}(0) = -15.51 - 15.90 = -31.41 V$$

Let's make an equivalent operational circuit for the free components

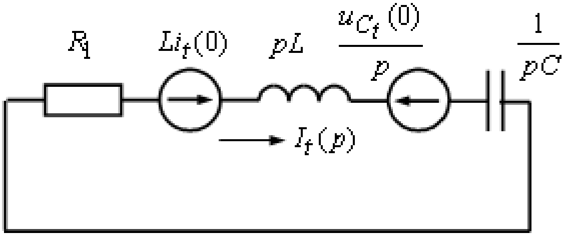


Fig. 6.30.

The equation for the transient current transform

$$I_t(p) = \frac{F_1(p)}{F_2(p)} = \frac{Li_t(0) - \frac{u_{C_t}(0)}{p}}{R_1 + pL + \frac{1}{pC}} = \frac{(pLi_t(0) - u_{Ct}(0))C}{p^2 LC + pCR_1 + 1}$$

Then we can find the solution for the transient current

$$F_2(p) = p^2 LC + pCR_1 + 1 = 0;$$

$$p^2 \cdot 0.1 \cdot 10^{-4} + p \cdot 10^{-4} \cdot 10 + 1 = 0;$$

From where

$$p_{1,2} = -50 \pm j312.2\ \text{s}^{-1};$$

$$F_1(p_1) = ((-50 + j312.2)\cdot 0.1\cdot(-0.411) - (-31.41))\cdot 10^{-4} =$$

$$= (33.47 - j12.83)\cdot 10^{-4};$$

$$F_2^{'}(p) = 2pLC + CR_1 = (2pL + R1)C;$$

$$F_2^{'}(p_1) = (2\cdot(-50 + j312.2)\cdot 0.1 + 10)\cdot 10^{-4} = j62.44\cdot 10^{-4}.$$

Now one can pass to the transient component of current original:

$$i_t(t) = 2\,\mathrm{Re}\left(\frac{F_1(p_1)}{F_2^{'}(p_1)}e^{p_1 t}\right) =$$

$$= 2\,\mathrm{Re}\left(\frac{(33.47 - j12.83)\cdot 10^{-4}}{j62.44\cdot 10^{-4}}e^{(-50+j312.2)t}\right) =$$

$$= 2\,\mathrm{Re}\left(0.574e^{-j111.0^o}e^{(-50+j312.2)t}\right) =$$

$$= 1.148e^{-50t}\cos(312.2t - 111.0^o) = 1.148e^{-50t}\sin(312 - 21^o)\ A.$$

Now we can determine the final form of the total solution for the current which flows in this circuit when a sinusoidal forcing function is applied. Thus

$$i(t) = i_{ss} + i_t.$$

The result in numbers is

$$i\ (t) = 1.581\sin(200\,t + 101.6^o) + 1.148e^{-50t}\sin(312\,t - 21^o)\ A.$$

The insignificant discrepancies given at the calculation of a transient current by the first and second methods are explained by the errors at a rounding off of the intermediate values.

Because we are dealing with linear circuits, the forced solution has the same form as the forcing function, i.e., sinusoidal.

However, the response differs in two respects: the amplitude and the argument. Of course the transient solution behaves as expected.

Note that the corresponding steady-state current is not zero at this instant. As time elapses after switching, the transient term decays to zero at a rate determined by the circuit time constant, and the total current then becomes identical to the steady-state current.

Summary review questions

1. In a linear network what is the meaning of settling time?

2. What is meant by the time constant of a first-order linear circuit?

3. How are settling time and time constant related in a first-order linear circuit?

4. Can you speak of a time constant of a higher-order circuit? Explain.

5.Give a general description of thfc relationship between the time constants and the settling time of a second-order linear circuit.

6. Is the nature of the transient solution dependent on the type of source function? Explain.

7. What is the general technique to be used to gain information about the initial values of the dependent variable in a linear circuit?

8. Describe the role that is served by the transient component in the complete solution to a first-order linear circuit driven by a constant source function.

9. A constant source voltage is applied to a series RC circuit with zero initial charge. Can the current change abruptly at $t = 0+$ following application of the source voltage? Explain.

10. Describe the basic notion involved in finding the pulse response of the series RC circuit. Assume that the dependent variable of interest is the voltage across the capacitor.

11. Explain why the application of an impulse function to a series RC circuit gives the appearance of voltage changing abruptly in a capacitor.

12. Identify the various cases of transient responses that can arise in a network that has two independent energy-storing elements.

13. Identify the general form of the forced solution of a series RL circuit when it is subject to a sinusoidal source function. Indicate where the response differs and is similar to the source function.

14. A sinusoidal source function is applied to an initially deenergized series RL circuit. The switch that applies the sinusoidal source to the circuit is closed at that instant in its cycle when the source voltage passes through zero in moving from the negative to the positive range of voltages. Is a transient solution generated? Explain.

15. To what percentage of its initial value does the transient component of a response function in a first-order circuit reach when the settling time is measured in terms of three time constants; in terms of four time constants; in terms of five time constants?

16. A constant source voltage is applied to a series *RC* circuit with zero initial charge. Can the current, change abruptly at $t = 0+$ following application of the source voltage? Explain.

17. Explain the effect, if any, of an initial charge on the capacitor in response to Question 11.

18. Describe the basic notion involved in finding the pulse response of the series *RC* circuit. Assume that the dependent variable of interest is the voltage across the capacitor.

19. Repeat Question 18 for the voltage across the resistor.

20. Define the impulse function mathematically and illustrate graphically.

21. Explain why the application of an impulse function to a series *RC* circuit gives the appearance of voltage changing abruptly in a capacitor.

22. Why does the impulse response of a circuit appear to be "free" of a forced solution?

23. Identify the various cases of transient responses that can arise in a network that has two independent energy-storing elements.

24. How is the damping ratio of an underdamped second-order circuit defined? What is the specific information about the character of the transient response that is conveyed by this ratio? Illustrate.

25. How is the equivalent time constant of an underdamped second-order circuit defined? What is the specific information about the character .of the ensuing transient response that is conveyed by this constant? Is this information likely to be slightly pessimistic or optimistic? Explain.

26. In a series *RLC* circuit define the critical resistance and describe its importance.

27. How is the natural frequency of an *RLC* network determined? Offer a physical explanation of why the expression for the natural frequency involves the particular circuit parameters cited.

28. Repeat Questions 24 and 25 for the parallel GLC circuit which is chosen as the exact dual of the series RLC circuit.

29. Identify the general form of the forced solution of a series RL circuit when it is subject to a sinusoidal source function. Indicate where the response differs and is similar to the source function.

30. Does the character of the transient response in a linear circuit subject to a step function differ from that obtained for a sinusoidal source function? Explain.

31. When a sinusoidal source is applied to a deenergized series *RL* circuit, is it possible to close the switch that connects the source to the circuit without incurring a transient current? Explain.

32. Describe briefly the role of a mathematical transform. What is a transformed quantity?

33. Identify the special quality in the definition of the Laplace transform of a time function that accounts for its usefulness in the solution of linear differential equations. How is this usefulness manifested?

34. State the restriction that is placed on the transformed variable л in the Laplace transform of a function and describe its importance.

35. How does the Laplace transform method of solving integrodifferential equations treat the initial conditions associated with energy-storing elements? Contrast this procedure with that of the classical method studied in this Chapter and identify the advantages and disadvantages of the two schemes.

36. The Laplace transform method is said to provide a systematic formulation of the solution process of linear differential equations. Explain and illustrate the meaning of this statement.

37. Describe and illustrate how the Laplace transform of a deterministic function is found. Does this result need to be found again? Explain.

38. Describe the importance of a fairly complete table of Laplace-transform pairs. Illustrate.

39. State the time-displacement theorem of Laplace-transform theory and explain its importance especially in electrical engineering.

40. State the final-value theorem of Laplace transform theory and tell why it is useful.

41. Explain what is meant by the statement "performing the inverse Laplace transformation via tables." Illustrate.

42. Once a solution of a problem in circuits is found in the transformed domain (i.e., the s domain), how do you recognize the

steady-state component of the solution in the resulting algebraic expression for the transformed solution?

43. Write the expression in the transformed domain for the current that flows in a series *RL* circuit that has an initial current *i*(0) and is driven by a unit step voltage. Identify the poles of the transformed current solution that are associated with the steady-state and transient components of the solution.

44. What is the operational impedance of an *RC* circuit? Describe its usefulness.

45. Compare the manner of finding the total solution of the current response in a series *RLC* circuit driven by a step voltage by the classical and Laplace-transform methods. Assume the presence of initial energy for both energy-storing elements.

46. Explain the meaning of the statement "the characteristic equation can be found by obtaining the transformed solution of any of the dependent circuit variables.

47. Distinguish between operational impedance and driving-point impedance.

48. Describe the general procedure for finding the characteristic equation of a network that is described by a linear integrodifferential equation.

49. Explain why the transient solution of a linear network is always characterized by the exponential function?

50. Describe the procedure that is involved in determining a specific operational network function.

Problems

6.1. In the circuit of Fig. P6.1, determine the expression for the source current for all time after closing the switch. Assume zero current through the coil when the switch is closed.

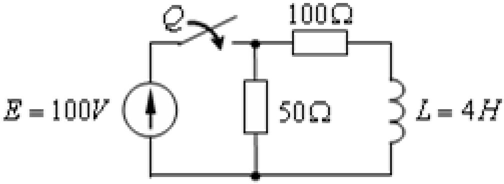


Fig. P6.1

6.2. (a) In the circuit of Fig. P6.2 with e.m.f. E = 100V find the complete expression of the current which flows through the coil when the switch is closed.

(b) What is the final value of the coil current?

(c) How long does it take for the coil current to reach 95% of its final value?

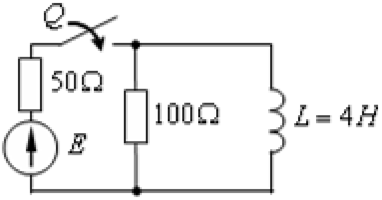


Fig. P6.2

6.3. Assuming the coil initially deenergized and the current source suddenly applied to the circuit of Fig. P6.3, find the total expression for the current through the energy- storing element.

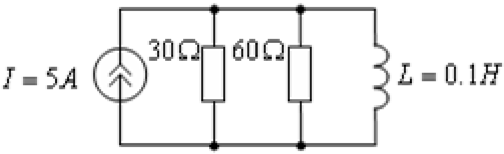


Fig. P6.3

6.4. The configuration of an *RC* circuit is as shown in Fig. P6.4. The initial-condition voltage on the capacitor is zero. Switch *Q* is then put to terminal *a*.

(a) Find the expression for the current through the capacitor.

(b) What is the time constant of the charging circuit? Ten seconds after switch *Q* has been at terminal *a* it is placed at terminal *b*.

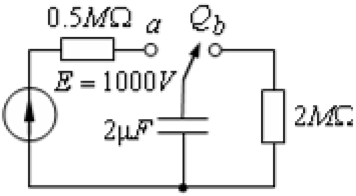


Fig. P6.4

(c) Find the current which flows through the 2-MΩ resistor.

(d) Compute the energy dissipated in this resistor after 2 s.

6.5. The circuit of Fig. P6.5 has been in steady state for a long time with the switch open. Determine the complete time expression for the current when switch is closed.

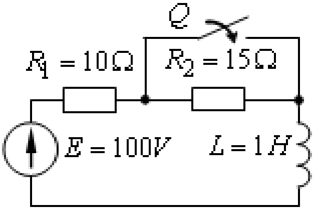


Fig. P6.5

6.6. The circuit of Fig. P6.6 is initially deenergized. The switch is then closed.

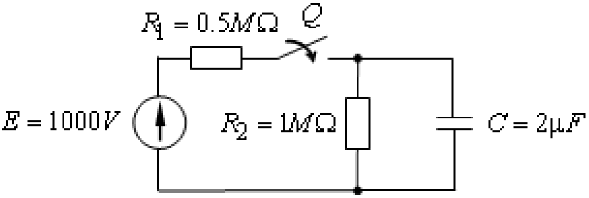


Fig. P6.6

(a) Find the expression for the current through R_2 as a function of time after the switch is closed.

(b) Repeat (a) for the current through C.

(c) What is the time constant of this circuit?

6.7. Refer to Fig. P6.7.

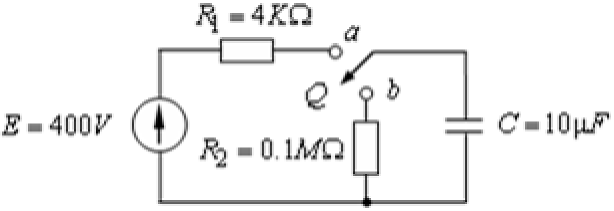


Fig. P6.7.

(a) Find the complete expression for the charging capacitor current when switch *Q* is put to position *a*.

(b) After a long time switch *Q* is placed at position *b*. Determine the expression for the current which flows through the 0.1-MΩ resistor.

(c) Find the time constant of the circuit of part (*b*).

6.8. For the circuit shown in Fig. P6.8 determine:

(a) The characteristic equation.

(b) The time constants at which transients decay in this network.

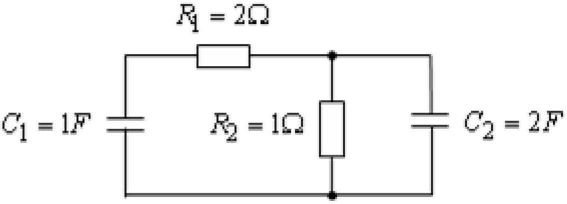


Fig. P6.8

6.9. The circuit of Fig. P6.9 has been in the condition shown for a long time. The switch is then suddenly closed.

(*a*) What is the value of the voltmeter *V* before the switch is closed?

(*b*) What is the value of the voltmeter *V* immediately after the switch is closed?

(c) Find the complete expression for u_C after the switch is closed.

(d) What is the value of the time constant of the transient term?

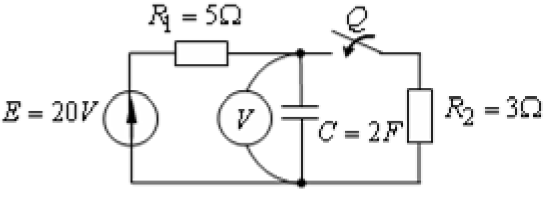


Fig. P6.9

6.10. The switch *Q* is initially in position *b* with zero initial conditions on *C* and *L*. The switch is then put to terminal *a* moving along the contacts (heavy lines) in Fig. P6.10.

(a) What is the initial value of the voltage appearing across the capacitor when *Q* is switched to *a*? Explain.

(b) Find the time expression for the current through the capacitor.

(c) How long does it take in seconds for the transient current to reach within 1% of its initial value? An approximate answer is acceptable. After a long time the switch *Q* quickly moved back to position *b*. (d) Obtain the p-domain solution for the current which flows in the *LC* circuit.

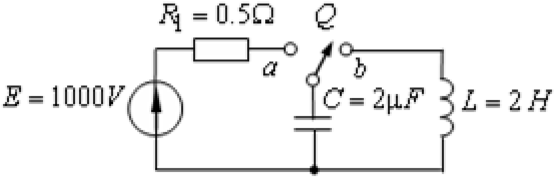


Fig. P6.10

6.11. After the circuit shown in Fig. P6.11 has been energized for a long time, switch *Q* is suddenly opened.

(a) Find the expression for the field winding current as a function of time.

(b) What is the voltage across the *a* - *b* terminals at the instant the switch is opened?

(c) What would be the voltage computed in part (*b*) if *R* were increased by 10 times?

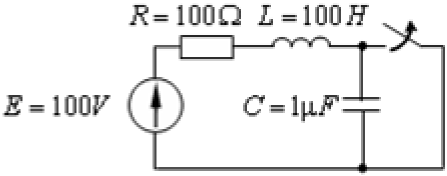


Fig. P6.11

6.12. The inductor and capacitor in the circuit of Fig. P6.12 are initially deenergized.

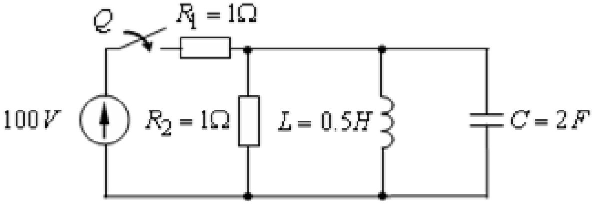


Fig. P6.12

(a) Find the time it takes for the battery current to reach within 99% of its final value after the switch is closed.

(b) What is the maximum value of the coil current during the transient state?

6.13. The Laplace transform of a time function is

$$F(p)=\frac{10}{p^2+4p+8}.$$

Find the time function.

6.14. Obtain the time function whose Laplace transform is given by

$$F(p)=\frac{10}{p(p^2+6p+8)}.$$

6.15. The Laplace transformed solution for the current in a circuit that contains two energy-storing elements is given by

$$I(p)=\frac{105}{p(p^2+10p+21)}$$

The initial conditions are known to be zero.

a) What type and what magnitude of forcing function was used?

b) What are the characteristic modes of the dynamic response?

c) What is the final value of the circuit?

6.16. The circuit parameters (Fig. 6.16) are $R_1=R_2=1\,\Omega$; $L=2\ H$; $E=100\ V$. By operator method determine the voltage $u_{12}(t)$ and the circuit current $i(t)$ when the key is switched out.

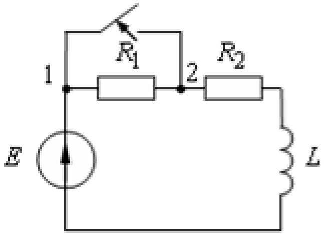


Fig. P6.16

6.17. The Laplace transform function is given by

$$F(p) = \frac{3p+8}{p^2+6p+25}.$$

What is the corresponding *f*(*t*)?

6.18. Compose general expressions for operational resistances of the following circuits described in Fig. P6.18.

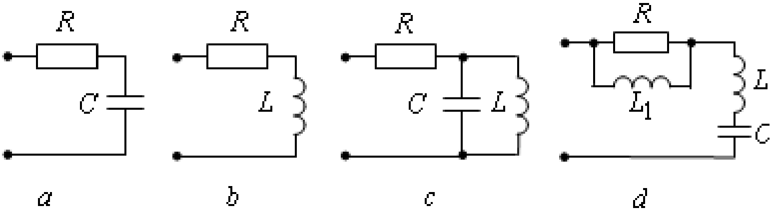


Fig. P6.18

6.19. Find the operational expression for the input current *I*(*p*) and its change in time *i*(*t*) when the e.m.f. source (see Fig. 6.19) is *a*) $e(t) = E = 100\ V$; *b*) $e(t) = 100e^{-200t} V$. The input operator impedance is $Z_{in}(p) = p + 100$.

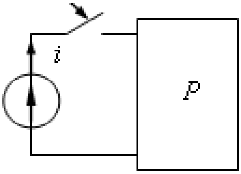


Fig. P6.19

6.20. Find inductive current and voltage after switching (Fig. 6.20) if: *a*) $e(t) = E = 100\,V$; $i(t) = J = 10A$; *b*) $e(t) = 200\sin 10^4 t$; $i(t) = J = 10\ A$. The circuit parameters are $R = 10\ \Omega$; $L = 1\,mH$.

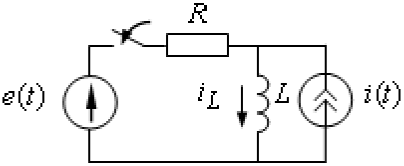


Fig. P6.20

Chapter 7

NON-LINEAR DIRECT-CURRENT CIRCUITS

Resistive elements for which the volt-ampere characteristic is other than a straight line are called *non-linear*, and so the electric circuits containing them are called *non-linear circuits*.

A non-linear resistor is one in which the terminal voltage-current characteristic (*U/I* in abbreviated form) is not a straight line or does not pass through the origin, such as is shown in Fig. 7.1. For nonlinear resistors, the terminal voltage and current are not linearly related. For example, $u = R\,i^2$ is a nonlinear relationship since the voltage does not depend directly on the current but depend on the square of the current.

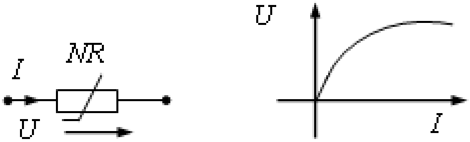


Fig. 7.1. A general nonlinear resistor and its volt-ampere characteristic.

Non-linear circuit elements may be resistive (non-linear resistances), inductive (non-linear inductances), or capacitive (non-linear capacitances). As distinct from linear resistances, non-linear ones have nonlinear volt-ampere (*U/I)* characteristics; hence their name. They may be controlled or non-controlled. Controlled non-linear resistances have an additional circuit whose current or voltage can be varied to change the *U/I* characteristic of the main circuit at will. The *U/I* relationships of non-controlled non-linear resistances are single curves. A controlled non-linear resistance has a set of *U/I* curves.

The group of non-controlled non-linear resistances includes incandescent lamps, the electric arc, barretters, gas-filled diodes, glow-discharge diodes, thyrite and vylite resistors, semiconductor diodes, etc.

The group of controlled non-linear resistances includes multi-element valves and transistors. Of course, the introduction of nonlinear resistors into an otherwise linear circuit means that the resulting circuit becomes a nonlinear one. The analysis of these nonlinear circuits becomes considerably more difficult than the analysis of linear circuits.

7.1. *U/I* Characteristics of Non-linear Resistive Elements

Eleven of the most commonly encountered *U/I* characteristics of non-controlled non-linear resistive elements are shown in Fig. 7.2.

At Fig. 7.2, *a* is the *U/I* characteristic of metal-filament incandescent lamps.

As the current through the filament increases, more heat is dissipated, and the resistance of the filament rises. This type of characteristic satisfies the identity

$$I(U) = -I(-U).$$

Non-linear resistive elements satisfying this requirement have a symmetrical *U/I* curve, and, by extension, are called *symmetrical*.

The curve at Fig. 7.2, *b* applies to thyrite and vylite resistors, some types of thermistors, and carbon-filament incandescent lamps.

Thyrite and vylite resistors are manufactured from finely divided graphite and carborundum. After suitable treatment, the mass is pressed into discsand sintered. Thyrite and vylite resistors are used in non-linear bridge circuits employed as automatic voltage deviation indicators and also in high tension (*H. T.*) power transmission lines in lightning arrestors and other protective devices. As the current flowing through this kind of elements increases, their resistance decreases. Their *U/I* characteristics are likewise symmetrical.

The curve at Fig. 7.2, *c* is associated with barretters. A barretter is essentially a sensitive metallic resistor whose resistance increases with temperature. It is usually enclosed in a glass container filled with hydrogen under a pressure of about 80 *mm* Hg to prevent environmental changes in its characteristics.

Barretters are used to stabilize the filament current of valves against variations in supply voltage, and also as measuring and control devices. Within a certain current range, the *U/I* characteristic of barretters is flat. It is also symmetrical.

As distinct from the three previous types of curves, the one at Fig. 7.2, *d* is unsymmetrical. It is typical of semi-conductor rectifiers (copper-oxide, selenium, silicon and germanium).

The curve at Fig. 7.2, *e* applies to an electric arc between dissimilar electrodes, gas-filled thermionic diodes, and some types of thermistors. As the voltage rises from zero, the current also rises but very slowly.

At some definite voltage, termed *firing voltage*, the current in the circuit suddenly increases while the voltage across the arc or the electrodes of a gas-filled diode drops.

The upper portion of the curve shows that the voltage across a non-linear resistive element decreases as the current through it keeps growing. This is what is called a *drooping* characteristic.

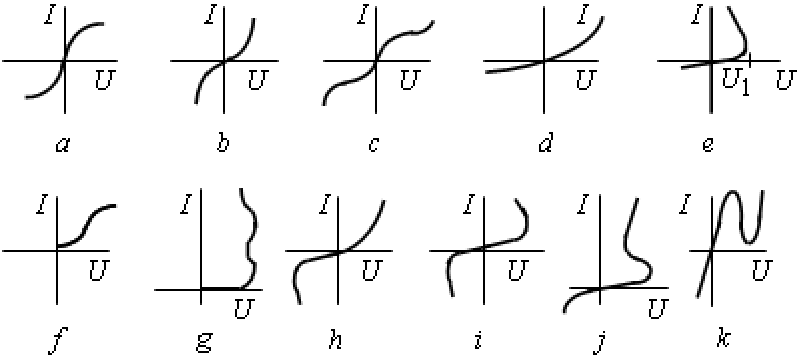


Fig. 7.2. *U/I* characteristics of non-controlled non-linear resistive elements

The curve at *f* is typical of vacuum rectifier diodes (or kenotrons). At the beginning, the curve obeys the *3/2* law:

$$I = aU^{3/2} .$$

The *U/I* characteristic of vacuum rectifier diodes is unsymmetrical because there is a flow of electrons from the cathode to the anode only when the latter is positive with respect to the former.

At *g* is the curve of glow-discharge tubes, such as diode stabilizers and neon lamps. The name "glow discharge" is due to the fact that the inert gas filling a tube (neon, argon, etc.) glows on discharge.

As long as the current in a tube remains within certain limits, the voltage across the tube stays almost constant.

The curve at *h* applies to some types of point-contact germanium and silicon diodes, while the curve at *i* is typical of the electric arc between similar electrodes.

The curve at *j* is true of four-layer semiconductor devices, such as trinistors, and the one at *k*, of tunnel diodes.

7.2. Analysis of Non-linear D. C. Circuits

In this text we shall examine elementary non-linear circuits containing non-linear resistive elements in series, parallel and series-parallel combinations, and voltage sources, and also complex networks containing only one non-linear resistive element (or networks reducible to such).

It should be noted that the linear portion of a non-linear complex multi-mesh network containing non-linear resistive elements may well be treated by any of the methods discussed in Chapter 3.

Of course, any conversions involved will be warranted if they simplify analysis or synthesis of a particular network. One such conversion, from a delta to a star, facilitating the determination of the input resistance of the linear portion in a complex network, is discussed in Sec.7.8.

Of the methods discussed in Chapter 3, the following will be applied to non-linear circuits in this chapter:

- the nodal-pairs method;
- the parallel-generator theorem;
- the Thevenin equivalent method.

Analysis of non-linear circuits involves knowledge of the respective *U/I* curves. As a rule, direct current non-linear circuits are solved graphically.

7.3. Series Non-linear Resistive Circuits

In the circuit of Fig. 7.3, *a*, the non-linear resistive element *NR* is placed in series with a linear resistance *R*. The circuit contains a voltage source *E*. We inquire about the current around the circuit. The *U/I* curve of *NR* is $I = f(U_{NR})$, (see the curve 1 in Fig. 7.3, *b*), that of *R* is a straight line, and that of the entire circuit current (see the curve 2 in Fig. 7.3, *b*) is

$$I = f(U_{NR} + U_R)\ .$$

Analysis is based on Kirchhoff's laws. We shall discuss two procedures illustrated in *b* and *c* of Fig. 7.3, respectively.

By the first procedure, the resultant *U/I* curve is plotted for the entire circuit on the assumption that the same current flows through *NR* and *R* connected in series. To construct it, we set an arbitrary current, as given by point *m*, draw a horizontal line through it (Fig. 7.3, *b*), and add the interval *mn* equal to the voltage across *NR* and the interval *mp* equal to the voltage across *R*: $\overline{mn}+\overline{mp}=\overline{mq}$.

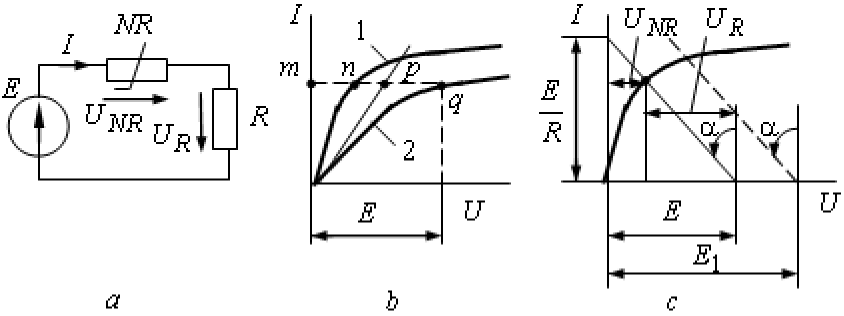


Fig. 7.3. The non-linear and linear resistive elements in series

The point *q* belongs to the *U/I* curve of the entire circuit. Other points of the resultant *U/I* curve are found similarly.

The current round the circuit for a given voltage *E* is found graphically from the resultant *U/I* curve (Fig. 7.3, *b*). The value of *E* is laid off as abscissa, and a perpendicular is erected from the point thus obtained. The intersection of the perpendicular and the resultant *U/I* curve gives the point *q*. The ordinate of *q* is the current we are seeking.

By the second procedure, the resultant *U/I* curve of the line portion need not be constructed. Instead, we draw a straight line described by the function $IR+U_{NR}=E$ from the point ($I = 0$, $U = E$) to the point ($I = E/R$, $U = 0$), as shown in Fig. 7.3, *c*. When the plot is drawn to an appropriate scale, $\tan\alpha$ is numerically equal to *R*. The intersection of the straight line and the *U/I* curve of *NR* gives the operating point of the circuit. As can be seen, at this point the same current flows through *NR* and *R*, and the sum of the voltage drops $U_{NR}+U_R=E$. As the applied voltage changes from *E* to E_1, the line $I = f(U_R)$ should be moved parallel to itself so that it originates at the point ($I=0, U=E_1$) (the dotted line in Fig. 7.3, *c*).

A similar procedure applies to circuits containing two or more non-linear resistive elements in series. The first step is to plot the *U/I* curve for any two non-linear resistive elements, then for the first two taken as one and the third, etc.

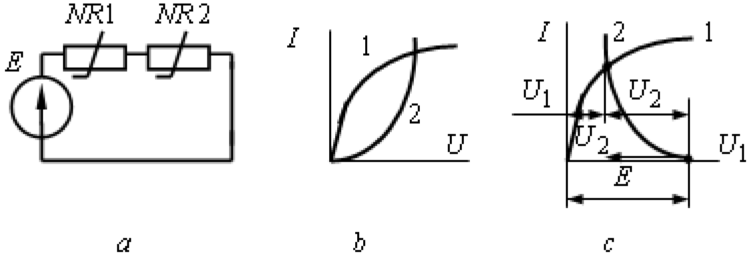


Fig. 7.4. Two non-linear resistive elements in series

We shall use the second procedure for the circuit of Fig. 7.4, *a* containing two different non-linear resistive elements *NR*l and *NR*2. The respective *U/I* curves, 1 and 2, are shown in Fig. 7.4, *b*. Since *NR*2 is non-linear, instead of the straight line $I = f(U_R)$ as in Fig. 7.3, *c*, we should draw the curve $I = f(U_2)$. Its origin is at the point (I = 0, $U_1 = E$) (Fig. 7.4, *c*). U_2 is positive to the left of the origin. Since U_{NR2} is positive to the right of the origin in Fig. 7.3, *b*, and to the left in Fig. 7.4, *c*, the $I = f(U_2)$ curve is a mirror image of curve 2 in Fig. 7.4, *b* about the vertical axis drawn through the point ($U_1 = E$).

7.4. Non-linear Resistive Elements in Parallel

Fig. 7.5, *a* shows two non-linear resistive elements in parallel. Their *U/I* curves are shown in Fig. 7.5, *b*. In plotting the resultant *U/I* curve, we start from the fact that the voltages across *NR*l and *NR*2 are equal by virtue of parallel connection, and that the total input current *I* is the sum of I_1 and I_2: $I_o = I_1 + I_2$. The resultant *U/I* curve is at 3 in Fig. 7.5, *b*. It is plotted as follows.

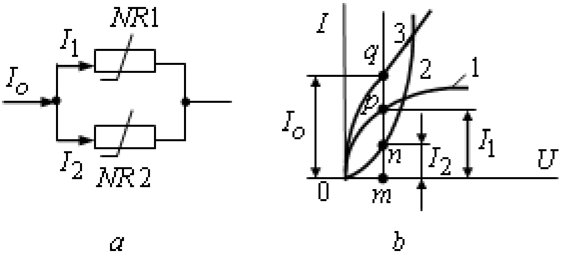


Fig. 7.5. Two non-linear resistive elements in parallel

On putting U equal to the distance. $0m$, we erect a perpendicular from the point m, and add together the interval mn equal to the current in $NR2$ and the interval mp equal to the current in $NR1$:

$$\overline{mn} + \overline{mp} = \overline{mq}\ ,$$

where m is equal to the total input current for the voltage $0m$. Other points of the resultant U/I curve are found similarly.

7.5. Non-linear Resistances in Series-parallel

In the circuit of Fig. 7.6, *a*, containing two non-linear resistances $NR1$ and $NR2$ connected in parallel, and a third one $NR3$ placed in series with them, we inquire about the branch currents.

The U/I curves of $NR1$, $NR2$ and $NR3$ (at 1, 2 and 3 in Fig. 7.6, *b*) and the value of E are specified.

The first step is to plot the U/I curve for $NR1$ and $NR2$ in parallel as explained in the previous paragraph. It is represented by curve (*1 + 2*) in Fig. 7.6, *b*. This reduces the circuit to a series connection of $NR3$ and a combined NR having the U/I curve (*1 + 2*).

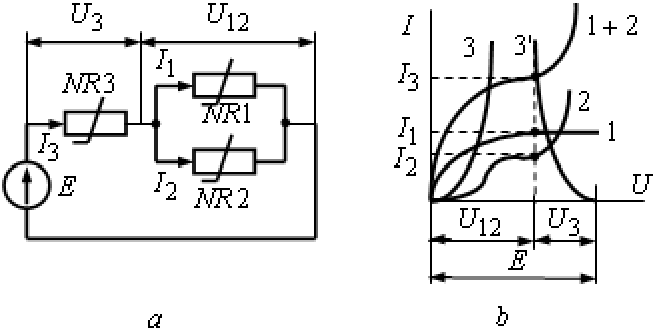


Fig. 7.6. The mixed connection of three non-linear resistances.

We apply the second procedure of Sec. 7.3. Curve *3*' in Fig. 7.6 *b* is a mirror image of the U/I curve of $NR3$, about the vertical line drawn through the point ($U = E$).

The point of intersection of curve *3*' and curve (*1 + 2*) satisfies the Kirchhoff's voltage law:

$$U_3 + U_{12} = E\ .$$

The sum of the currents $I_1 + I_2 = I_3$.

7.6. Application of the Nodal-pairs Method to Non-linear Resistive Circuits

Circuits having only a pair of nodes or reducible to such can be solved by the nodal-pairs method.

We consider its application, taking the circuit of Fig. 7.7 as an example. The circuit contains three non-linear resistances and three voltage sources.

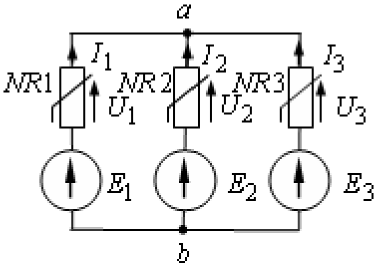


Fig. 7.7. The circuit with a pair of nodes.

The *U/I* curves of the non-linear resistances are at *a*, *b* and *c* in Fig. 7.8.

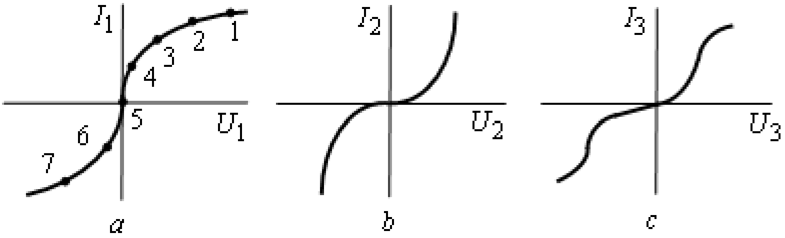


Fig. 7.8. The *U/I* curves of the non-linear resistances.

To make the problem more specific, we assume E1 > E2 > E3 and all currents being positive when they flow towards, say, node *a*. Then, by the Kirchhoff's current law

$$I_1 + I_2 + I_3 = 0 \quad . \tag{7.1}$$

Each of the currents is a non-linear function of the voltage drop across the respective non-linear resistive element. Thus, I_1 is a function of U_1, I_2 is a function of U_2; and I_3 is a function of U_3.

Now we write all the currents as functions of a single variable, the voltage U_{ab} across the nodal pair *a* and *b*, and not of different variables (U_1, U_2, U_3). This may be done, because

$$U_1 = E_1 - U_{ab}; \tag{7.2}$$

$$U_2 = E_2 - U_{ab}; \tag{7.3}$$

$$U_3 = E_3 - U_{ab}. \tag{7.4}$$

Thus the problem is to transform the curve $I_1 = f(U_1)$ into a curve $I_1 = f(U_{ab})$, the curve $I_2 = f(U_2)$ into a curve $I_2 = f(U_{ab})$, and so on. Let the curve $I_1 = f(U_1)$ of Fig. 7.7, *a* be transformed to the curve $I_1 = f(U_{ab})$ of Fig. 7.9.

For point 5 the coordinates are $I_1 = 0$ and $U_1 = 0$, and $U_{ab} = E_1$ (see Eq. 7.2). In other words, the origin of the curve $I_1 = f(U_{ab})$ is shifted to the point $U_{ab} = E_1$. When $U_1 > 0$, U_1 increases with decreasing U_{ab}. For point 2 and $U = E_1$, the voltage $U_{ab} = 0$. When $U_1 < 0$, $|U_1|$ increases with increasing U_{ab}, and $U_{ab} > E_1$.

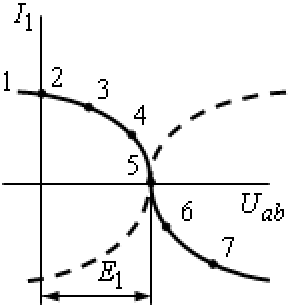


Fig. 7.9

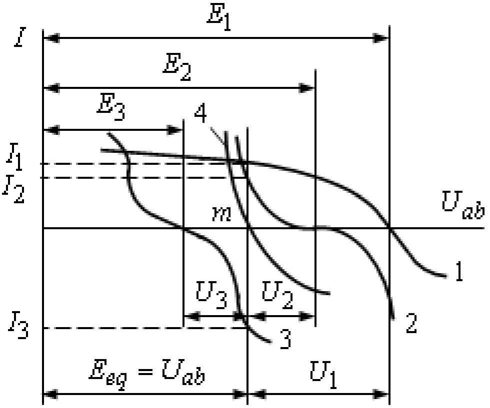


Fig.7.10

To sum up, the following steps may be recommended for the transformation of the curves:

1. Shift the curve $I_1 = f(U_1)$ parallel to itself so that its origin is at the point $U_{ab} = E_1$. The shifted curve is shown by the dotted line in Fig. 7.9.

2. Draw a vertical line through the point $U_{ab} = E_1$ and construct a mirror image of the dotted curve relative to the vertical.

The same procedure is to be followed for the curves of the other branches. Now we plot the curves $I_1 = f(U_{ab})$, $I_2 = f(U_{ab})$ and $I_3 = f(U_3)$ together, as shown in Fig. 7.10 (where they are marked 1, 2 and 3, respectively) and draw the curve $I_1 + I_2 + I_3 = f(U_{ab})$ marked 4 in the same figure, by adding together the ordinates of curves 1, 2 and 3. The intersection of curve 4 and the *x*-axis (point *m*) gives the value of U_{ab} which satisfies Eq. (7.1). Now erect a perpendicular from this point to the abscissa axis. The ordinates of the intersections between the perpendicular and curves 1, 2 and 3 identify the currents I_1, I_2 and I_3 both in magnitude and in direction.

7.7. Application of the Parallel-generator Theorem to Networks Containing Non-linear Resistances and E.M.F.s

Let there be a network of several parallel branches containing non-linear resistances and e. m. f.s, which is part of a more complex network not shown in the figure (Fig. 7.11, *a*).

We inquire about the e.m.f. and the *U/I* curve of a non-linear resistance that would form an equivalent circuit (Fig. 7.11, *b*) for the original network of Fig. 7.11, *a*. If the single branch of Fig. 7.11, *b* is to be equivalent to the network of Fig. 7.11, *a*, the current *I* in the common part of the network of Fig. 7.11, *a* should be equal to the current *I* in the circuit of Fig. 7.11, *b* for any value of U_{ab}.

Let us make use of the curves of Fig. 7.10. Curve 4 in that figure is described by $I_1 + I_2 + I_3 = f(U_{ab})$. In other words, curve 4 is the resultant *U/I* curve of three parallel branches. The branch of Fig. 7.11, *b* should have a similar *U/I* characteristic. If the current *I* in the circuit of Fig. 7.11, *b* be zero, then $U_{ab} = E_{eq}$. Consequently, E_{eq} of Fig. 7.10 is decided by the value of U_{ab} at which curve 4 crosses the *x*-axis. To find the *U/I* characteristic of the non-linear resistance equivalent *NRE* in

Fig. 7.11, *b*, one should reflect curve 4 of Fig. 7.10 about the perpendicular erected from point *m*.

The *U/I* curve of *NRE* is shown in Fig. 7.11, *c*. It is important to note that application of voltage sources in the parallel branches makes the *U/I* curve of *NRE* unsymmetrical, although those of the original non-linear resistances in Fig. 7.7 are symmetrical.

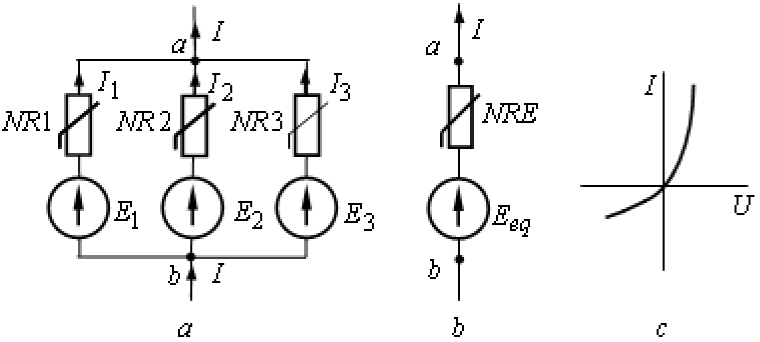


Fig. 7.11. Illustration to application of the parallel-generator theorem.

In other words, by varying the voltages in the parallel branches of a network, it is possible to control the resultant *U/I* curve of the network at will and produce any forms of characteristics.

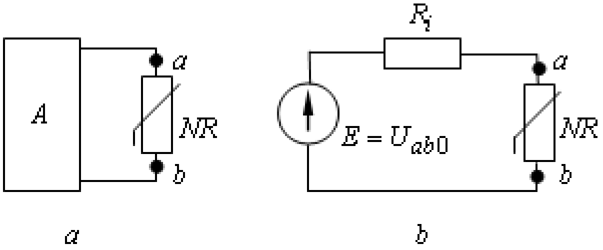
7.8. Application of Thevenin's Theorem to Non-linear Networks

If a complex network contains a branch with a non-linear resistance, the current in that branch can be found by use of Thevenin's theorem. The first step is to remove the branch and to replace the remaining network by an active two-terminal circuit (Fig. 7.12, *a*).

As stated in the equivalent-generator method, an active two-terminal network can be replaced by the Thevenin equivalent which consists of a voltage source in series with a source resistance.

The source voltage is equal to U_{abo} appearing across the terminals *ab* of the original network under open circuit, and the source resistance *R*, is equal to R_i or that presented by the box to the terminals *ab* with all independent sources removed, e.m.f. sources short-circuited and current sources open-circuited (Fig. 7.12, *b*).

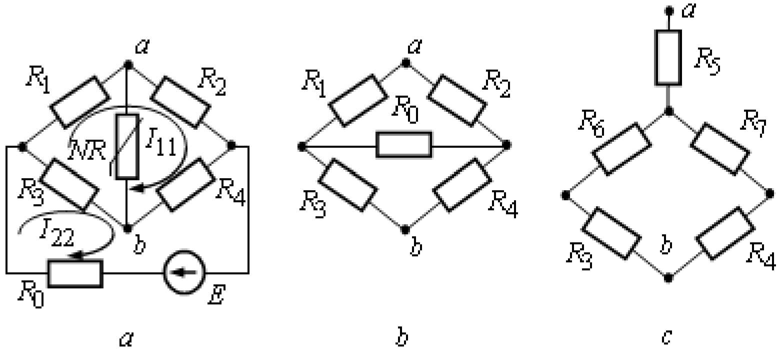
The current in the network of Fig. 7.12, *b* is easy to determine by reference to the section where the equivalent-generator method was explained (see Ch. 3.14).



a *b*

Fig. 7.12. Active two-terminal circuit

Example 7.1. Find the current in the branch *ab* of the network of Fig. 7.13, *a* by Thevenin's theorem for $R_1 = R_0 = 9\ \Omega$, $R_2 = 36\ \Omega$, $R_3 = 27\ \Omega$, $R_4 = 18\ \Omega$, $E = 60\ V$. The *U/I* curve of the nonlinear resistance *NR* is shown in Fig. 7.14.



a *b* *c*

Fig.7.13

Solution: Imagine the terminals *ab* open-circuited and find the open-circuit voltage. For this purpose we use the mesh-current method and write the system of two equations for two independent contours:

$$\begin{aligned} &I_{11}(R_1 + R_2 + R_3 + R_4) - I_{22}(R_3 + R_4) = 0 \\ &-I_{22}(R_3 + R_4) + I_{22}(R_3 + R_4 + R_0) = E \end{aligned}.$$

On solving the system of equations we get mesh currents $I_{11} = 0.95\ A$; $I_{22} = 1.9\ A$. Then actual currents $I_1 = 0.95\ A$; $I_2 = 1.9\ A$.

Now one can find the open-circuit voltage

$$U_{ab_{oc}} = R_3 I_3 - R_1 I_1 = 17 \text{ volts.}$$

To determine the resistance R presented by the linear portion to the terminals *ab*, it is necessary to transform the delta formed by $R_1,\ R_2,\ R_0$ in Fig. 7.13, *b* into an equivalent star (Fig. 7.13, *c*) by use of Eqs. (3.28):

$$R_5 = \frac{R_1 \cdot R_2}{R_1 + R_2 + R_0} = \frac{9 \cdot 36}{9 + 36 + 9} = 6\ \Omega;$$

$$R_6 = \frac{R_1 \cdot R_0}{R_1 + R_2 + R_0} = 15\ \Omega;$$

$$R_7 = \frac{R_2 \cdot R_0}{R_1 + R_2 + R_0} = 6\ \Omega.$$

Then we can determine the input resistance by the following equation:

$$R_{in} = R_5 + \frac{(R_6 + R_3)(R_7 + R_4)}{R_6 + R_3 + R_7 + R_4} = 19\ \Omega$$

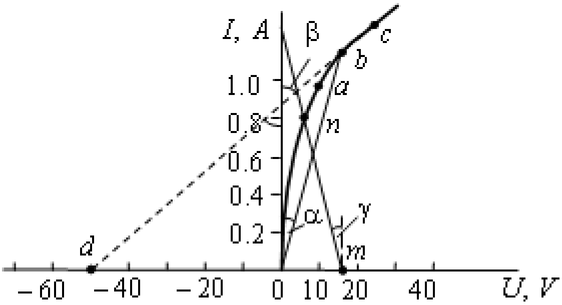


Fig. 7.14. A volt-ampere curve.

To find the current in branch *ab* (see Fig. 7.13, *a*), we draw a line *mn* (see Fig. 7.14) from the point *m* ($U_{o.c.} = 17$ volts) at an angle γ from the vertical, whose tangent (with the scales of abscissa and ordinates suitably chosen) is equal to R_i.

The point *n* at which the line *mn* intersects the *U/I* curve of the non-linear element gives the operating current of the network $I = 0.8\ A$.

7.9. Static and Incremental Resistances

A complete description of a non-linear resistive element or circuit can be given either in the form of its *U/I* curve or by its *static* and *incremental resistances* as functions of current or voltage.

The static resistance R_{st} of a non-linear resistive element is the ratio of the voltage across, to the current through, the element

$$R_{st} = \frac{U}{I} . \tag{7.5}$$

Numerically, the static resistance is equal to tan α in Fig. 7.14. On moving from point to point on the *U/I* curve of a non-linear resistive element, its R_{st} varies describing the behaviour of the element under conditions of a steady-state current.

The incremental resistance R_d is the ratio of a small (theoretically infinitesimal) increment in the voltage across a non-linear resistive element to a similar increment in the current through the element

$$R_d = \frac{dU}{dI} \tag{7.6}$$

where *dU* is the incremental voltage, and *dl* is the incremental current.

Numerically, the incremental resistance is equal to $\tan\beta$ (Fig. 7.14) and describes the response of a non-linear resistive element to very small departures from a previous state.

The above relation can be restated in order to relate the incremental voltage across a non-linear resistive element to the incremental current through it

$$dU = R_d dI \tag{7.7}$$

If the *U/I* curve is drooping (that is, when the voltage is increased by *dU*, the current decreases by *dI*), as in the electric arc (see Fig. 7.2, *e*), the incremental resistance is negative and one speaks of a negative-resistance element.

Of the two quantities, R_{st} and R_d the latter is used more. It appears in the equivalent circuits of non-linear resistive elements (see the following section) and also in stability studies of non-linear circuits.

7.10. The Thevenin Equivalent of a Non-linear Resistive Element

When it is known in advance that a given non-linear resistive element is to operate only within a limited range of its *U/I* curve and that this range may, to a good approximation, be replaced by a straight line, the element in question may, for design purposes, be represented by a voltage source in series with an equivalent linear resistance (the Thevenin equivalent).

Let the operating point of a non-linear resistive element be within the interval *ab* of the curve in Fig. 7.14 (see also Fig. 7.15). For this interval

$$U = U_o + I \tan\beta = U_o + I\,R_d \tag{7.8}$$

Equation (7.8) is satisfied by the circuit of Fig. 7.16 where $E = -U_0$ and the linear resistance $R = R_d$.

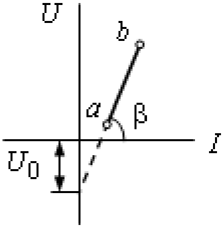


Fig.7.15

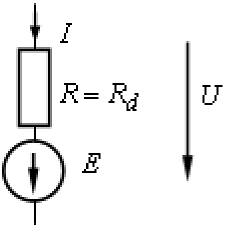


Fig. 7.16

An attractive feature about the replacement of a non-linear element or circuit by the Thevenin equivalent is that the transform circuit becomes linear and can be solved by methods applicable to linear circuits.

Of course, it is essential that the operating point remain within the linear portion of the *U/I* curve.

Example 7.2. Present the interval *bc* of the *U/I* curve in Fig. 7.14 analytically.

From Fig. 7.14 one can find $U_o = -50\ V$ (point "*d*") and

$$R_d = \tan\beta = {}^{50}\!\big/_{0.88} = 57\ \Omega\,.$$

Consequently, $U = -50 + 57I$ (approximately).

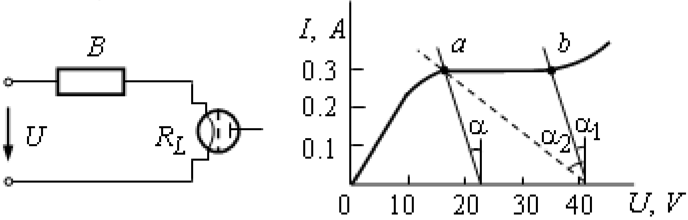
7.11. Current Stabilizers

Non-linear resistive elements can and do impart to electric circuits properties unattainable with linear elements. Some of these properties are current and voltage regulation, *dc* amplification, logarithmic and exponential function generation, etc.

The term current stabilizer (or regulator) applies to a device capable of maintaining a constant load current against variations in the load resistance or the input voltage of the entire network.

A variety of circuits exist for current regulation. The most commonly used and, at the same time, the simplest circuit is the one shown in Fig. 7.17, *a*.

In this circuit a non-linear resistive element of the barretter type, *B*, is placed in series with the load R_L Fig. 7.17, *b* shows the *U/I* curve of the some barretter Type 0.3 B17-35. The first numeral in the type designation gives the amperes the barretter will maintain constant, while the numerals "17-35" give the volts across the barretter over the current regulation range.



a *b*

Fig. 7.17. The non-linear barretter and its *U/I* curve.

Example 7.3. A Type 0.3 B 17-35 barretter is used to regulate the filament current of a valve. The rated filament current is 0.3 ampere and the filament voltage is 6 volts.

Find the range of the input voltage for which the filament current will remain practically constant at 0.3 ampere.

Solution: The filament resistance of the valve

$$R_u = \frac{6}{0.3} = 20\,\Omega.$$

We draw two straight lines through the points *a* and *b* (Fig. 7.17 *b*) bounding the current regulation range, so that they make an angle a with the vertical (tan α = R_u = 20 ohms). From Fig. 7.17, *b* it can be seen that the input voltage *U* can be varied from 23 to 41 volts.

Example 7.4. A series resistance R_1 is placed in the circuit of *Example 7.3*. Holding the input voltage constant at 41 volts, find the maximum value of R_1 at which the current will remain constant.

Solution: If R_1 = 0 and *U* = 41 volts, the operating point of the circuit will be at *b* (Fig. 7.17, *b*).

As R_1 is increased, the operating point on the *U/I* curve shifts towards the point *a*. In the limit at the point *a*

$$R_{1\max} + R_u = \tan \alpha_2 \frac{m_U}{m_I} = 80\,\Omega,$$

$$R_{1\max} = 80 - 20 = 60\,\Omega.$$

7.12. Voltage Stabilizer

By *voltage stabilizer* (or *regulator*) is meant a device the output voltage U_L of which is held constant or almost constant against variations in the load resistance R_L or the input voltage U_1.

An elementary voltage regulator is shown in Fig. 7.18, *a*. It uses a stabilivolt as the voltage-regulating non-linear element, and a ballast resistor R_b. The volt-ampere curve of a stabilivolt 150C5-30 is shown in Fig. 7.18, *b*.

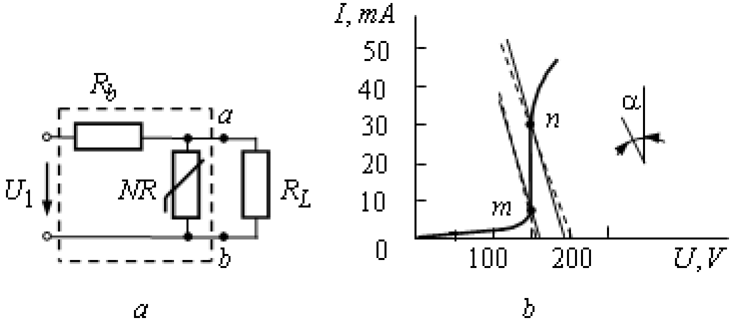


Fig. 7.18. A voltage regulator and its volt-ampere curve

An analysis of voltage-regulator operation involves the determination of the range within which U_1 may vary when R_L = constant and also its operation when both U_1 and R_L vary.

It is usual to evaluate the operation of voltage regulators in terms of the stabilization factor which is the ratio of the fractional increment in the input voltage $(\Delta U_1 / U_1)$ to the fractional increment in the output voltage $(\Delta U_L / U_L)$ Consider two numerical cases.

Example 7.5. In the circuit of Fig. 7.18, *a* R_L = 8 kilohms, R_b = 2 kilohms. The *U/I* curve of the stabilivolt is shown in Fig. 7.18, *b*. Find the range of U_1 within which the stabilizer will maintain a constant voltage of 120 volts.

Solution. By Thevenin's theorem, the open-circuit voltage of the stabilivolt is

$$U_{ab0} = \frac{U_1 R_L}{R_L + R_b} = 0.8 U_1 .$$

The resistance of the linear part of the network (Fig. 7.18, *a*) looking into the terminals *ab*

$$R_i = \frac{R_L R_b}{R_L + R_b} = 1600\,\Omega.$$

We draw two straight lines (shown by the solid lines in Fig. 7.18, *b*) through the points *m* and *n* on the *U/I* curve at an angle α to the vertical so that $\tan\alpha = R_i = 1600\ \Omega$.

The segments cut off by the straight lines on the axis of abscissa give the open-circuit voltage U_0. Referring to the diagram, $0.8 U_{1\min} = 120$ volts, and $U_{1\min} = 150$ volts.

Similarly, $0.8 U_{1\max} = 180$ volts, and $U_{1\max} = 225$ volts. Consequently, U_1 may vary from 180 to 225 volts.

Example 7.6. For the network of Fig. 7.18, *a*, R_b = 2 kilohms, the volt-ampere curves of the stabilivolt as shown in Fig. 7.18, *b*, and U_1 = 200 volts, find the range of R_L within which the stabilizer will perform its function.

Solution. By Thevenin's theorem, the open-circuit voltage of the stabilizer is

$$U_0 = \frac{U_1 R_L}{R_L + R_b} = \frac{200\, R_L}{R_L + 2000}$$

The resistance presented to the terminals of the network is

$$R_i = \tan\alpha = \frac{R_L \cdot R_b}{R_L + R_b} = \frac{200\,0 R_L}{R_L + 2000}$$

Now we inquire about the values of R_L at which the straight lines representing R_i will pass through the points *m* and *n* on the *U/I* curve of the stabilivolt.

In our example, neither $\tan\alpha$ nor the abscissa from which the lines are to be drawn are specified. So, our approach will be one of trial and error. Setting various values to R_L, we compute the respective values of U_0 and R_i

R_L, *kilohms*	2	3	4	5	6	7
U_0, *volts*	100	120	133	143	150	156
R_i, *ohms*	1000	1200	1330	1430	1500	1556

Referring to the table, we draw several straight lines and select those passing through the points *m* and *n* on the volt-ampere curve (shown by the dotted lines in Fig. 7.18, *b*). Accordingly, $R_{L\min}$ = 2.4kilohms and $R_{L\max}$ = 4.2 kΩ.

7.13. D.C. Voltage Amplifier

A d*irect current voltage amplifier* is a device the incremental output voltage of which is greater than its incremental input voltage. D.c voltage amplifiers often use controlled non-linear resistive elements, such as vacuum triodes and transistors.

Fig. 7.19 shows the anode characteristic of the 6C2C vacuum triode. It relates the anode current I_a of the valve to its anode voltage U_a for several values of its grid voltage U_g.

In the d.c. voltage amplifier of Fig. 7.20, the input (signal) voltage is applied to the valve grid, while the load resistance R_L is placed across the output of the amplifier (terminals *ab*).

The grid, being closer to the cathode than is the anode, has a greater effect than the anode field on the electron flow from the cathode.

Therefore, even relatively small variations in the grid potential bring about marked changes in the anode current and the output (load) voltage. For the anode circuit

$$E_a = U_a + I_a R_L .$$

The relationship between the load voltage ($I_a R_L = E_a - U_a$) and the input voltage U_g is represented by a set of curves in Fig. 7.18.

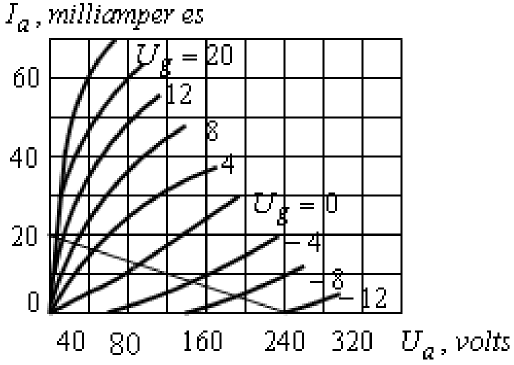


Fig.7.19. The anode characteristic of the 6C2C vacuum triode

Example 7.7. Plot, the curve $U_L = f(U_g)$ for the circuit of Fig. 7.20, *a*, if R_L = 12 kΩ and E_a = 240 volts. The valve is a 6C2C triode.

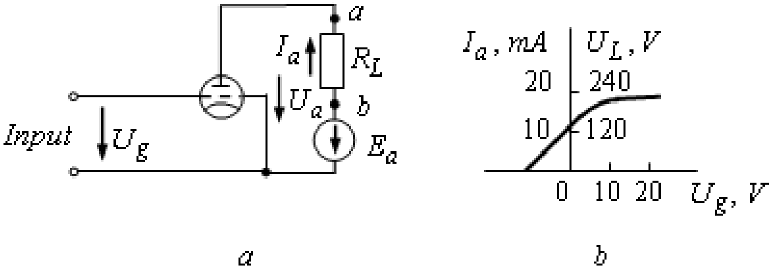


Fig. 7.20. The voltage amplifier

Solution. We draw a line from the point ($I_a = 0$, $U_a = E$) at an angle α from the vertical (tan α = 12 ohms) as shown in Fig. 7.20, *b*.

Its intersections with the anode curves give the respective values of I_a and U_g.

The curve $U_L = f(U_g)$ differs from the curve $I_a = f(U_g)$ of Fig. 7.20, *b* only in scale ($U_L = I_a R_L$, with R_L = constant).

Summary review questions

1. Describe in some details what do you know about non-linear elements.
2. Explain the difference between controlled and non- controlled non-linear resistances. Give the examples of controlled resistances. What non- controlled resistances do you know?
3. Which resistive elements have a symmetrical volt-ampere curve?
4. What methods of calculation might be applied to non-linear circuits?
5. Describe the procedure of the calculation of series non-linear resistive circuits. What laws is this analysis based?
6. What can you say about the application of Thevenin's theorem to non-linear network?
7. What elements does the Thevenin's equivalent consist of?
8. How are non-linear circuits usually solved?
9. What is called a static resistance? How can one determine it?
10. What is called an incremental resistance? How can one determine it?
11. Is the incremental resistance negative or positive if the V/A curve is drooping?
12. Define the current transfer ratio of the semiconductor diode as used in the common-base circuit configuration. Why is current transfer ratio of the semiconductor diode always less than unity?
13. What device is called a current stabilizer? What do you know about this device? What is a current regulator used for?
14. What device is called a voltage stabilizer? What does the analysis of voltage-regulator operation involve?
15. What is called a stabilization factor?
16. What device is called a d.c. voltage amplifier? When is it appropriate to do amplifier analysis by linear models?
17. Draw the anode characteristic of a vacuum triode.
18. Draw the linear equivalent circuit of the amplifier and identify the meaning of each circuit parameter and each circuit variable.

Problems

7.1. Find the current in the branch *ab* of the network of Fig. 7.13, *a* by Thevenin's theorem for $R_1 = R_0 = 2\ \Omega$, $R_2 = 8\ \Omega$, $R_3 = 4\ \Omega$, $R_4 = 6\ \Omega$, $E = 50$ volts. The *U/I* curve of the nonlinear resistance *NR* is shown in Fig. 7.14.

7.2. For the network of Fig. 7.18, *a*, $R_b = 2k\Omega$. The *V/A* curves of the stabilivolt is shown in Fig. 7.18, *b*, and $U_1 = 250$ volts. Find the range of *RL* within which the stabilizer will perform its function.

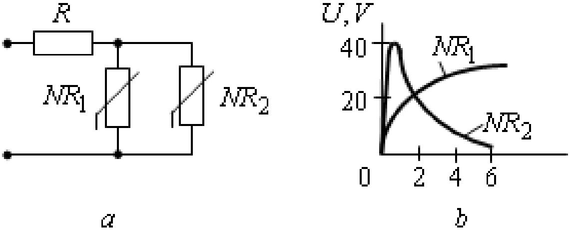


Fig. P7.1

7.3. Draw a volt-ampere characteristic $U = f(I)$ of the circuit described in Fig. P7.1, *a*. U/I characteristics of non-linear elements *NR*1 and *NR*2 are shown in Fig. P7.1, *b*. The value of linear resistor *R* = 5Ω.

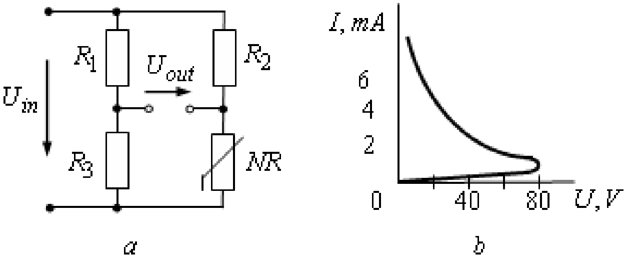


Fig. P7.2

7.4. In Fig. P7. 2, *a* there is the circuit being used for definition of a time constant of the thermistor which is switched on in a branch of the bridge (a volt-ampere characteristic of the thermistor is presented in Fig P7. 2, *b*), if $R_1 = 30\ k\Omega$, $R_2 = 10\ k\Omega$, $U_{in} = 120\ V$.

One must find:

a) the resistance R_3 at which the bridge is balanced;

b) output voltage U_{out} if the input one U_{in} decreases by 20 volt.

In both cases find static R_{st} and incremental R_d resistances of the thermistor.

7.5. In the circuit (see Fig. P7.3) one must define all the currents and the voltage across *NR*. The volt-ampere characteristic of non-linear resistance is depicted in Fig. P7.3, *b* (curve 1). The circuit parameters: $E_1 = 18\ V$; $E_2 = 6\ V$; $J = 1\ A$; $R_1 = 3\ \Omega$; $R_1 = 6\ \Omega$.

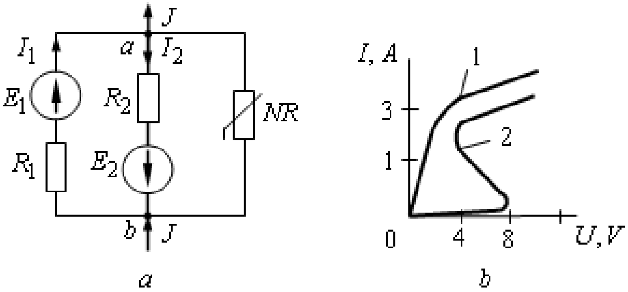


Fig. P7.3

7.6. Find the current through non-linear resistance *NR* in the circuit in Fig. 7.4 and the values of static R_{st} and incremental R_d resistances if this current flows through the resistance. The *V/A* curve of *NR* is in Fig. P7.3, *b* (curve 2). The circuit parameters: $E = 20V$; $J = 5A$; $R_1 = 3\Omega$; $R_2 = 2\Omega$. How will the current through *NR* and the values of R_{st} and R_d vary if the polarity of a current source changes?

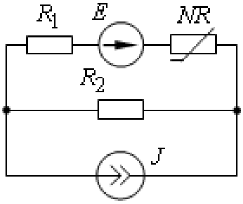


Fig. P7.4

7.7. Calculate the current through non-linear resistance *NR* (see Fig. 7.5, *a*), if the circuit is supplied:

a) from the e.m.f. source $E = 24\ V$;

b) from the current source $J = 2\ A$.

The *V*/*A* characteristic of *NR* is given in Fig. 7.5, *b*. The circuit parameters: $R_1 = R_2 = \quad = 4\ \Omega$; $R_3 = 3\ \Omega$; $R_4 = 1\ \Omega$. Determine the working rate of NR, find the currents in all the branches of the circuit.

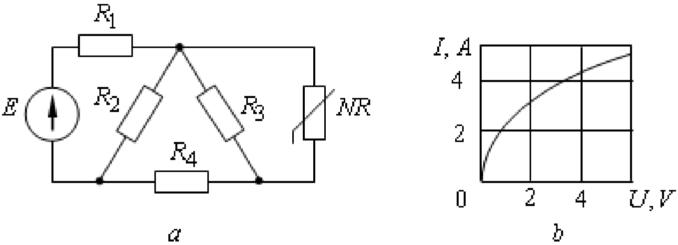


Fig. P7.5

7.8. In the following circuit in Fig. 7.6 calculate the currents in all the branches. The volt-ampere characteristic of two similar non-linear elements is depicted in Fig. 7.3, *b* (curve 1). The circuit parameters: $E = 20\ V$; $J = 3\ A$; $R_1 = R_2 = R_3 = 2\ \Omega$.

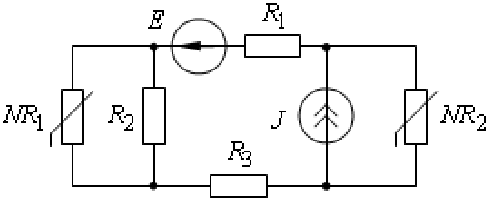


Fig. P7.6

7.9. Two incandescent lamps are connected in a parallel through the resistor $R = 50\,\Omega$, and attached to the voltage source $E = 0.68V$, Fig. P7.7. The *V*/*A* characteristic of lamps is possible to express analytically: $I_1 = 0.06U_1 + 10^{-3}U_1^2$; $I_2 = 0.04U_2 + 0.6 \cdot 10^{-3}U_2^2$. Find currents and voltage in lamps.

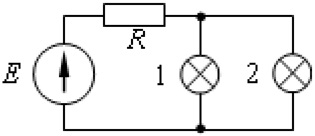


Fig. P7.7

Chapter 8

PERIODICAL NON-SINUSOIDAL CURRENTS IN LINEAR ELECTRICAL CIRCUITS

8.1. Periodical Non-sinusoidal Currents and Voltages Defined

In practice one often encounters alternating currents which are periodic but whose waveform is other than sinusoidal.

Non-sinusoidal waveforms can be produced in four fundamentally different sets of circuit conditions.

In the first case, a voltage or current source generates a non-sinusoidal electromotive force or current, while other circuit elements (resistances, inductances and capacitances) are linear, that is, independent of the magnitude of current.

In the second case, a voltage or current source generates a sinusoidal electromotive force (e.m.f.) or current in a circuit with one or several nonlinear elements.

In the third case, the source generates a non-sinusoidal e. m. f. or current, and the circuit incorporates one or several non-linear impedances.

In the fourth case, the source generates a direct current (d.c.) or sinusoidal e.m.f., while one or several circuit elements vary periodically.

This chapter deals with the design and performance of linear electrical circuits carrying non-sinusoidal voltages or currents, which fall under the first set of circuit conditions. The second and, partly, the third case and the fourth are discussed and examined further in this book.

The calculation of linear electric circuits with non-sinusoidal periodic voltage is based on the superposition theorem. Namely, currents and voltages are determined separately for a constant and each harmonious component.

Then their resultants of function are defined as the sum of constant and harmonic components.

The calculation of a constant component is made by known rules for calculation of direct current circuits, and the calculation of harmonic components – by the rules for calculation of sinusoidal current circuits but separately for each harmonic.

8.2. Fourier Analysis

As is known from mathematics, any function *f*(*x*) which is periodic with period 2π and satisfies the conditions established by Dirichlet, can be represented by a series of cosine and sine terms, called a *Fourier series* and *harmonics*, respectively.

Dirichlet conditions are: (1) Any periodic function may have only a finite number of discontinuities in a period; (2) A periodic function may oscillate with only a finite number of maxima and minima in a period; (3) A periodic function must have a finite average value. Most of the functions met with in circuit theory satisfy these conditions, so that they need not be checked for compliance.

The period variable *x* is related to time thus

$$x = \omega t = 2\pi \frac{t}{T},$$

where *T* is the time period of the function. Thus the period of the function in terms of *x* is 2π, and in terms of time, *T*.

A Fourier series has the form

$$\begin{aligned} f(x) &= A_o + A_1^{'} \sin x + A_2^{'} \sin 2x + A_3^{'} \sin 3x + A_4^{'} \sin 4x + \ldots \\ &\ldots + A_1^{''} \cos x + A_2^{''} \cos 2x + A_3^{''} \cos 3x + A_4^{''} \cos 4x \ldots \end{aligned} \qquad (8.1)$$

where A_o is a constant term;

$A_1^{'}$ is the amplitude of the sine fundamental (or first harmonic);

$A_1^{''}$ is the amplitude of the cosine fundamental;

$A_2^{'}$ is the amplitude of the second harmonic, etc.

$$A_o = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx \qquad (8.2)$$

$$A_1^{'} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin x\, dx; \quad A_1^{''} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos x\, dx \qquad (8.3)$$

……………………………………………………………

$$A_k^{'} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin kx\, dx; \quad A_k^{''} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos kx\, dx; \qquad (8.3)$$

Since

$$A_k^{'} \sin kx + A_k^{''} \cos kx = A_k \sin(kx + \psi_k)$$

where

$$A_k = \sqrt{\left(A_k^{'}\right)^2 + \left(A_k^{''}\right)^2} = \text{harmonic amplitudes}$$

$$\psi_k = -arc\tan\left(A_k^{''} / A_k^{'}\right) = \text{phase angles,}$$

A second form of a Fourier series (8.1) having only sine terms may be used

$$f(x) = A_o + A_1 \sin(x + \psi_1) + A_2 \sin(2x + \psi_2) + \ldots =$$
$$= A_0 + \sum_{k=1}^{\infty} A_k \sin(kx + \psi_k) \qquad (8.4)$$

The harmonics for which k is an odd number are *odd harmonics*; the harmonics for which k is an even number are *even harmonics*. Hence comes another name for Fourier analysis: *harmonic analysis*.

8.3. Some Properties of Symmetric, Periodic Wave-forms

Figures 8.1 and 8.2 show three wave-forms which have certain specific properties.

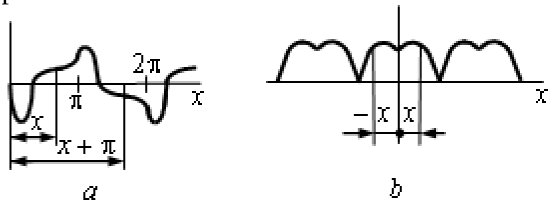


Fig. 8.1. Wave-forms with certain specific properties.

Thus the wave-form of Fig. 8.1, *a* is described by the equation

$$-f(x) = f(x).$$

Wave-forms answering this equation are called symmetrical with respect to the *x*-axis. If we move the wave-form of Fig. 8.1, *a* a half-period along, and fold it back on itself about the *x*-axis, it will coincide with the curve *f*(*x*). Such wave-forms contain no constant component or even harmonics, that is $A_0 = A_2^{'} = A_2^{''} = A_4^{'} = A_4^{''} = \ldots = 0$.

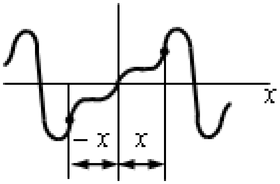


Fig. 8.2. A wave-form with certain specific properties.

Therefore, a Fourier series for this type of wave-form is

$$f(x) = A'_1 \sin x + A''_1 \cos x + A'_3 \sin 3x + A''_3 \cos 3x + ...$$

Each term of this series satisfies the condition $-f(x+\pi) = f(x)$. Thus, $-\sin(x+\pi) = \sin x$.

The wave-form of Fig. 8.1, *b* is symmetrical about the *y*-axis and satisfies the condition

$$f(-x) = f(x).$$

If the curve to the left of the *y*-axis is transposed about this axis to produce its mirror image, the latter will coincide with the curve lying to the right of the *y*-axis. This type of wave-form contains only the cosine terms and the constant component, and no sine terms, that is

$$A'_1 = A'_2 = A'_3 = ... = 0:$$

The wave-form of Fig. 8.1, *b* satisfies the condition

$$f(x) = A_0 + A''_1 \cos x + A''_2 \cos 2x + A''_3 \cos 3x + ...$$

The wave-form of Fig. 8.2 satisfies the condition $-f(-x) = f(x)$, and is symmetrical about the origin of coordinates. Its Fourier series has the form

$$f(x) = A'_1 \sin x + A'_2 \sin 2x + A'_3 \sin 3x + ...$$

8.4 Fourier Series

Much is said in the preceding sections about the response of electric circuits to a periodic forcing function having a sinusoidal variation. This attention is merited, since the bulk of the electric power in the world is generated, transmitted, and consumed as a sinusoidally varying quantity at constant frequency. At this point, however, it is appropriate to ask: How can the response of electric circuits be obtained when the forcing function is not sinusoidally varying but is periodic? There are numerous applications in engineering where non-sinusoidal forcing functions such as rectangular, triangular, trapezoidal, or other wave shapes are used. Moreover, in communication engineering situations exist which involve the summation of sinusoidal signals of many frequencies, the resultant wave shape of which is extremely non-sinusoidal albeit periodic. This condition is illustrated in Fig. 8.3, which shows how it is possible to represent three entirely different waveforms by means of two sinusoidal components. Note that in each case one of the components is assumed to have a frequency three times greater than the other. For the condition depicted in Fig. 8.3, *a* the resultant waveform may be expressed mathematically as

$$f(t) = f_1(t) + f_3(t) = F_1 \sin \omega t + F_3 \sin 3\omega t \tag{8.5}$$

where $f_1(t)$ is the fundamental sinusoidal component of frequency ω, and $f_3(t)$ is the third harmonic component of frequency 3ω. Both the fundamental and the third harmonic terms are called periodic functions because they satisfy the condition

$$f(t) = f(t+T) \tag{8.6}$$

where T denotes the period of the fundamental wave. Observe that the period of the third harmonic is one-third that of the fundamental and that the resultant wave has the same period as the fundamental.

The equation which describes the waveform of Fig. 8.3, *b* is

$$f(t) = f_1(t) - f_3(t) = F_1 \sin \omega t - F_3 \sin 3\omega t \tag{8.7}$$

By merely reversing the third harmonic term the resultant waveshape changes from one which is essentially flat-topped (Fig. 8.3, *a*) to one which is peaked.

The resultant waveshape of Fig. 8.3, *c* is obtained by shifting the third harmonic term of Fig. 8.3, *a* 90° in the lead direction. The corresponding mathematical formulation is

$$f(t) = F_1 \sin \omega t + F_3 \sin\left(\omega t + \frac{\pi}{2}\right) \tag{8.8}$$

A little thought should make it apparent that even with only two sinusoidal components manipulated in the manner just described it is possible to represent a great number of different waveshapes.

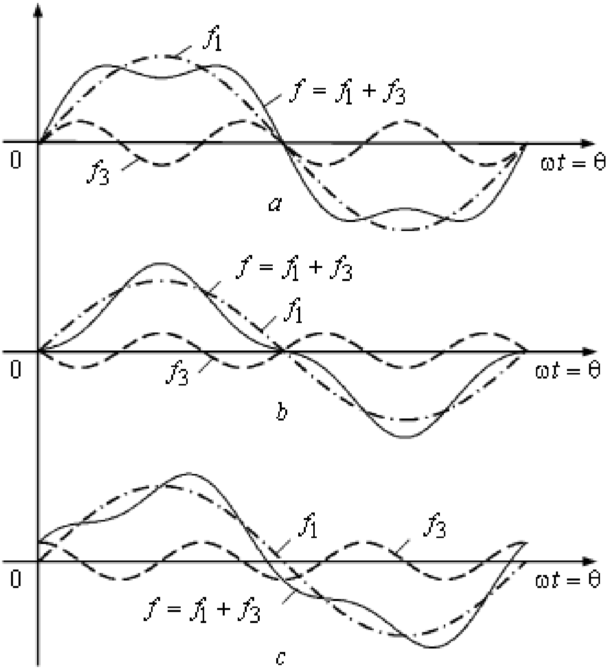


Fig. 8.3. (*a*) Addition of fundamental sine term plus positive third harmonic; (*b*) addition of fundamental plus negative third harmonic; (*c*) addition of fundamental sine term and a third harmonic cosine term.

A general formulation of this situation is provided by the Fourier series theorem, which is stated here without proof. *A periodic function* $f(t)$ *with period T can be represented by the sum of a constant term, a fundamental of period T, and its harmonics*. Mathematically, we have

$$f(\theta) = \left(\frac{a_0}{2}\right) + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots + \\ + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots \tag{8.9}$$

where $\theta = \omega t$ and $\omega = 2\pi/T$, where *T* is the period of the function *f*(0).

Thus Eq. (8.9) states that any periodic function of whatever shape can be replaced by the sum of a constant term ($a_0/2$) and sine and/or cosine terms involving odd and/or even harmonics. The *a*- and the *b*-coefficients are evaluated from the shape of the *f*(0) function.

General expressions can readily be derived for evaluating these coefficients, as shown below. In this connection it is useful to keep in mind the following identities. Consider first the evaluation of the constant term of Eq. (8.9). The waveform of $f(0)$ is assumed to be known for all values of θ and to be periodic. To evaluate a_0 it is necessary merely to integrate both sides of Eq. (8.9) over one period and do necessary transformations. Since the average value over one period of sine and cosine functions is zero, all terms on the right side of the last equation are zero with the exception of the first one. Hence we have

$$\int_0^{2\pi} f(\theta)\ d\theta = \frac{a_0}{2}(2\pi)$$

or

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\ d\theta = \frac{2}{T}\int_0^{T} f(t)\ dt \tag{8.10}$$

The strategy to be employed in evaluating a_1, is to manipulate Eq. (8.9) in such a way as to cause all terms to drop out with the exception of a_1. A little thought reveals that this is readily achieved by multiplying each term of Eq. (8.9) by cos θ and then integrating over one period. The result is

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos\theta\ d\theta = \frac{2}{T}\int_0^{T} f(t)\cos\omega t\ dt \tag{8.11}$$

To evaluate b_1 a similar procedure is used, but now since we are dealing with the coefficients of the sine term in the Fourier Series it is necessary to multiply each term of Eq. (8.9) by sin θ and then to integrate over T. Performing the integration leads to

$$\int_0^{2\pi} f(\theta)\ d\theta = b_1\pi$$

or

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin\theta\ d\theta = \frac{2}{T}\int_0^{T} f(t)\sin\omega t\ dt \tag{8.12}$$

By following this procedure for the coefficients of each of the higher harmonic term it can be shown that the general expressions for the *a*- and *b*-coefficients for the *n*th harmonic are given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta = \frac{2}{T}\int_0^{T} f(t)\cos n\omega t\, dt \tag{8.13}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta = \frac{2}{T}\int_0^{T} f(t)\sin n\omega t\, dt \tag{8.14}$$

It is interesting to note that by writing the constant term in Eq. (8.9) as $a_0/2$ it is permissible to use Eq. (8.13) to evaluate a_0 as well as the coefficients of the cosine components in the wave.

Example 8.1. Find the Fourier series representation of the waveform depicted in Fig. 8.4.

Solution: A check of the symmetry conditions indicates that none is satisfied. Hence we can expect both even and odd harmonics to exist as well as sine and cosine terms. The average value of the function is found by evaluating Eq. (8.13) for $n = \theta$.The periodic function in the first period may be expressed as

$$f(\theta) = \frac{U_m}{\pi}\theta, \quad 0 \le \theta \le \pi;$$

$$f(\theta) = 0, \qquad \pi \le \theta \le 2\pi.$$

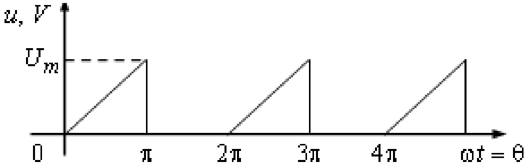


Fig. 8.4. Waveforms for Example 8.1.

Accordingly we get

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, d\theta = \frac{1}{\pi}\int_0^{\pi}\frac{U_m}{\pi}\, d\theta = \frac{U_m}{2\pi^2}\left[\theta^2\right]_0^{\pi} = \frac{U_m}{2}$$

$$\left(\frac{a_0}{2}\right) = \frac{U_m}{4} = average\ \ value$$

This average value of the function can also be found in this case by inspection. How? The general expression for the coefficient a_n of the cosine terms is found as follows

$$a_n = \frac{1}{\pi}\int_0^{\pi} \frac{U_m}{\pi} \cos n\theta \; d\theta = \frac{U_m}{\pi^2}\left[-\theta\frac{\sin n\theta}{n} + \frac{\cos n\theta}{n^2}\right]_0^{\pi} =$$

$$= \frac{U_m}{\pi^2 n^2}(\cos n\pi - 1).$$

The limits of the integral for a_n are taken as zero to π rather than as zero and 2π because the contribution of the function from π to 2π is obviously zero. Inserting consecutive integer values for n yield

$$a_1 = \frac{U_m}{\pi^2}(-1-1) = -\frac{2U_m}{\pi^2}$$

$$a_2 = \frac{U_m}{4\pi^2}(1-1) = 0 = a_4 + a_6 + a_8 + \ldots$$

$$a_3 = \frac{U_m}{9\pi^2}(-1-1) = -\frac{2U_m}{9\pi^2}$$

$$a_5 = -\frac{2U_m}{25\pi^2}$$

…………………………

The evaluation of the coefficients of the sine terms follows in a similar fashion. Thus

$$b_n = \frac{1}{\pi}\int_0^{\pi} \frac{U_m}{\pi}\theta \sin n\theta \; d\theta = \frac{U_m}{\pi^2}\left[\frac{-\theta \cos n\theta}{n} + \frac{\sin n\theta}{n^2}\right]_0^{\pi} =$$

$$= \frac{U_m}{\pi}\,\frac{\cos n\pi}{n}.$$

Therefore,

$$b_1 = -\frac{U_m}{\pi}(-1) = \frac{U_m}{\pi}$$

$$b_2 = -\frac{U_m}{\pi}$$

$$b_3 = -\frac{U_m}{3\pi}$$

$$b_4 = -\frac{U_m}{4\pi}$$

Hence the Fourier series representation of the waveform is

$$u(\theta) = u(\omega t) = \frac{U_m}{4} + \frac{U_m}{\pi}\left[\sin\ \theta - \frac{1}{2}\sin\ 2\theta + \frac{1}{3}\sin\ 3\theta - (-1)^n\frac{1}{n}\sin\ n\theta\right]$$

$$= -\frac{U_m}{\pi^2}\left[\cos\ \theta + \frac{1}{9}\cos\ 3\theta + \frac{1}{25}\cos 25\theta + \frac{1}{k^2}\cos\ k\theta\right]$$

where n = 1, 2, 3, 4,…all integers,
……...k = 1, 3, 5, 7,…odd integers.

The last equation shows that a non-sinusoidal periodic function can be expressed entirely in terms of sines and cosines. It should be apparent then that if this waveform is used as the forcing function in a series *RL* circuit, the corresponding current response can be found by a systematic application of the sinusoidal steady-state theory developed in the preceding sections.

The current response at each frequency is found in the accustomed manner for each sine and cosine term as well as the constant term in $u(\theta)$ and then summed. The waveform of the current response in such a case, however, will be different than that of $u(\theta)$ because at each frequency the amount of phase lag caused by the inductance is different.

The Fourier series representation of a periodic time function has one other useful property which is worth noting. From the treatment so far it should be clear that an infinite number of terms in the series is required to get an exact representation of the original function. If we use only the first N terms of the series, then instead of an exact representation we have an approximation of the original time function yielding an error.

Here is the interesting aspect of this result; it can be shown that by using the coefficients of the Fourier series (i.e., a_n and b_n) for any given N the error is minimized. In other words, any other choice of coefficients results in a larger error.

8.5. Fourier Analysis of Regular and Irregular Wave-forms

The periodic wave-forms encountered in electrical engineering may be classed into two groups. One group includes regular periodical wave-forms, such as trapezoidal, triangular, rectangular waves and so on. Their Fourier series are given in the table on page 362, where ωt is taken instead of x.

The other group covers irregular periodical wave-forms. In most cases they are specified in graphic form. They are expanded into Fourier series also graphically.

8.6. Harmonic Analysis by a Graphical Method

One of the best organized graphical methods of harmonic analysis is the *Range schedule technique*, based on the replacement of the definite integrals by the summation of a finite number of terms. To this end, the period of the function $f(x)$ equal to 2π is divided into n equal parts Δx

$$\Delta x = \frac{2\pi}{n}$$

and the integrals are replaced by the sums as follows. By definition, the constant term is

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \approx \frac{1}{2\pi}\sum_{p=1}^{p=n} f_p(x)\Delta x = \frac{1}{2\pi}\sum_{p=1}^{n} f_p(x)\frac{2\pi}{n},$$

or

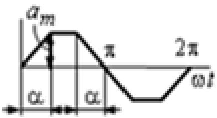
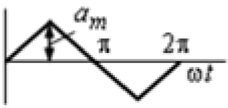
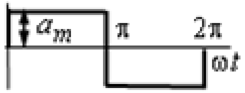
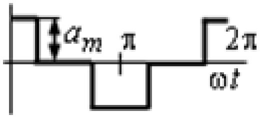
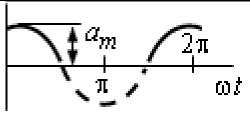
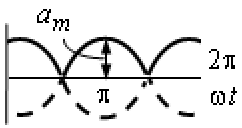
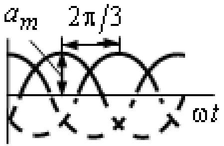
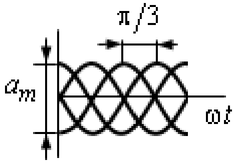
$$A_0 = \frac{1}{n}\sum_{p=1}^{n} f_p(x), \tag{8.15}$$

where p is the actual index taking on values from 1 to n, and $f_p(x)$ is the value of the function $f(x)$ for $x = (p - 0.5)\Delta x$, that is, in the middle of the p-th range. The amplitude of the sine term of the first harmonic is

$$A_k' = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx \approx 2\frac{1}{2\pi}\sum_{p=1}^{n} f_p(x)\frac{2\pi}{n}\sin_p kx,$$

or

$$A_k' = \frac{2}{n}\sum_{p=1}^{n} f_p(x)\sin_p kx. \tag{8.16}$$

	$f(\omega t)=\frac{4a_m}{\alpha\pi}(\sin\alpha\sin\ \omega t+{}^{1}\!/_{9}\sin 3\alpha\sin 3\omega t$ $+{}^{1}\!/_{25}\sin 5\alpha\sin 5\omega t+...)$
	$f(\omega t)=\frac{8a_m}{\pi^2}(\sin\ \omega t-{}^{1}\!/_{9}\ \sin 3\omega t+$ $+{}^{1}\!/_{25}\sin 5\omega t-{}^{1}\!/_{49}\sin 7\omega t...)$
	$f(\omega t)=\frac{4a_m}{\pi}(\sin\ \omega t+{}^{1}\!/_{3}\ \sin 3\omega t+$ $+{}^{1}\!/_{5}\sin 5\omega t+{}^{1}\!/_{7}\sin 7\omega t...)$
	$f(\omega t)=\frac{4a_m}{\pi}(\sin\frac{\alpha\pi}{2}\ \omega t+$ $\frac{1}{3}\ \sin\frac{3\alpha\pi}{2}\cos 3\omega t+\frac{1}{5}\sin\frac{5\alpha\pi}{2}\cos 5\omega t+...)$
	$f(\omega t)=\frac{2a_m}{\pi}(\frac{1}{2}+\frac{\pi}{4}\cos\omega t-\frac{1}{1\times 3}\cos 2\omega t-$ $\frac{1}{3\times 5}\cos 4\omega t+\frac{1}{5\times 7}\cos 6\omega t-...)$
	$f(\omega t)=\frac{2a_m}{\pi}(\frac{1}{2}+\frac{\pi}{4}\cos\omega t-\frac{1}{1\times 3}\cos 2\omega t-$ $\frac{1}{3\times 5}\cos 4\omega t+\frac{1}{5\times 7}\cos 6\omega t-...)$
	$f(\omega t)=\frac{3\sqrt{3}a_m}{\pi}(\frac{1}{2}+\frac{1}{2\times 4}\cos 3\omega t-$ $\frac{1}{5\times 7}\cos 6\omega t+\frac{1}{8\times 10}\cos 9\omega t-...)$
	$f(\omega t)=\frac{3a_m}{\pi}\left(1+\frac{2\cos 6\omega t}{5\times 7}-\frac{2\cos 12\omega t}{11\times 13}+\right.$ $\left.+\frac{2\cos 18\omega t}{17\times 19}-...\right)$

The amplitude of the cosine term of the *k*-th harmonic is

$$A_k' = \frac{2}{n}\sum_{p=1}^{n} f_p(x)\cos_p kx\,. \tag{8.17}$$

In Eqs. (8.16) and (8.17) $\sin_p kx$ and $\cos_p kx$ are the values of the functions sin *kx* and cos *kx* for $x = (p–0.5)\Delta x$, that is, in the middle of the *p*-th range. In analysis by Eqs. (8.15) through (8.17) it will suffice to divide, the period into $n = 24$ or 18 parts. In some cases, even fewer parts will do.

Before carrying out graphical analysis, it will save much work to inspect the wave to be analysed for symmetry (see Sec 8.3). Thus, if the wave *f*(*x*) is symmetrical about the *x*-axis, it contains no constant term or even harmonics, and also the sum $\sum f_p(x)\sin kx$ over the first half-cycle is equal to the sum $\sum f_p(x)\sin_p kx$ over the second half-cycle.

The sign of the angles φ_k in Eq. (8.4) depends on those of A_k' and A_k''. In drawing harmonic curves on a common graph, the scale of, say, the *k*-th harmonic should be *k* times greater than that of the first harmonic.

For example, let some interval on the *x*-axis represent the angle π/3 for the first harmonic. Then for the third harmonic the same interval on the *x*-axis will represent three times this angle, or 3(π/3) = π.

8.7. Calculation of Non-sinusoidal Currents and Voltages

Before getting down to the actual calculation, the e. m. f. should be represented as a Fourier series. According to the superposition principle, the instantaneous current of any branch in a network equals the sum of instantaneous currents due to the various harmonics.

Similarly, the instantaneous voltage across any portion of a network equals the sum of instantaneous voltages due to the various harmonics across that part.

For each harmonic the current and voltage are found by the method discussed in the previous chapters of this book.

As the first step, one finds the currents and voltages due to the constant component of the e. m. f. Then they are found for the first harmonic, then for the second harmonic, etc.

In determining the currents and voltages due to the constant component of the electromotive force, one should remember that the voltage drop across an inductance *L* due to a direct current is zero, and also that no direct current can flow through a capacitance *C*.

Still another point to bear in mind is that the inductive reactance increases with frequency, and so the inductive reactance X_{Lk} for the *k*-th harmonic is *k* times that for the first harmonic, X_{L1}:

$$X_{Lk} = k\omega L = kX_{L1}; \qquad X_{L1} = \omega L. \tag{8.18}$$

Capacitive reactance decreases with increasing frequency, so that the capacitive reactance X_{Ck} for the *k*-th harmonic is 1/*k*-th of the capacitive reactance, X_{C1}, for the first harmonic:

$$X_{Ck} = \frac{1}{k\omega C} = \frac{X_{C1}}{k}; \qquad X_{C1} = \frac{1}{\omega C}. \tag{8.19}$$

For each harmonic one can draw a vector diagram of its own. However, one cannot lay off the currents and voltage drops due to different frequencies on the same plot or perform vector addition of such currents or voltage drops, because the angular velocities of the vectors are different for different frequencies. If the frequency is low, resistances may be deemed independent of the frequency.

Example 8.2. The e. m. f. in the network of Fig. 8.5, *a* is

$$e(t) = E_{1m}\sin(\omega t + \psi_1) + E_{3m}\sin(3\omega t + \psi_3),$$

where E_{1m} = 25.9 volts, E_{3m} = 6 volts, $\psi_1 = -12^o$, $\psi_3 = 55^o$, *R* = 10 ohms, 1/*ωC* = 9 ohms; *ωL* = 6 ohms.

Find expressions for the instantaneous currents in the branches and the total current of the network.

Solution: Referring to Fig. 8.5, *a*, the first harmonic of the current i_I is

$$\frac{E_{1m}}{R}\sin(\omega t + \psi_1) = 2.59\sin(\omega t - 12^o).$$

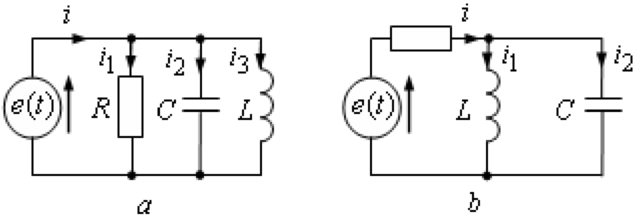


Fig.8.5

The third harmonic of the current i_I is

$$\frac{E_{3m}}{R}\sin(3\omega t+\psi_3)=0.6\sin(3\omega t+55^o)\,.$$

The total current in the first branch is

$$i_I=2.59\sin(\omega t-12^o)+0.6\sin(3\omega t+55^o)\,.$$

The first harmonic of the current i_{II} is

$$\frac{E_{1m}}{\dfrac{1}{\omega C}}\sin(\omega t+\psi_1+90^o)=2.87\sin(\omega t+78^o)\,.$$

The third harmonic of the current i_{II} is

$$\frac{E_{3m}}{\dfrac{1}{3\omega C}}\sin(3\omega t+\psi_3+90^o)=2\sin(3\omega t+145^o)\,.$$

The total current in the second branch is

$$i_{II}=2.87\sin(\omega t+78^o)+2\sin(3\omega t+145^o)\,.$$

The first harmonic of the current i_{III} is

$$\frac{E_{1m}}{\omega L}\sin(\omega t+\psi_1-90^o)=4.32\ \sin(\omega t-102^o)\,.$$

The third harmonic of the current i_{III} in is

$$\frac{E_{3m}}{3\omega L}\sin(3\omega t+\psi_3-90^o)=0.33\ \sin(3\omega t-35^o)\,.$$

The total current in the third branch is

$$i_{III}=4.32\sin(\omega t-102^o)+0.33\sin(3\omega t-35^o)\,.$$

The total current of the network is

$$i = i_I + i_{II} + i_{III} .$$

The first harmonic of the current *i* equals the sum of the first harmonic of the branch currents. Each harmonic is written as a complex number. Harmonics of the same order are summed by adding complex numbers or their vectors in a complex plane.

Similarly, the third harmonic of the current I equals the sum of the third harmonics of the branch currents. Finally,

$$i = 3\sin(\omega t - 40^o) + 1.8\sin(3\omega t + 125^o) \text{ amperes.}$$

Example 8.3. Find the instantaneous currents and voltages of the network shown in Fig. 8.5, *b* where

$$e(t) = 100 + 80\sin(\omega t + 30^o) + 60\sin(3\omega t + 20^o) + \\ + 50\sin\ (5\omega t + 45^o)\ volts,$$

R = 3 ohms, 1/*ωC* = 27 ohms; *ωL* = 3 ohms.

Solution: As the first step, find the complex impedance presented to the terminals *bc* for the first, third and fifth harmonics:

$$Z_{bc1} = \frac{j3(-j27)}{j3 - j27} = j3.38;\ \ Z_{bc3} = \frac{j9(-j9)}{j9 - j9} = \infty\ ;$$

$$Z_{bc5} = \frac{j15(-j5.4)}{j15 - j5.4} = -j8.5\,\Omega.$$

The total impedance of the network for the first harmonic is

$$Z_1 = R + Z_{bc1} = 3 + j3.38 = 4.5e^{j48^o}\ \Omega.$$

The total impedance of the network for the third harmonic is

$$Z_3 = \infty\ .$$

The total impedance of the network for the fifth harmonic is

$$Z_5 = 3 - j8.5 = 9e^{-j70^o}\ \Omega.$$

The direct current resistance of the network is 3 ohms (in the branch *bc* the direct current flows through the inductance *L* whose resistance is taken to be zero).

Now find the current components for the common portion of the network. The direct component of the current is

$$\frac{100}{3} = 3.33 \text{ amperes.}$$

The complex current of the first harmonic is

$$\frac{80e^{j30^o}}{4.5e^{j48^o}} = 17.8e^{-j18^o} \text{ amperes.}$$

At the third harmonic, the impedance of two parallel branches of the network is ∞, and the current is equal to *0*, which is a condition known as parallel (or current) resonance. In other words, there is no third-harmonic component in the current *i*, although it is present in the currents i_1 and i_2.

The complex current of the fifth harmonic is

$$\frac{50e^{j45^o}}{9e^{-j70^o}} = 5.55e^{j115^o} \text{ amperes.}$$

Now we find the voltage drop across *bc* due to each of the harmonics. The voltage drop due to the direct current component is zero.

The complex amplitude of the voltage drop due to the first-harmonic current is

$$\underline{U}_{bc1m} = 17.8e^{-j18^o} \times j3.38 = 60.2e^{j72^o} .$$

Similarly, as instantaneous voltage $u_{bc3} = 60\sin(3\omega t + 20^o)$, its complex amplitude $\underline{U}_{bc3m} = 60e^{j20^o}$.

Analogically, the complex amplitude of the voltage drop due to the fifth-harmonic current is

$$\underline{U}_{bc5m} = 5.55e^{j115^o} \times 8.5e^{-j90^o} = 47.2e^{j25^o} .$$

The instantaneous voltage u_{bc} is

$$u_{bc} = 60.2\sin(\omega t + 72^o) + 60\sin(3\omega t + 20^o) + \\ + 47.2\sin(5\omega t + 25^o) \ volts, \quad .$$

The next step is to find the complex amplitudes of the first, third and fifth-harmonic currents through the capacitance.

The first-harmonic current:

$$\frac{60.2e^{j72^o}}{-j27} = 2.23e^{j162^o} ;$$

the third-harmonic current:

$$\frac{60\,e^{j20^o}}{-\,j9}=6.67e^{j110^o}\ ;$$

the fifth-harmonic current:

$$\frac{47.2e^{j25^o}}{5.4\,e^{-j90^o}}=8.72\,e^{j115^o}\ \text{amperes;}$$

The instantaneous current through the capacitance is

$$i_1=2.23\sin(\omega t+162^o)+6.67\sin(3\omega t+110^o)+$$

$$+8.72\ \sin(5\omega t+115^o)\ \mathit{amperes}.$$

Finally, we determine the components of the current flowing through the inductance.

The direct current (*dc*) component is 3.33 amperes. The first-harmonic current is

$$\frac{60.2e^{j72^o}}{j3}=20.1e^{-j18^o}\ \text{amperes (approx).}$$

The third-harmonic current is

$$\frac{60\,e^{j20^o}}{j9}=6.67e^{-j70^o}\ \text{amperes;}$$

the fifth-harmonic current is

$$\frac{47.2e^{j25^o}}{j15}=3.13\,e^{-j65^o}\ \text{amperes;}$$

and the total current through the impedance is

$$i_2=33.3+20.1\sin(\omega t-18^o)+6.67\sin(3\omega t-70^o)+$$

$$+3.13\ \sin(5\omega t-65^o)\ \mathit{amperes}.$$

The total current round the network is

$$i=33.3+17.8\sin(\omega t-18^o)+5.55\ \sin(5\omega t+115^o)\ \text{amperes.}$$

8.8. Resonance Phenomena with Non-sinusoidal Currents

In defining the resonance phenomena in series and parallel *RLC* circuits, it was stated that they take place when the current and the applied voltage at the input of such a circuit are in phase.

This definition may now be extended to cover non-sinusoidal currents and voltages by stating that if the applied voltage is non-sinusoidal, a resonance (series or parallel) may occur not only at the first harmonic, but also at the higher harmonics.

By resonance at the *k*-th harmonic will be meant a condition under which the *k*-th-harmonic current at the circuit input is in phase with the *k*-th harmonic of the applied voltage (other current harmonics being out of phase with the respective voltage harmonics).

If one takes into account the resistance of inductances, then the condition for resonance at any harmonic is that the reactance term of the impedance at that harmonic must vanish.

As often as not, resonance phenomena with non-sinusoidal currents are investigated setting the resistance of the coils equal to zero. Then for parallel resonance the impedance of the circuit should be equal to infinity; for series resonance it must be zero.

It may be noted that at, or very near resonance at any higher harmonic the currents and/or voltages of this harmonic may be greater than those due to the first harmonic in the same parts of the circuit, although the respective harmonic of the applied voltage may be only a fraction of its first harmonic.

Example 8.4. In the parallel resonant circuit of Fig. 8.6 the inductance L_2 is fixed.

Setting the resistance of L_2 equal to zero, find the values of C_1 and C_2 for which the impedance of the circuit is zero at the first harmonic and infinity at the ninth harmonic.

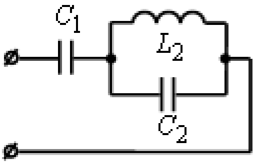


Fig.8.6.

Solution: Writing down the impedance of the circuit at the first harmonic and equating it to zero, we get

$$Z_1 = \frac{-j}{\omega C_1} + \frac{j\omega L_2 \left(\dfrac{-j}{\omega C_2} \right)}{j \left(\omega L_2 - \dfrac{1}{\omega C_2} \right)} = 0 .$$

Equating the impedance of the circuit at the ninth harmonic to infinity, we obtain

$$Z_9 = 9 \frac{-j}{\omega C_1} + \frac{j 9\omega L_2 9 \left(\dfrac{-j}{\omega C_2} \right)}{j \left(9\omega L_2 - \dfrac{1}{9\omega C_2} \right)} = \infty .$$

Solving the two equations simultaneously gives

$$\frac{1}{\omega C_2} = 81 \omega L_2 \text{ and } \frac{1}{\omega C_1} = \frac{81}{80} \omega L_2 .$$

8.9. R.M.S. Values of Non-sinusoidal Current and Voltage Wave-forms

By definition (see Sec. 4.2), the square of the effective current I can be expressed in terms of the instantaneous current i thus

$$I^2 = \frac{1}{T} \int_0^T i^2 dt .$$

If the instantaneous current is

$$i = I_0 + I_{1m} \sin(\omega t + \varphi_1) + I_{2m} \sin(2\omega t + \varphi_2) + \dots ,$$

then we can get the following equations:

$$i^2 = I_0^2 + \sum_{k=1}^{\infty} I_{km}^2 \sin^2(k\omega t + \varphi_k) +$$

$$+ \sum_{\substack{p=0 \\ q=0,\ p \neq q}}^{\infty} I_{pm} I_{qm} \sin(p\omega t + \varphi_p) \sin(q\omega t + \varphi_q).$$

But

$$\left.\begin{aligned}&\int_0^T \sin^2(k\omega t+\varphi_k)\,dt=\frac{T}{2},\\&\int_{\substack{0\\p\neq q}}^T \sin(p\omega t+\varphi_p)\,\sin(q\omega t+\varphi_q)\,dt=0\end{aligned}\right\}\tag{8.20}$$

Therefore

$$I^2=I_0^2+\frac{I_{1m}^2}{2}+\frac{I_{2m}^2}{2}+\frac{I_{3m}^2}{2}+\ldots,$$

and

$$I=\sqrt{I_0^2+\frac{I_{1m}^2}{2}+\frac{I_{2m}^2}{2}+\frac{I_{3m}^2}{2}+\ldots}\ .$$

Since the amplitude of the *k*-ih harmonic of the current I_{km} is $\sqrt{2}$ times the effective value of the *k*-th harmonic of the current I_k, we have

$$\frac{I_{km}^2}{2}=\frac{I_{km}}{\sqrt{2}}\frac{I_{km}}{\sqrt{2}}=I_k^2$$

and

$$I=\sqrt{I_0^2+I_1^2+I_2^2+I_3^2+\ldots}\ .\tag{8.21}$$

Thus, the root mean square (r.m.s.) value of a non-sinusoidal current is the square root of the sum of the squared values of the direct component and the r.m.s. values of the individual harmonic components. The r.m.s. value is independent of the phase angles φ_k.

Similarly, the r.m.s. value of a non-sinusoidal voltage is the square root of the sum of the squared values of the direct component and the r.m.s. values of the individual harmonic components, or

$$U=\sqrt{U_0^2+U_1^2+U_2^2+U_3^2+\ldots}\ .\tag{8.22}$$

Example 8.5. Find the r.m.s. values of the total current *i* and the applied e.m.f. *e* for *Example* 8.2.

Solution:

$$I = \sqrt{33.3^2 + \frac{17.87^2}{2} + \frac{5.55^2}{2}} = 35.5 \text{ amperes;}$$

$$E = \sqrt{100^2 + \frac{80^2}{2} + \frac{60^2}{2} + \frac{50^2}{2}} = 127.2 \text{ volts.}$$

8.10. The Average Value of Non-sinusoidal Function

The average value of non-sinusoidal function is given by

$$\frac{1}{2\pi}\int_0^{2\pi} \left| f(\omega t) \right| d\omega t \qquad (8.23)$$

As distinct from the r.m.s. value, the average value of a non-sinusoidal function depends on the phase angle φ_k.

Example 8.6. Find the average value of the function

$$i = I_{1m}\sin(\omega t + \varphi_1) + I_{3m}\sin(3\omega t + \varphi_3) + I_{5m}\sin(5\omega t + \varphi_5) + \ldots$$

assuming that the sign does not change during each half-cycle.

Solution: On integration, we get

$$I_{av} = \frac{2}{\pi}\left(I_{1m}\ \cos\varphi_1 + \frac{1}{3} I_{3m}\ \cos\varphi_3 + \frac{1}{5} I_{5m}\ \cos\varphi_5 + \ldots \right) \qquad (8.24)$$

8.11. Instruments for Non-sinusoidal Currents and Voltages

Different types of instruments may be used to measure non-sinusoidal currents and voltages, and their action is likewise different. Moving-coil, electrodynamic and thermal instruments measure the r.m.s. values of currents or voltages. Moving-coil instruments with a rectifier measure the average values, and those without a rectifier the direct component of currents or voltages. Valve voltmeters measure the peak values of voltages.

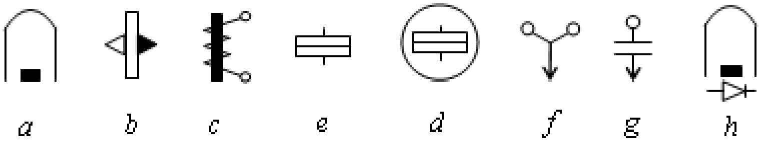


Fig. 8.7. Instruments for non-sinusoidal values

The conventions for the instruments adopted in our country are shown in Fig. 8.5, where at *a* is the symbol of moving-coil instruments;

at *b*, of the moving-magnet type; at *c* of the moving-iron type; at *d* of the electrodynamic type; at *e* of the ferrodynamic (iron-cored electrodynamic) type; at *f* of the hot-wire type; at *g* of the electrostatic 'type; and at *h* of the moving-coil type with a rectifier.

8.12. Power for Non-sinusoidal Voltages and Currents

The average power P in a circuit fed a non-sinusoidal voltage u and a non-sinusoidal current i is

$$P = \frac{1}{T}\int_0^T u\,i\,dt \ \text{(watts)}.$$

Expanding u and i into Fourier series, we get

$$u = U_o + U_{1m}\sin(\omega t + \psi_1) + U_{2m}\sin(\omega t + \psi_2) + $$
$$+ U_{3m}\sin(3\omega t + \psi_3) + ...;$$

$$i = I_0 + I_{1m}\sin(\omega t + \psi_1 - \varphi_1) + I_{2m}\sin(2\omega t + \psi_2 - \varphi_2) +$$
$$+ I_{3m}\sin(3\omega t + \psi_3 - \varphi_3)...$$

By virtue of Eq. (8.10)

$$P = U_0 I_0 + U_1 I_1 \cos\varphi_1 + U_2 I_2 \cos\varphi_2 + U_3 I_3 \cos\varphi_3 + ... \quad (8.25)$$

Thus, harmonics produce power separately, and the *average power* due to non-sinusoidal voltages and currents is equal to *the sum, of the average powers due to the individual harmonics*.

The volt-amperes (or the apparent power, S) in a circuit is

$$S = U\,I\,, \quad (8.26)$$

where U and I stand for the r.m.s. values of the non-sinusoidal voltage and current in the circuit, or

$$U = \sqrt{U_0^2 + U_1^2 + U_2^2 + U_3^2 + ...}$$

$$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + I_3^2 + ...}$$

Example 8.7. Find the average power P and the apparent power S, if

$$u = 25.9\sin(\omega t - 11^o) + 6\sin(3\omega t + 54^o) \ \text{volts;}$$

$$i = 3\sin(\omega t - 40^o) + 0.9\sqrt{2}\sin(3\omega t + 125^o) \ \text{amperes.}$$

Solution:

$$U_1 = \frac{25.9}{\sqrt{2}} = 18.3 \text{ volts; } U_3 = \frac{6}{\sqrt{2}} = 4.26 \text{ volts;}$$

$$I_1 = 2.13 \text{ amperes; } I_3 = 0.9 \text{ ampere;}$$

$$\varphi_1 = -11^o - (-40^o) = 29^{o;} \;\; \varphi_3 = -71^o;$$

$$P = 18.3 \times 2.13 \cos 29^o + 4.26 \times 0.9 \cos(-71^o) = 35.5 \text{ watts;}$$

$$U = \sqrt{U_1^2 + U_3^2} = 18.55 \text{ volts; } I = \sqrt{I_1^2 + I_3^2} = 2.32 \text{ amperes;}$$

$$S = U\,I = 18.55 \times 2.32 = 42.8 \text{ volt-amperes.}$$

8.13. Substitution of Sinusoidal for Non-sinusoidal Wave-forms

In dealing with some of the most elementary properties of nonlinear networks (which will be discussed in Chapter Nine) it is customary to substitute sinusoidal wave-forms for non-sinusoidal ones. This is done so that the root mean square (r.m.s.) values of the sinusoidal current and voltage be equal to the r.m.s. values of the non-sinusoidal current and voltage being replaced. The phase angle φ_{eq}, between the equivalent sinusoidal voltage and current is determined from the ratio

$$\cos\varphi_{eq} = \frac{P}{U\,I}, \tag{8.27}$$

which is the power factor, so that the power P be the same in both cases.

Example 8.8. Replace the complex current and voltage of *Example* 8.2 with sinusoidal ones and find the phase angle, φ_{eq}, between them.

Solution: The root mean square. value of the complex voltage is U = 18.55 volts. The r.m.s. value of the complex current is I = 2.32 amperes.

$$\cos\varphi_{eq} = \frac{35.5}{18.55 \times 2.32} = 0.828, \;\; \varphi_{eq} = 34^o.$$

8.14. The Effect of Triplen Harmonics on Three-phase Systems

The phase voltages of a three-phase transformer or a three-phase alternator are often non-sinusoidal and can therefore be expanded into their Fourier series in which the direct component is usually absent.

Let the *k*-th harmonic of phase *A* be

$$u_{kA} = U_{km} \sin(k\omega t + \varphi_k)\,.$$

The phase voltages (u_A, u_B and u_C) are displaced in time relative to each other through 120°, or one-third of a cycle, *T/3*. Thus u_B lags behind, and u_C leads, u_A by 120°. Accordingly, the *k*-th harmonics of u_B and u_C are

$$u_{kB} = U_{kB} \sin\left[k\omega\left(t - \frac{T}{3}\right) + \psi_k\right] =$$

$$= U_{kB} \sin((k\omega t - 120^o k + \varphi_{k)};$$

$$u_{kC} = U_{kC} \sin\left[k\omega\left(t + \frac{T}{3}\right) + \psi_k\right] =$$

$$= U_{kC} \sin((k\omega t + 120^o k + \varphi_{k)},$$

because $k\omega\dfrac{T}{3} = k\dfrac{2\pi T}{T3} = k\dfrac{2\pi}{3} = 120^o k$.

As can be seen, when $k = 1,\ 4,\ 7,\ 10$, the *k*-th harmonic of u_B lags 120° behind the *k*-th harmonic of u_A, so that the 1st, 4th, 7th and 10th harmonics form a positive-phase-sequence system. When $k = 2,\ 5,\ 8,\ 11$, the *k*-th harmonic of u_B leads the *k*-th harmonic of u_A by 120°, and the 2nd, 5th, 8th and 11th harmonics form a negative-phase-sequence system. The third-harmonic terms are all in phase, and this is true of all triplen harmonics, i.e., those harmonics whose order is an integral multiple of 3. Triplen harmonics form a zero-phase-sequence system (for phase sequences, refer to Sec. 5.14. It will be useful to represent the case of triplen harmonics graphically.

In Fig. 8.7 the voltages u_A, u_B and u_C are the phase voltages of a three-phase alternator. They are of rectangular wave-form and are displaced with respect to each other through one-third of the cycle of the fundamental term. The same figure shows the first and third harmonics of each voltage. As is seen, the third harmonics are all in phase.

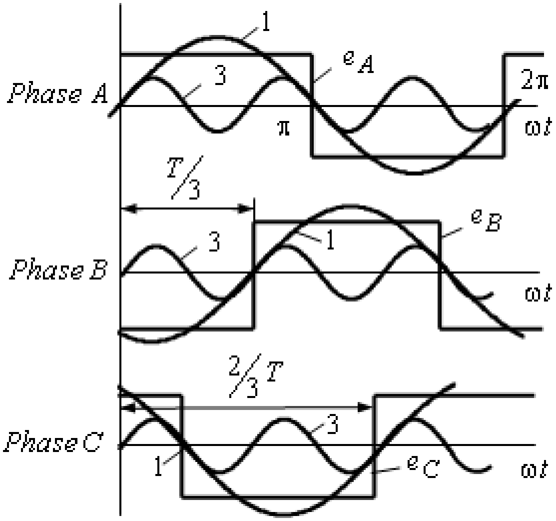


Fig.8.7. The case of triplen harmonics.

The effects of triplen harmonics on the operation of three-phase systems may be summed up as follows.

1. In a delta-connected three-phase alternator or transformer (Fig. 8.8, *a*), there will be a current flowing in their windings due to triplen harmonics even at no-load.

The algebraic sum of the third harmonics of the voltages in a delta-connected three-phase system is $3U_3$. The algebraic sum of the first and all other harmonics, except triplen harmonics, is zero. So, at no-load, these harmonics do not contribute to the current flowing around the closed delta.

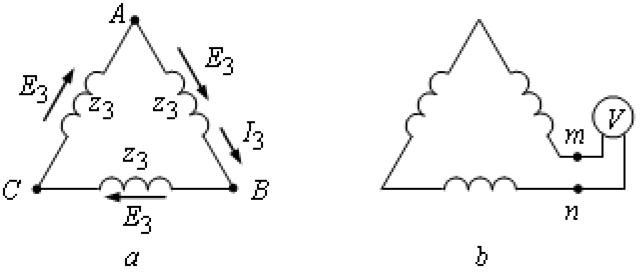


Fig.8.8. Delta-connected three-phase systems.

Let the impedance of each phase to the third-harmonic current be Z_3. Then the third-harmonic current around the delta is

$$\underline{I}_3 = \frac{3\underline{U}_3}{3\underline{Z}_3} = \frac{\underline{U}_3}{\underline{Z}_3}.$$

Similarly, the sixth-harmonic current is.

$$\underline{I}_6 = \frac{\underline{E}_6}{\underline{Z}_6},$$

where $\underline{E}_6$ is the r.m.s. value of the sixth-harmonic phase voltage and Z6 is the impedance the phase offers to the sixth harmonic component.

The r.m.s. value of the current around the closed delta of Fig. 8.8, *a* is

$$I = \sqrt{I_3^2 + I_6^2 + I_9^2 + ...}$$

2. In an open-delta three-phase system (Fig. 8.8, *b*), the triplen harmonics appearing in the phase voltages give rise to a voltage across the terminals m and n, equal to the sum of the triplen harmonic voltages:

$$U_{mn} = 3U_{3m} \sin\,(3\omega t + \varphi_3) + 3U_{6m} \sin\,(6\omega t + \varphi_6) + ...\ .$$

The voltmeter in the circuit of Fig. 8.8, *b* reads

$$U = 3\sqrt{U_3^2 + U_6^2 + ...}$$

3. No triplen harmonics appear in the line voltage in either star- or delta-connected three-phase systems.

As a proof, consider a delta-connected system (Fig. 8.8, *a*) at no-load, although the statement holds for operation under load as well.

Let the point *A* be at a third-harmonic potential φ_{A3} and the point *B* at a third-harmonic potential φ_{B3}. Then

$$\underline{\varphi}_{A3} = \underline{\varphi}_{B3} - \underline{U}_3 + \underline{I}_3\underline{Z}_3$$

But $\underline{U}_3 = \underline{I}_3\underline{Z}_3$. Hence,

$$\underline{\varphi}_{A3} = \underline{\varphi}_{B3}.$$

In a star-connected system (Fig. 8.9), the third-harmonic line voltage is equal to the difference between the respective phase voltages. Since the third-harmonic phase voltages are in phase, they are subtracted in writing the difference.

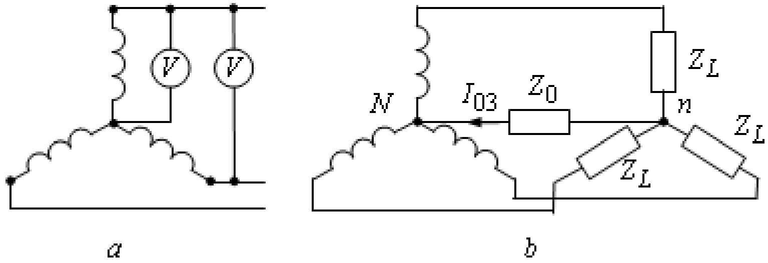


Fig. 8.9. Star connections.

All harmonics may appear in the phase voltage (the direct component is usually absent), so that the r.m.s. value of the phase voltage is

$$U_p = \sqrt{U_1^2 + U_2^2 + U_3^2 + U_4^2 + \ldots}$$

As has been shown, triplen harmonics do not contribute to the line voltage, and so the r.m.s. value of the line voltage is

$$U_l = \sqrt{3}\sqrt{U_1^2 + U_2^2 + U_4^2 + \ldots}$$

The ratio U_l / U_p is less than $\sqrt{3}$ when triplen harmonics are present.

When an alternator and a balanced load are connected into a star without a return wire, no currents due to triplen or other zero-phase-sequence harmonics can flow in the line wires. Therefore, the potential difference (p.d.) between the neutral point *n* of the load and the neutral point *N* of the alternator (Fig. 8.9, $Z_0 = \infty$) is

$$U_{nN} = U_{3m} \sin\,(3\omega t + \varphi_3) + U_{6m} \sin\,(6\omega t + \varphi_6) + \ldots$$

Its root mean square (r.m.s.) value is

$$U_{nN} = \sqrt{\frac{U_{3m}^2}{2} + \frac{U_{6m}^2}{2} + \ldots}$$

5. In a balanced wye-wye system, the third-harmonic current in the neutral wire is

$$\underline{I}_{03} = \frac{\underline{U}_3}{\underline{Z}_{03} + \frac{\underline{Z}_{L3}}{3}}$$

where $\underline{Z}_{03}$ is the impedance presented to the third-harmonic current by the neutral wire and $\underline{Z}_{L3}$ is the impedance presented to the third-harmonic current by the load. The third-harmonic current is $\underline{I}_{03}/3$. Other triplen- harmonic currents are found in a similar way.

Example 8.9. The e.m.f. of phase A of the circuit in Fig.8.10 is $e_A = 170 \sin \omega t + 80 \cos 3\omega t + 34 \cos 9\omega t\ V$. R = 9 Ω; $\omega L = 2\,\Omega$. Find readings of the instruments (which are of the dynamic type).

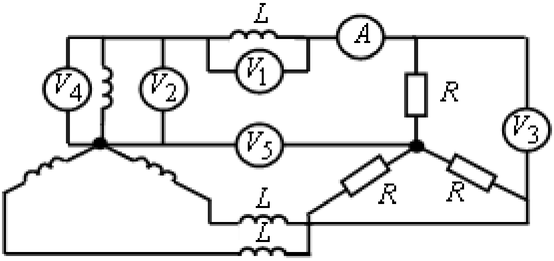


Fig. 8.10

The r.m.s. values

$$E_1 = \frac{170}{\sqrt{2}} = 121\,V; \quad E_3 = 56.5\,V; \quad E_9 = 24.2\,V.$$

The first-harmonic line current: $I_1 = \dfrac{E_1}{\sqrt{R^2 + (\omega L)^2}} = \dfrac{121}{9.2} = 13.2\ A.$

The voltmeter V_1 reads $\sqrt{E_1^2 + E_3^2 + E_9^2} = 136\ V.$

The voltmeter V_2 reads $R_1 \cdot I_1 = 13.2 \cdot 9 = 118.8\ V.$

The voltmeter V_3 reads $\sqrt{3} \cdot 118.8 = 205.8\ V.$

The voltmeter V_4 reads $\omega L I_1 = 2 \cdot 13.2 = 26.4\ A.$

The voltmeter V_5 reads $\sqrt{E_3^2 + E_9^2} = 62.3\ V.$

Example 8.10. The phase voltages of the alternator in Fig. 8.11 are of a trapezoidal wave-form; $a_m = 220\,V$; $\alpha = 10^o$. The load is balanced $R = 6\,\Omega$; $\omega L = 0.5\,\Omega$; $1/\omega C = 12\,\Omega$.Find the instantaneous current in the neutral wire, neglecting harmonics higher than the seventh one.

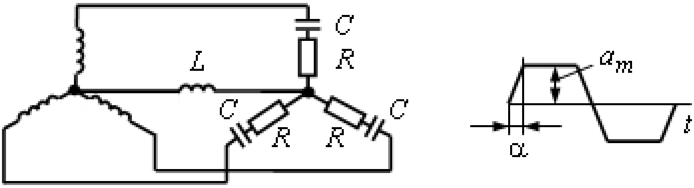


Fig. 8.11

Referring to the table on page 362, the Fouier series for the trapezoidal wave is

$$U_A = \frac{4a_m}{\alpha\pi}(\sin\alpha \sin\ \omega t + \frac{1}{9}\sin 3\alpha \sin 3\omega t +$$

$$+\frac{1}{25}\sin 5\alpha \sin 5\omega t + \frac{1}{49}\sin 7\alpha \sin 7\omega t) =$$

$$= \frac{4\times 220}{\frac{\pi}{18}\pi}(\sin 10^o\ \sin\ \omega t + \frac{1}{9}\sin 30^o\ \sin 3\omega t +$$

$$+\frac{1}{25}\sin 50^o\ \sin 5\omega t + \frac{1}{49}\sin 70^o\ \sin 7\omega t).$$

Consequently,

$$u_A = 274\ \sin\omega t + 89.3\ \sin 3\omega t + 49.5\ \sin 5\omega t + 30.9.\ \sin 7\omega t\ .$$

The current in the neutral wire is solely a third-harmonic current

$$\underline{I}_{03} = \frac{\underline{U}_3}{\underline{Z}_{03} + \frac{\underline{Z}_{L3}}{3}};$$

Then we can determine the voltage

$$\underline{U}_3 = \frac{89.3}{\sqrt{2}} = 63.3\ V;$$

$$\underline{Z}_{03} = j1.5; \quad \underline{Z}_{L3} = 6 - j4; \quad \frac{\underline{Z}_{L3}}{3} = 2 - j1.33;$$

$$\underline{I}_{03} = \frac{63.3}{j1.5 + 2 - j1.33} = 31.8e^{-j5^o}\ A.$$

The instantaneous current

$$i_{03} = 44.8\ \sin\ (3\omega t - 5^o)\ A.$$

Summary review questions

8.1. How can non-sinusoidal waveforms be produced?

8.2. On what theorem is the calculation of linear electric circuits with non-sinusoidal periodic voltage based?

8.3. How are currents and voltages determined for non-sinusoidal waveforms?

8.4. How can a periodic function with period 2π be represented?

8.5. Describe Dirichlet conditions for periodic functions.

8.6. Describe specific properties for different wave-forms.

8.7. How is it possible to represent three entirely different waveforms by means of two sinusoidal components?

8.8. State the Fourier series theorem and comment its importance.

8.9. How many groups may the periodic wave-forms encountered in electrical engineering be classed into? What kinds of periodical wave-forms do these groups include?

8.10. When may the resonance phenomena in non-sinusoidal circuits occur? At what conditions?

8.11. What is the root mean square (r.m.s.) value of a non-sinusoidal current or voltage? What is the r.m.s. value independent?

8.12. What types of instruments may be used to measure non-sinusoidal currents and voltages?

8.13. What can you say about the value of impedance of the circuit for series and for parallel resonance?

8.14 How can we define the power for non-sinusoidal circuit?

8.15. Explain the effects of triplen harmonics on the operation of three-phase systems.

8.16. How does one take advantage of the symmetry properties of periodic waves? Illustrate.

8.17. When can one use Fourier series in which the direct component is usually absent?

8.18. How can be determine the r.m.s. value of the current around the closed delta connection?

8.19. What can you say about the calculation the third-harmonic line voltage in a star-connected system?

8.20. What can you say about the current in the neutral wire in the case of triplen harmonics? How may it be determined?

Problems

8.1. Find the average and effective values of each wave shape depicted in Fig. P8.1.

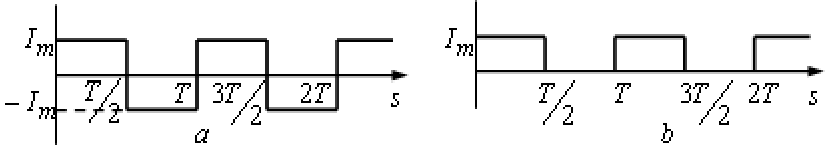


Fig. P8.1

8.2. In each of the sketches of Fig. P8.2 the curves are sine waves or parts thereof. Find average and effective values for each wave shape.

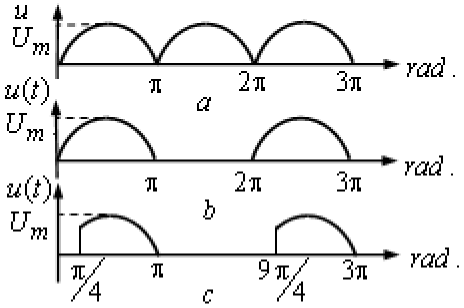


Fig. P8.2

8.3. Compute the average and effective values for each wave shape shown in Fig. P8.3.

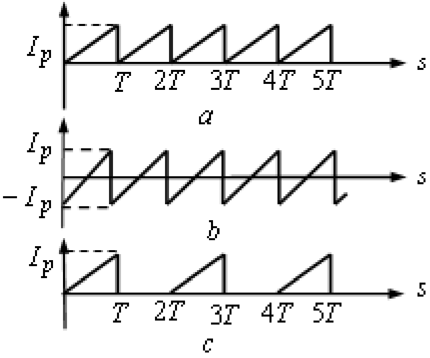


Fig. P8.3

8.4. Find the Fourier series for the waveform of Fig. P8.4, *a*.

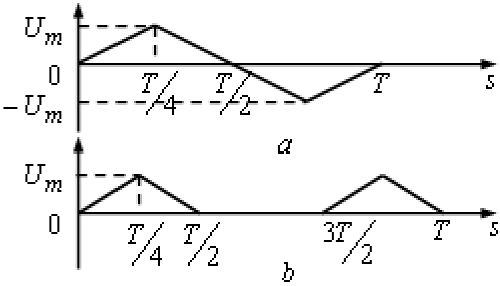


Fig. P8.4

8.5. Determine the Fourier series of the rectified waveform shown in Fig. P8.2, *a*.

8.6. Find the Fourier series of the sawtooth waveform of Fig. P8.3, *a*.

8.7. For each waveshape shown in Fig. P8.4 find the average and effective values.

8.8. The AM radio dial spreads over a frequency range 570 to 1560 kHz.

(a) Calculate the range of capacitance that is needed in series with a 20-μH inductance to tune over the entire frequency band.

(b) If the quality factor for a radio station operating at 570 kHz is 100, what is the bandwidth of the tuning circuit?

(c) What is the resistance of the tuning circuit?

(d) Determine the value of the quality factor at the upper end of the radio band.

8.9. Compute the average and effective values of each of the waveshapes shown in Fig. 8.49.

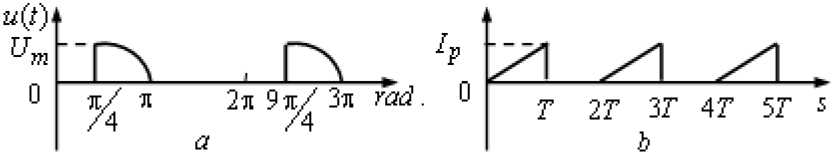


Fig. 8.9.

8.10. A voltage wave has the form depicted in Fig. P8.10. Obtain the Fourier series representation of this periodic function.

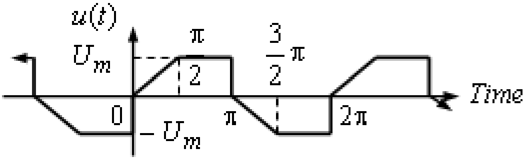


Fig. P8.10

8.11. The voltage $U_m = 314\ V$ being changed as it is shown in Fig. P8.11, *b* is applied to the circuit (see Fig. P8.11, *a*). Having limited to three harmonious components of voltage one must define instantaneous and effective values of a circuit current, and voltages across elements. Calculate active, reactive and apparent powers, developed by the source; $\omega = 10^3\ s^{-1};\ R = 10\,\Omega;\ L = 5\,mH;\ C = 66.7\,\mu F$

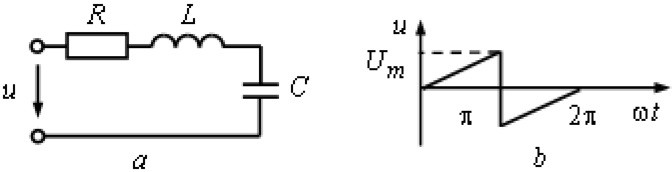


Fig. P8.11

8.12. The voltage $u = 45 - 60\sin\omega t + 30\sin 3\omega t$ is applied to the circuit in Fig. P8.12.

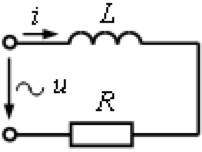


Fig. P8.12

Circuit parameters: $R = 15\Omega$, $L = 15.92\ mH$, the frequency of the first harmonic is $f = 50\ H$. One must find: a time function of the current $i(t)$ and depict it together with the function of the voltage $u(t)$; effective values of the current and the voltage.

8.13. A voltage wave has the variation shown in Fig. P8.13. (a) Find the average and the effective value of the voltage. (b) If the voltage of part (*a*) is applied to a 10-Ω resistance, find the dissipated power in watts.

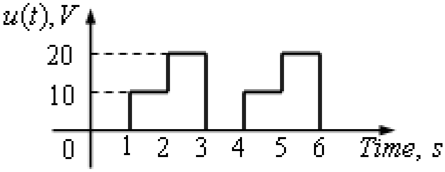


Fig. P8.13

8.14. Find current *i* and effective values of the current *I* and the voltage *U*, and the consumed active power for the circuit in Fig. P8.14, if its parameters are $R1 = 18\Omega$, $R2 = 12\Omega$, $L = 25.5\ mH$, $C = 398\ mkF$. The input voltage $u = 60 + 21.8\sin\omega t + 11.45\sin 2\omega t\ \ V$.

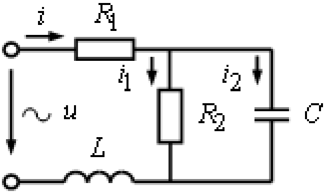


Fig. P8.14

8.15. The circuit in Fig. P8.15 has two energy sources.

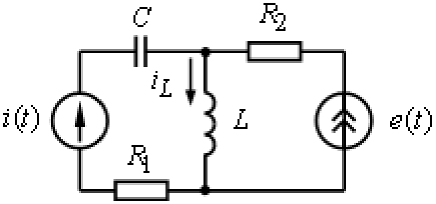


Fig P8.15

Calculate current i_L if $R_1 = 5\,\Omega$, $R2 = \omega L = 1/\omega C = 10\,\Omega$; $e(t) = 100 + 50\sin(\omega t - 30^o)\ V$ and $i(t) = 10 + 5\sin 2\omega t\ \ A$.

Chapter 9

MAGNETIC THEORY AND CIRCUITS

An understanding of electromagnetism is essential to the study of electrical engineering because it is the key to the operation of a great part of the electrical apparatus found in industry as well as the home. All electric motors and generators depend upon the electromagnetic field as the coupling device permitting interchange of energy between an electrical system and a mechanical system and vice versa. Similarly, static transformers provide the means for converting energy from one electrical system to another through the medium of a magnetic field. Other important devices - for example, circuit breakers, automatic switches, and relays - require the presence of a confined magnetic field for their proper operation. It is the purpose of this chapter to provide the reader with background so that he can identify a magnetic field and its salient characteristics and more readily understand the function of the magnetic field in electrical equipment.

As has been previously pointed out, the science of electrical engineering is founded on a few fundamental laws derived from basic experiments. In the area of electromagnetics, it is Ampere's law that concerns us, and, in fact, serves as the starting point of our treatment. It should be here inferred that this is the only starting point in developing a quantitative theory of the magnetic circuit.

Faraday's law of induction is equally valid as a starting point, and is preferred when the goal is the development of an electromagnetic wave theory rather than a theory leading to the treatment of electromechanical energy conversion.

On the basis of the results obtained by Ampere in 1820, in his experiments on the forces existing between two current- carrying conductors, such quantities as magnetic flux density, magnetic field intensity, permeability, and magnetic flux are readily defined.

Once this base is established, attention is then directed to a discussion of the magnetic properties of certain useful engineering materials as well as to the idea of a "magnetic circuit" to help simplify the computations involved in analyzing magnetic devices.

9.1. Ampere's Law - Definition of Magnetic Quantities

Appearing in Fig. 9.1 is a simplified modification of Ampere's experiment. The configuration consists of a very long conductor 1 carrying a constant-magnitude current I_1 and an elemental conductor of length l carrying a constant-magnitude current I_2 in a direction opposite to I_1. When taken together the elemental conductor and the current I_2 constitute a current element $I_2 l$.

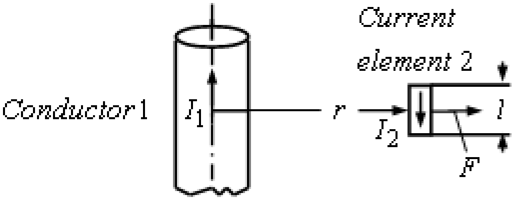


Fig. 9.1 Illustrating the force existing between a current element and a very long conductor carrying current, as described by Ampere's law.

The elemental conductor 2 is actually part of a closed circuit in which I_2 flows, but for simplicity and convenience the details of the circuit are omitted except for the length l.

Moreover, it is assumed that conductors 1 and 2 lie in the same horizontal plane and are parallel to each other. In accordance with Ampere's law it is found that with this configuration there exists a force on the elemental conductor directed to the right.

Furthermore, the magnitude of the force is found to be directly proportional to I_1, I_2, l and the medium surrounding the conductors as well as inversely proportional to the distance between them. The magnitude of this force can be shown to be given by

$$F = \frac{\mu I_1}{2\pi r} I_2 l \tag{9.1}$$

where, I_1 and I_2 are expressed in amperes, l and r in meters, and μ is a property of the medium. A further interesting revelation about this experiment is that if the elemental conductor 2 is used as an exploring device to find those points in space where the force is of constant magnitude and outwardly directed, the locus is found to be a circle of radius r and centered along the axis of conductor 1. In other words it is possible to identify a field having constant lines of force. In this connection it is useful at this point to rewrite Eq. (9-1) as follows:

$$F = I_2 \, l \, B \tag{9.2}$$

where

$$B = \frac{\mu I_1}{2\pi r} . \tag{9.3}$$

Magnetic induction (or *magnetic flux density*) *B* is one of the main parameters of a magnetic field. It determines the e.m.f. induced in the conductor that is moving through the field at that point. From Eq. (9.2) it is obvious that the defined quantity *B* carries the units of force per current-length. This may be expressed as

$$B \approx \frac{newtons}{ampere \times metre} \tag{9.4}$$

where ≈ denotes "is proportional to." A closer examination of this relationship reveals an interesting and very useful aspect of the significance of *B*. Recalling that

$$newton = \frac{energy}{length} = \frac{watt \times \sec onds}{metre} = \frac{volts \times ampere \times \sec onds}{metre} \tag{9.5}$$

and inserting into expression (9-4) allows *B* to be expressed alternatively as

$$\mathrm{B} \approx \frac{volts \times \sec onds \times ampere}{ampere \times metre^2} = \frac{volts \times \sec onds}{metre^2} . \tag{9.6}$$

From Faraday's law ($d\phi = e\,dt$) we note further that the unit of (volts x sec) is equivalent to that of flux. Therefore, we can properly conclude that *B* is in fact a *flux density* since it has the units of flux per square meter. When the flux is expressed in webers, *B* then has the units of webers/meter^2 or *teslas*. Equation (9.3) indicates the factors that determine the magnitude of *B* but it serves also as a means of identifying the manner in which the current I_1 influences the force field about the current element. It is significant to note that as long as I_2 is not zero, the force field and the magnetic field have the same characteristics - both have a circular locus and both are vector quantities possessing magnitude and direction. However, because of the way *B* is defined, the magnetic field exists as long as I_1 is not zero irrespective of the value of I_2.

In our study of electrical machinery, which comes later, conductor 1 will be referred to as the field winding because it sets up the working magnetic field, whereas the total circuit of which the elemental conductor is a part is called the *armature winding*.

The direction of the magnetic field is readily determined by the *right-hand rule* which states that if the field winding conductor (1 in this case) is grasped in the right hand with the thumb pointing in the direction of current flow, the lines of flux (or flux density) will be in the direction in which the fingers wrap around the conductor.

A glance at Eq. (9.1) shows that all the factors are known in this equation with the exception of the proportionality factor μ, which is a characteristic of the surrounding medium. Upon repeating the experiment of Fig. 9-1 in iron rather than air, it is found that the force is many times greater for the same values of I_1, I_2, l, and *r*.

Therefore, it follows that μ may be defined from Eq. (9.1) for various media because it is the only unknown quantity. Moreover, because of the way magnetic flux density was defined, Eq. (9.3) indicates that the effect of the surrounding medium may be described in terms of the degree to which it increases or decreases the magnetic flux density for a specified current I_1.

Thus, when iron rather than free space is the medium, it can be said that the iron provides a greater penetration of the magnetic field in a given region, i.e., there is a greater flux density. This property of the surrounding medium in which the conductors are embedded is called *permeability*.

When the conductors of Fig. 9.1 are assumed placed in a vacuum (free space) and the force is measured for specified values of I_1, I_2, l, and *r*, the solution for the permeability of free space obtained from Eq. (9.1) and expressed in SI units comes out to be

$$\mu_o = 4\pi \times 10^{-7} . \tag{9.7}$$

The unit of permeability also follows from Eq. (9-1). Thus

$$\mu_o = \frac{newtons}{ampere^2}.$$

But

$$newton - meter = joule = volts \times amperes \times \sec ond$$

or

$$\mu_o = \frac{volts \times amperes \times \sec onds}{amperes^2 \times meter} = \frac{volts \times \sec onds}{amperes \times meter}.$$

However, volts × seconds/ampere is the unit of inductance expressed in henrys. Accordingly, permeability is expressed in units of henrys/meter.

In those cases where the surrounding medium is other than free space, the absolute permeability is again readily found from Eq. (9.1).

A comparison with the result obtained for free space then leads to a quantity called *relative permeability*, μ_r. Expressed mathematically, we have

$$\mu_r = \frac{\mu}{\mu_o}. \tag{9.8}$$

Equation (9-8) clearly indicates that relative permeability is simply a numeric which expresses the degree to which the magnetic flux density is increased or decreased over that of free space.

According to their magnetic properties all materials may be divided into *diamagnetic*, *paramagnetic* and *ferromagnetic*.

Diamagnetic materials are those which tend to move away from a stronger magnetic field and their relative permeability μ_r is slightly less than unity (for bismuth it is 0.99983).

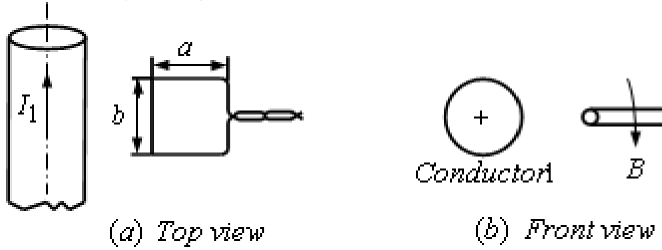
Paramagnetic materials are those which have high values of relative permeability μ_r slightly greater than unity (for platinum it is 1.00036).

Ferromagnetic materials are those which have high values of relative permeability μ_r up to 10^4 or even 10^6 such as iron, nickel, cobalt, etc.

For purposes of electrical engineering it will suffice to class them simply as ferromagnetic and non-ferromagnetic. The former include materials of relative permeability μ_r many times greater than unity, while the latter have relative permeability practically equal to unity. Most ferromagnetic materials, however, have values of μ_r in the hundreds or thousands.

9.2. Magnetic Flux

It is reasonable to expect that since B denotes magnetic flux density, multiplication by the effective area that B penetrates should yield the total *magnetic flux*.



(a) Top view (b) Front view

Fig. 9.2. Associating an area with a magnetic field *B* to identify a magnetic flux.

To illustrate this point refer to Fig. 9.2, which shows a coil of area *ab* lying in the same horizontal plane containing conductor 1. We already know that when a current flows through this conductor a magnetic field is created in space and specifically it is described by Eq. (9.3).

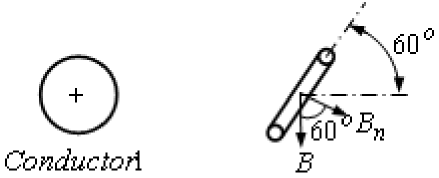


Fig. 9.3. Same as Fig. 9.2, *b* except that the coil is tilted 60° relative to the horizontal plane

To find the total flux penetrating the coil it is necessary merely to perform an integration of *B* over the surface area involved. Of course, if *B* were a constant over the area of concern, the flux would be simply the product of B and the area ab. Next consider that the plane of the coil is tilted with respect to the plane of conductor 1 by 60°, as depicted in Fig. 9.3. Clearly now the total flux penetrating the coil is reduced by a factor of one-half. If the coil is oriented to a position of 90° with respect to the horizontal plane, no flux threads the coil.

On the basis of these observations, then, the magnetic flux through any surface is more rigorously defined as the surface integral of the normal component of the vector magnetic field B. Expressed mathematically, we have

$$\Phi = \int_{s} B_n \, dS \qquad (9.9)$$

where s stands for surface integral, S represents the cross-section area of the coil, and B_n is the normal component of magnetic induction B to the coil area.

From expression (9.6) we know that magnetic flux must have the dimensions of volt-seconds. However, this is more commonly called *webers*. The volt-seconds unit of flux is better understood in terms of Faraday's law of induction.

9.3. Magnetic Field Intensity

Often in magnetic circuit computations it is helpful to work with a quantity representing the magnetic field which is independent of the medium in which the magnetic flux exists.

This is especially true in situations such as are found in electrical machinery where a common flux penetrates several different materials, including air.

A glance at Eq. (9.3) discloses that division of B by μ identifies such a quantity. Accordingly, magnetic field intensity is defined as

$$H = \frac{B}{\mu} \qquad (9.10)$$

and has the units $$\frac{\dfrac{newtons}{ampere \times metre}}{\dfrac{newtons}{ampere^2}} = \frac{ampere}{metre}$$

Thus H is dependent upon the current that produces it and also on the geometry of the configuration but not the medium. For the system of Fig. 9.1 the value of the magnetic field intensity immediately follows from Eq. (9.3) and is given by

$$H = \frac{B}{\mu} = \frac{I_1}{2\pi r} \,. \tag{9.11}$$

More generally the units for H are ampere-turns/meter rather than amperes/meter. This is apparent whenever the field winding is made up of more than just a single conductor.

9.4. Ampere's Circuital Law

Now that the magnetic field intensity has been defined and shown to have dimensions of ampere-turns/meter, we shall develop a very useful relationship. Recall that H is a vector having the same direction as the magnetic field B. For the configuration of Fig. 9.1 H has the same circular locus as B. A line integration of H along any given closed circular path proves interesting. Of course the line integral is considered because H involves a per unit length dimension. Thus

$$\oint H\,dl = \int_0^{2\pi r} \frac{I_1}{2\pi r} dl = I_1 \text{ amperes.}$$

(Again keep in mind that the units here would be ampere-turns if more than one conductor were involved in Fig. 9.1.) The previous equation states that the closed line integral of the magnetic field intensity is equal to the enclosed current (or ampere-turns) that produces the magnetic field lines. This relationship is called *Ampere's circuital law* and is more generally written as

$$\oint H\,dl = F \tag{9.12}$$

where the quantity F is also known as the *magnetomotive force* and frequently abbreviated as m.m.f.

Magnetomotive force (m.m.f.), F, due to a current-carrying coil is given by the product of the turns, N, of the coil and the current, I, linked by the coil turns. The magnetomotive force F gives rise to a magnetic flux in a magnetic circuit, much as an e.m.f. gives rise to an electric current in an electric circuit. This is a scalar quantity. The unit of m.m.f. is measured in ampere-turns.

Ampere's circuital law establishes communication between electrical and magnetic quantities. The law may be also written as

$$\oint H \cdot dl = \sum I \, . \tag{9.13}$$

where the scalar quantity $\sum I$ is a total current that is numerically equal to an algebraic sum of the currents embraced by a contour.

Eq. (9.13) is also referred to as the total current law. It is applied to the calculation of magnetic intensities due to current-carrying conductors.

The *magnetic potential difference* (m.p.d.) or the *magnetic voltage* between some two points *a* and *b* in a magnetic field is the line integral of the magnetic intensity between these two points

$$U_{Mab} = \int_a^b Hdl \, . \tag{9.14}$$

This quantity is related to magnetomotive force and also is measured in amperes or in ampere turns.

If *H* between the two points is constant and is in the same direction as the path element *dl,* then $Hdl = Hdl\cos 0^{o}$, and H may be placed outside the integral sign. Then

$$U_{Mab} = H\int_a^b dl = H \cdot l_{ab} \, ,$$

where l_{ab} is the path between the points a and b in the magnetic field.

9.5. Derived Relationships

In the preceding pages the fundamental magnetic quantities - flux density, flux, field intensity and, permeability - are defined starting with Ampere's basic experiment involving two current-carrying conductors. By the appropriate manipulation of these quantities additional useful results can be obtained.

Equation (9.10) is a vector equation describing the magnetic field intensity for a given geometry and current. If the total path length of a flux line is assumed to be l, then the total magnetomotive force associated with the specified flux line is

$$F = Hl = \frac{B}{\mu} l . \tag{9.15}$$

Now in those situations where B is a constant and penetrates a fixed, known area S, the corresponding magnetic flux may be written from Eq. (9.9) as $\Phi = BS$. Inserting the last equation into Eq. (9.15) yields

$$F = Hl = \Phi \left(\frac{l}{\mu S} \right) . \tag{9.16}$$

The quantity in parentheses in this last expression is interesting because it bears a very strong resemblance to the definition of resistance in an electric circuit.

Refer to Eq. (9.16). Recall that the resistance in an electric circuit represents an impediment to the flow of current under the influence of a driving voltage.

An examination of Eq. (9.16) provides a similar interpretation for the magnetic circuit. We are already aware that F is the driving magnetomotive force which creates the flux penetrating the specified cross-sectional area S.

However, this flux is limited in value by what is called the *magnetic resistance* or, preferably, the *reluctance* of the magnetic circuit, which is defined as

$$R_m = \frac{l}{\mu S} . \tag{9.17}$$

Equation (9.17) reveals that the impediment to the flow of flux which a magnetic circuit presents is directly proportional to the length and inversely proportional to the permeability and the cross-sectional area - results which are entirely consistent with physical reasoning.

The reciprocal of the reluctance is known as the *permeance*

$$G_m = \frac{1}{R_m} = \frac{\mu_0 \mu_r S}{l} .$$

Since the flux-magnetic potential difference (m.p.d.) curves of magnetic circuits are, in the general case, non-linear, both R_m and G_m

are functions of the magnetic flux. Because of this, the relationships for R_m and G_m hold only as long as the magnetic circuit or part of it remains unsaturated.

Most often, this is true of magnetic circuits with a sufficiently large air gap which straightens the respective flux-m.p.d. characteristics.

In a way, the reluctance of the magnetic circuit may be compared to the static resistance of non-linear resistor. Among other things, R_m may be used in the qualitative considerations of flux distribution between two parallel magnetic branches, much as the static resistance figures in the solution of electric network.

Inserting Eq. (9.17) into Eq. (9.16) yields

$$F = \Phi R_m \tag{9.18}$$

which is often referred to as the Ohm's law for the magnetic circuit.

It is important to keep in mind, however, that these manipulations in the forms shown are permissible as long as magnetic field B and cross-sectional area S are fixed quantities.

9.6. Ampere's Law for Various Orientations of the Current Element

In Fig. 9.1 the assumption was made that the current element was located parallel to conductor 1 and lying in the same plane. Because this orientation was sufficient for the purpose at hand - to define the fundamental magnetic quantities - it was pursued as a matter of convenience.

However, in the interest of furnishing a more complete picture of the experiment, we shall now consider the effect on the force of placing the current element, $I_2 l$, in two additional different orientations.

Consider first that the current element is no longer placed parallel to conductor 1 but continues to be located in the same horizontal plane. Refer to Fig. 9.4. The dots in this figure indicate that the magnetic field is directed outward on the left side of conductor 1 and inward (with respect to the plane of the paper) on the right side of the conductor as indicated by the right-hand rule.

The results of this experiment show that the magnitude of the force is the same as that found by using the configuration of Fig. 9.1.

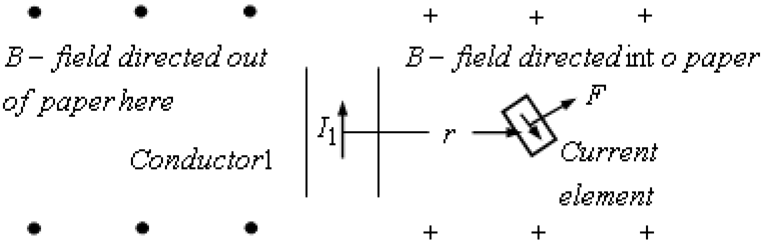


Fig. 9.4. Showing the direction of the force when the current element is no longer located parallel to conductor 1 but remains in the same plane.

This conclusion is not surprising because the value of the magnetic flux density as well as I_2 and I remain unchanged so that Eq. (9.2) is still valid in describing the force. The only change is the direction of the force. However, as Fig. 9.4 indicates, the force continues to be normal to the current element. It is worthwhile to keep this point in mind. Presently, a general rule for establishing the force direction for all configurations will be described. Next let us consider that orientation of the current element which places it parallel to conductor 1 but inclined at an angle θ = 30° with respect to the vertical. A side-view projection of the configuration is depicted in Fig. 9.5.

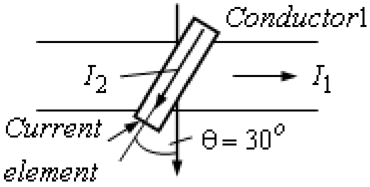


Fig. 9.5. Side-view projection of Fig. 9.1 but with the current element tilted relative to the horizontal plane. Force is directed out of paper.

Note that the magnetic field is directed downward along the vertical for this view. Actually, of course, the locus of *B* is circular, but in Fig. 9.5 we are looking at just that small portion of the *B*-field about the plane containing conductor 1. With this configuration the force on the current element is found to have the same direction but one half the magnitude of that obtained with the orientation of Fig. 9.1.

It follows, then, that the angle between the current vector $I_2 l$ and the flux density B affects the magnitude of the force. As a matter of fact, further experimentation reveals that the general expression for the force is

$$F = I_2 lB \sin\theta . \qquad (9.19)$$

Equation (9.19) conveys information solely about the magnitude of the force and not its direction. It is possible, however, by employing the notation of vector analysis, to rewrite Eq. (9.19) so that information about magnitude as well as direction is present.

This result is readily accomplished by the use of the cross-product notation between two vectors, yielding, a third vector having magnitude and direction. Thus Eq. (9.19) is more completely expressed as

$$\overline{F} = I_2 \overline{l} \times \overline{B} . \qquad (9.20)$$

The cross symbol must always be understood to involve the sine of the angle between the two vectors $\overline{B}$ and $\overline{l}$ (or the direction of I_2, which is determined by the orientation of $\overline{l}$).

Moreover, whenever the cross product is involved, the direction of the resultant vector is always normal to the plane containing the vectors $\overline{B}$ and $\overline{l}$ in the sense determined by the direction of advance of a right- hand screw as $\overline{l}$ is turned into $\overline{B}$ through the smaller of the two angles made by the vectors.

Accordingly, in the configuration of Fig. 9.5 the direction of the force is found by turning $I_2 \overline{l}$ into $\overline{B}$ and then noting that this would cause a right- hand screw to advance out of the plane of the paper. Hence the force is directed outward, and this corresponds with the experimentally established result.

It is also possible to determine the direction of the force by means of another right-hand rule, which requires that the forefinger be put in the direction of the current and the middle finger in the direction of $\overline{B}$ and the two assumed lying in the same plane.

The thumb of the right hand then points in the direction of the force when placed perpendicular to the other two fingers.

9.7. Theory of Magnetism

In order to understand the magnetic behaviour of materials, it is necessary to take a microscopic view of matter. A suitable starting point is the composition of the atom, which Bohr described as consisting of a heavy nucleus and a number of electrons moving around the nucleus in specific orbits. Closer investigation reveals that the atom of any substance experiences a torque when placed in a magnetic field; this is called a *magnetic moment*.

More precisely, the term magnetic moment normally refers to a system's magnetic dipole moment, which produces the first term in the multipole expansion of a general magnetic field. The dipole component of an object's magnetic field is symmetric about the direction of its magnetic dipole moment, and decreases as the inverse cube of the distance from the object. The magnetic moment of a magnet is a quantity that determines the force that the magnet can exert on electric currents and the torque that a magnetic field will exert on it.

The resultant magnetic moment of an atom depends upon three factors - the positive charge of the nucleus spinning on its axis, the negative charge of the electron spinning on its axis, and the effect of the electrons moving in their orbits.

The magnetic moment of the spin and orbital motions of the electron far exceeds that of the spinning proton. However, this magnetic moment can be affected by the presence of an adjacent atom.

Accordingly, if two hydrogen atoms are combined to form a hydrogen molecule, it is found that the electron spins, the proton spins, and the orbital motions of the electrons of each atom oppose each other so that a resultant magnetic moment of zero should be expected.

Although this is almost the case, experiment reveals that the relative permeability of hydrogen is not equal to one but rather is very slightly less than unity. In other words, the molecular reaction is such that when hydrogen is the medium there is a slight decrease in the magnetic field compared to free space.

This behaviour occurs because there is a precessional motion of all rotating charges about the field direction, and the effect of this precession is to set up a field opposed to the applied field regardless of the direction of spin or orbital motion.

Materials in which this behaviour manifests itself are called diamagnetic for obvious reasons. Besides hydrogen, other materials possessing this characteristic are silver and copper.

Continuing further with the hydrogen molecule, let us assume next that it is made to lose an electron, thus yielding the hydrogen ion. Clearly, complete neutralization of the spin and orbital electron motions no longer takes place.

In fact when a magnetic field is applied, the ion is so oriented that its net magnetic moment aligns itself with the field, thereby causing a slight increase in flux density.

This behaviour is described as paramagnetism and is characteristic of such materials as aluminium and platinum. Paramagnetic materials have a relative permeability slightly in excess of unity.

So far we have considered those elements whose magnetic properties differ only very slightly from those of free space. As a matter of fact, the vast majority of materials fall within this category. However, there is one class of materials - principally iron and its alloys with nickel, cobalt, and aluminium - for which the relative permeability is very many times greater than that of free space.

These materials are called ferromagnetic and are of great importance in electrical engineering. We may ask at this point why iron (and its alloys) is so very much more magnetic than other elements. Essentially, the answer is provided by the domain theory of magnetism. Like all metals, iron is crystalline in structure with the atoms arranged in a space lattice.

However, domains are subcrystalline particles of varying sizes and shapes containing about 10^{15} atoms in a volume of approximately 10^{-9} cm^3. *The distinguishing feature of the domain is that the magnetic moments of its constituent atoms are all aligned in the same direction*.

Thus in a ferromagnetic material not only must there exist a magnetic moment due to a non-neutralized spin of an electron in an inner orbit, but also the resultant spin of all neighbouring atoms in the domain must be parallel.

It would seem by the explanation so far that if iron is composed of completely magnetized domains then the iron should be in a state of complete magnetization throughout the body of material even without the application of a magnetizing force.

Actually, this is not the case, because the domains act independently of each other, and for a specimen of unmagnetized iron these domains are aligned haphazardly in all directions so that the net magnetic moment is zero over the specimen.

Figure 9.6 illustrates the situation diagrammatically in a simplified fashion. Because of the crystal lattice structure of iron the "easy" direction of domain alignment can take place in any one of six directions - left, right, up, down, out or in - depending upon the direction of the applied magnetizing force.

Figure 9.6, *a* shows the unmagnetized configuration.

Figure 9.6, *b* depicts the result of applying a force from left to right of such magnitude as to effect alignment of all the domains.

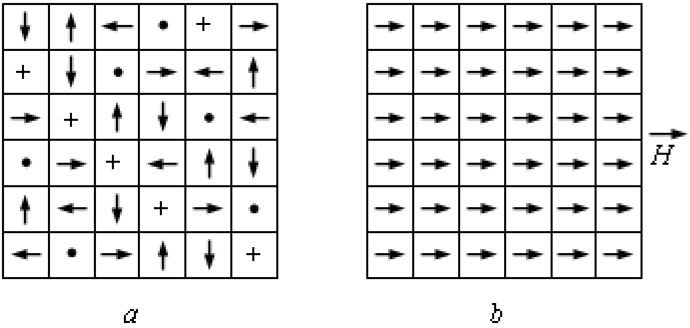


Fig. 9.6 Representation of a ferromagnetic crystal: *a* unmagnetized and *b* fully magnetized by the field *H*.

When this state is reached the iron is said to be saturated - there is no further increase in flux density over that of free space for further increases in magnetizing force.

Large increases in the temperature of a magnetized piece of iron bring about a decrease in its magnetizing capability. The temperature increase enforces the agitation existing between atoms until at a temperature of 750°C the agitation is so severe that it destroys the parallelism existing between the magnetic moments of the neighboring atoms of the domain and thereby causes it to lose its magnetic property. The temperature at which this occurs is called the *curie point*.

9.8. Magnetization Curves of Ferromagnetic Materials

If the experiment of Fig. 9.1 is repeated with iron or steel as the medium for increasing values of the field winding current I_1, and the corresponding values of μ computed, it is found that the relative permeability varies considerably with the magnetizing force that establishes the operating flux density. A typical variation of μ_r for cast steel appears in Fig. 9.7. Here μ_r is plotted versus flux density rather than the magnetizing force because it is the onset of the realignment of more and more domains that brings about the change in permeability.

Unfortunately, the state of development of the theory of magnetism is not so far advanced that it allows the prediction of the magnetic properties of a material on a purely theoretical basis even though the exact composition of the material is known.

For example, with the present theory it is not possible to say exactly what the flux density will be in a given specimen of iron for a specified value of the magnetizing force.

Rather, it is customary to obtain this information by consulting technical and descriptive bulletins where the measured magnetic properties of a representative sample of the specimen are published. These bulletins are made available to users by the manufacturers of magnetic steels and they include information on such varied shapes and forms as sheets, wires, bars, and even castings weighing up to hundreds of tons.

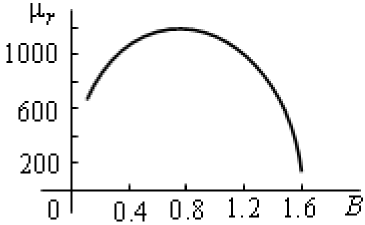


Fig. 9.7 Graph of relative permeability versus flux density.

Usually, the published magnetic characteristics of the various iron and steel samples are presented as plots of flux density B as a function of the magnetic field intensity H. In the interest of presenting a complete graphical picture of the functional relationship existing between these two quantities as well as to define additional terms used in this connection, refer to Fig. 9.8.

Assume that the steel specimen is initially unmagnetized and is in the form of a toroidal ring with a coil of *N* turns wrapped around it. Assume too that the coil can be energized from a variable voltage source capable of furnishing current flow in either direction in the coil.

As the current *I* is increased from zero in the positive direction (current flowing into the top terminal of the coil), an increasing magnetic flux *Φ* can be measured as taking place within the body of the toroid in the clockwise direction.

For any fixed value of I there is a specific value of flux. Then by Eq. (9.15) the corresponding flux density is determined since the toroidal cross-sectional area is known.

Moreover, the magnetomotive force *F* can be replaced by *Hl* in accordance with Ampere's circuital law [Eq. (9.13)] where *l* is the mean length of path of the toroid (as shown by the broken line circle in Fig. 9.8). The two fundamental quantities involved in this arrangement then are flux density *B* in teslas and magnetic field intensity *H* expressed in ampere-turns per meter.

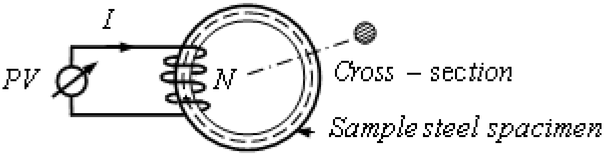


Fig. 9.8 Obtaining the magnetization curve of a sample steel specimen.

Magnetic field intensity *H* is the quantity we want to deal with rather than magnetomotive force because for the same flux density, doubling the mean magnetic length will not change *H* but will require doubling the magnetomotive force.

The conclusion to be drawn is that a plot of *B* versus *H* is a universal plot for the given material because it can be extended to any geometry of cross-sectional area and length. In contrast, a plot of *Φ* versus *F* is limited to a single geometrical configuration.

Therefore, a plot of the magnetic characteristics of a material always involves plotting *B* versus *H*. For the virgin sample of Fig. 9.8 the graph of *B* versus *H* follows the curve *0a* of Fig. 9.9 for field intensities up to H_a. Take note of the nonlinear relationship existing between these two quantities.

Another interesting characteristic of ferromagnetic materials is revealed when the field intensity, having been increased to some value, say H_a, is subsequently decreased.

It is found that the material opposes demagnetization and, accordingly, does not retrace along the magnetizing curve *0a* but rather along a curve located above *0a*. See curve *ab* in Fig. 9.9. Furthermore, it is seen that when the field intensity is returned to zero, the flux density is no longer zero as was the case with the virgin sample.

This happens because some of the domains remain oriented in the direction of the originally applied field. The value of *B* that remains after the field intensity *H* is removed is called residual flux density. Moreover, its value varies with the extent to which the material is magnetized. The maximum possible value of the residual flux density is called *retentivity* and results whenever values of *H* are used that cause complete saturation.

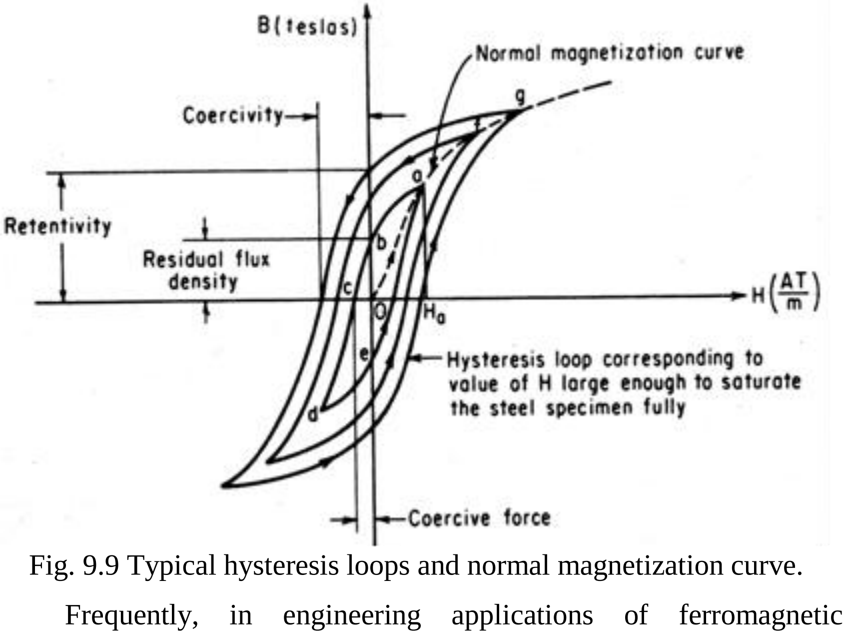


Fig. 9.9 Typical hysteresis loops and normal magnetization curve.

Frequently, in engineering applications of ferromagnetic materials, the steel is subjected to cyclically varying values of *H* having the same positive and negative limits. As *H* varies through many identical cycles, the graph of *B* versus *H* gradually approaches a fixed closed curve as depicted in Fig. 9.9. The loop is always traversed in the direction indicated by the arrows.

Since time is the implicit variable for these loops, note that B is always lagging behind H. Thus, when H is zero, B is finite and positive, as at point b, and when B is zero, as at c, H is finite and negative, and so forth.

This tendency of the flux density to lag behind the field intensity when the ferromagnetic material is in a symmetrically cyclically magnetized condition is called *hysteresis* and the closed curve *abcdea* is called a *hysteresis loop*.

Moreover, when the material is in this cyclic condition, the amount of magnetic field intensity required to reduce the residual flux density to zero is called the *coercive force*.

Usually, the larger the residual flux density, the larger must be the coercive force. The maximum value of the coercive force is called the *coercivity*.

A glance at the hysteresis loops of Fig. 9.9 makes it quite evident that the flux density corresponding to a particular field intensity is not single-valued. Its value lies between certain limits depending upon the previous history of the ferromagnetic material.

However, since in many situations involving magnetic devices this previous history is unknown, a compromise procedure is used in making magnetic calculations by working with a single-valued curve called the *normal magnetization curve*. This curve is found by drawing a curve through the tips of a group of hysteresis loops generated while in a cyclic condition. Such a curve is *0afg* in Fig. 9.9.

Typical normal magnetization curves of commonly used ferromagnetic materials appear in Fig. 9.10.

A final observation is in order at this point. By Eq. (9.10) the permeability of a material may be expressed as a ratio of B to H. Coupling this with the nonlinear variation existing between B and H (see Fig. 9.9), the variation of permeability with flux density as already cited in connection with Fig. 9.7 is verified.

As a matter of fact, for a material in a cyclic condition, the permeability is nothing more than the ratio of B to H for the various points along the hysteresis loop.

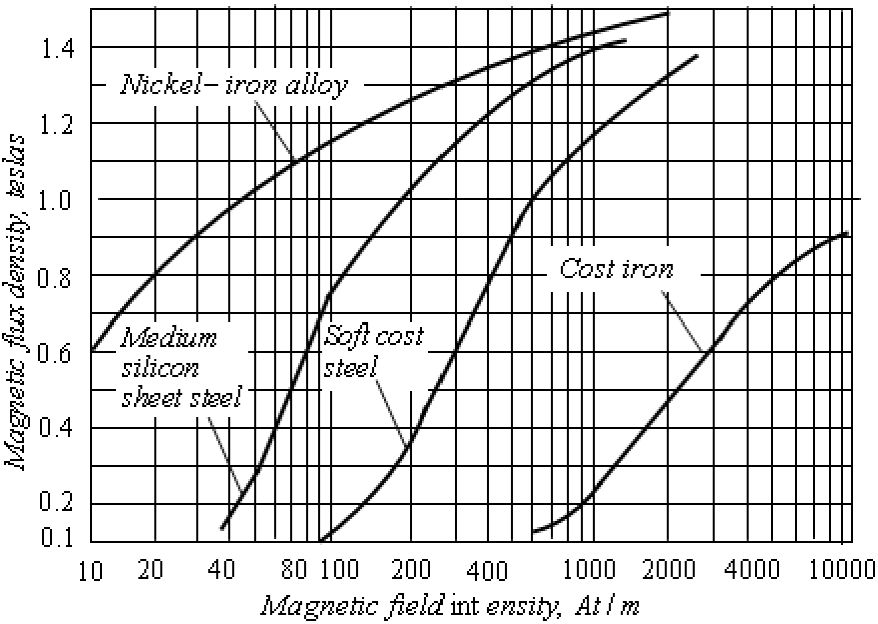


Fig. 9.10. Magnetization curves of typical ferromagnetic materials.

9.9. The Magnetic Circuit: Concepts and Analogies

The term magnetic circuits applies to a closed path of magnetic flux, the path having the direction of the magnetic induction at every point. The path may be made up of ferromagnetic materials or any other substances and media.

In general, problems involving magnetic devices are basically field problems because they are concerned with quantities such as Φ and B which occupy three-dimensional space. Fortunately, however, in most instances the bulk of the space of interest to the engineer is occupied by ferromagnetic materials except for small air gaps which are present either by intention or by necessity. For example, in electromechanical energy-conversion devices the magnetic flux must permeate a stationary as well as a rotating mass of ferromagnetic material, thus making an air gap indispensable. On the other hand, in other devices an air gap may be intentionally inserted in order to mask the nonlinear relationship existing between B and H. But in spite of the presence of air gaps it happens that the space occupied by the magnetic field and the space occupied by the ferromagnetic material are practically the same.

Usually, this is because air gaps are made as small as mechanical clearance between rotating and stationary members will allow and also because the iron by virtue of its high permeability confines the flux to itself as copper wire confines electric current or a pipe restricts water. On this basis the three-dimensional field problem becomes a one-dimensional circuit problem and in accordance with Eq. (9.17) leads to the idea of a magnetic circuit.

Thus we can look upon the magnetic circuit as consisting predominantly of iron paths of specified geometry which serves to confine the flux; air gaps may be included. Figure 9.11 shows a typical magnetic circuit consisting chiefly of iron.

Note that the magnetomotive force of the coil produces a flux which is confined to the iron and to that part of the air having effectively the same cross- sectional area as the iron. Furthermore, a little thought reveals that this magnetic circuit may be replaced by a single-line equivalent electric circuit as depicted in Fig. 9.12. As suggested by Eqs. (9.16) and (9.17) the equivalent circuit consists of the magnetomotive force driving flux through two series-connected reluctances - the reluctance of the iron, and, the reluctance of the air.

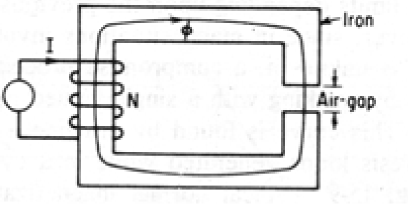


Fig. 9.11 Typical magnetic circuit involving iron and air.

This analogy of the magnetic circuit with the electric circuit carries through in many other respects. For the sake of completeness these details are presented below for the case of a toroidal copper ring and a toroidal iron ring having the same mean radius r and cross-sectional area S.

To distinguish the *electric field intensity* and *applied battery voltage* the last quantity is marked out with a capital letter ε (only in this section) in given table of correspondence between the electric circuit and the magnetic circuit on page 409.

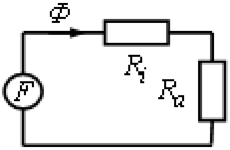
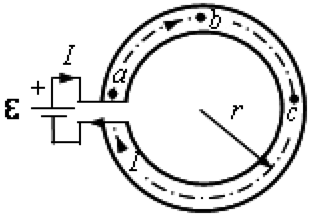
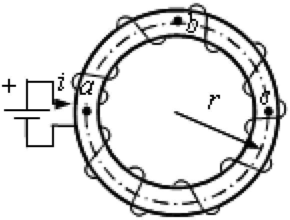


Fig. 9.12 Single-line equivalent circuit of Fig. 9.11.





ELECTRIC CASE

The toroidal copper ring is assumed open by an infinitesimal amount with the ends connected to a battery; a current of I amperes flows through the ring.

MAGNETIC CASE

The toroidal iron ring is assumed wound with *N* turns of wire so that with a current *i* flowing through it the magnetomotive force creates the flux *Φ*.

Driving Force

applied battery voltage = ε

applied ampere-turns = F

Response

$$current = \frac{driving\ \ force}{electric\ \ resistance}$$

$$flux = \frac{driving\ \ force}{magnetic\ \ reluctance}$$

$$or\quad I = \frac{\varepsilon}{R}$$

$$\Phi = \frac{F}{R_m}$$

Impedance

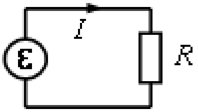
Impedance is a general term used to indicate the impediment to a driving force in establishing a response.

$$resistance = R = \rho\frac{l}{S}$$

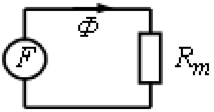
$$reluctance = R_m = \frac{l}{\mu S}$$

where $l = 2\pi r =$ mean length of turn of the toroid and *S* is the toroidal cross-sectional area.

Equivalent Circuit



$$E = I\,R$$



$$F = \Phi\,R_m$$

Electric Field Intensity

With the application of the voltage ε to the homogeneous copper toroid, there is produced within the material an electric potential gradient given by

$$E = \frac{\varepsilon}{l} = \frac{\varepsilon}{2\pi r}\ \text{V/m}$$

This electric field must occur in a closed path if it is to be maintained. It then follows that the closed line integral of *H* is equal to the battery voltage *E*. Thus

$$\oint E dl = \varepsilon$$

Voltage Drop

If it is desired to find the voltage drop occurring between two points - *a* and *b* - of the copper toroid, we may write:

$$U_{ab} = \int_a^b H dl = \frac{E}{l}\int_a^b dl = \frac{IR}{l}l_{ab} =$$

$$= \frac{I}{l}\rho\frac{l}{S}l_{ab} = I\rho\frac{l_{ab}}{S} = I\,R_{ab}\ ,$$

i.e., $U_{ab} = IR_{ab}$, where R_{ab} is the resistance of the copper toroid between points *a* and *b*.

Magnetic Field Intensity

When a magnetomotive force is applied to the homogeneous iron toroid, there is produced within the material a magnetic potential gradient given by

$$H = \frac{F}{l} = \frac{F}{2\pi r}\ \text{At/m}$$

As already pointed out in connection with Ampere's circuital law, the closed line integral of H equals the enclosed magnetomotive force. Thus

$$\oint H dl = F$$

m.m.f. = magnetic voltage

Magnetomotive force is related to magnetic voltage. The portion of the applied m.m.f. appearing between points *a* and *b* is found similarly:

$$F_{ab} = \int_a^b H dl = \frac{F}{l}l_{ab} = \frac{\hat{O}R_m}{l}l_{ab} =$$

$$= \frac{\hat{O}}{l}\frac{l}{\mu S}l_{ab} = \hat{O}\frac{l_{ab}}{\mu S} = \hat{O}\,R_{ab} = U_{Mab}$$

i.e., $F_{ab} = \Phi R_m$, where R_m is the reluctance of the iron toroid between points *a* and *b*.

Current Density

Current density is the amount of amperes per unit area. Thus

$$J = \frac{I}{S} = \frac{E}{SR} = \frac{Hl}{S\rho\left(l/S\right)} = \frac{H}{\rho}$$

or $H = \rho J$. This last expression is often referred to as the *microscopic* form of Ohm's law.

Flux Density

By definition, flux density is expressed as webers per unit area. Thus

$$B = \frac{\Phi}{S} = \frac{H}{SR_m} = \frac{Hl}{S\left(l/\mu S\right)} = \mu H$$

or $H = \dfrac{B}{\mu}$

It should not be inferred from the foregoing that electric and magnetic circuits are analogous in all respects.

For example, there are no magnetic insulators analogous to those known to exist for electric circuits.

Also, when a direct current is established and maintained in an electric circuit, energy must be continuously supplied. An analogous situation does not prevail in the magnetic case, where a flux is established and maintained constant.

9.10. Units for Magnetic Circuit Calculation

Magnetic circuit calculations can be carried out by use of any one of several different systems of units. These various systems arose initially because it was thought that the phenomena of electricity and magnetism were unrelated - thereby leading to the development of a separate system of units for each - and secondly, because of the desire to deal with practical values of the units once the relationship was discovered. Up to now attention has been given exclusively to the mks (meter-kilogram-second) system of units as developed by Giorgi about the turn of the twentieth century. This policy is prompted by the acceptance in 1960 of the mks system of units as the standard for scientific work and now referred to as SI units (System International Unite's). However, a good part of the past literature is written in terms of the units of the CGS (centimetre-gram-second) system. Furthermore, many of the present-day computations are carried on in terms of the mixed system employing such units as ampere-turns/inch, maxwells/inch^2, and ampere-turns because of the convenience they offer in dealing with dimensions that are expressed in inches.

Table 9.1. Magnetic Units

Quantity	Symbols	CGS	
		Unit	Relation
m.m.f.	F	gilberts	$F = 0.4\pi NI$
Permeability: Free space Abs. norm. permiability	μ_o		$\mu_o = 1$ $\mu = \mu_o \mu_r = \mu_r$
Length Area	l S	cm cm^2	
Reluctance	R_m		$R_m = \frac{l}{\mu_r S}$
Flux	Φ	maxwell = = lines	$\Phi = \frac{0.4\pi NI}{\frac{l}{\mu_r S}}$
Magnetic field intensity	H	gilberts/cm = =oersteds	$H = \frac{0.4\pi NI}{l}$
Flux density	B	lines/cm^2 = = gauss	$B = \mu_r H$

Furthermore, many of the present-day computations are carried on in terms of the mixed system employing such units as ampere-turns/inch, maxwells/inch^2, and ampere-turns because of the convenience they offer in dealing with dimensions that are expressed in inches. For these reasons the units of all three systems are shown in Tables 9.1, 9.2, 9.3.

The weber, which is the unit of flux in the SI system, is equal to 10^8 *maxwells* (or lines) where the maxwell is the unit of flux in the CGS system. The *gilbert* is the CGS unit for m.m.f. and is equal to 0.4π times the number of ampere-turns. The CGS unit for magnetic field intensity H is the *oersted* (or gilbert/cm) and the CGS unit for flux density B is the *gauss* (or lines/cm2). The relationships existing for the same quantity between the various systems of units are given below.

The conversion factors for tables:

Gradibert = 10 gilberts;

At = 0.4π gilberts = 1.257 gilberts;

1 metre = 39.4 inches;
1 metre^2 = 1550 inches$^{2;}$
1 centimetre = 0.01 metre;
1 weber = 10^8 lines;
1. oersted = 79.6 At/m;
1 praoersted = 1000 oersted;
$1\frac{At}{in} = \frac{0.4\pi}{2.54} = 0.495$ oersted;
$1\frac{At}{in} = \frac{2.02}{1000}$ praoersted;
1 gauss = 6.45 lines/in^2;
1 tesla = 64,500 lines/in^2;
1 tesla = 10,000 gauss.

Table 9.2. Magnetic Units

Quantity	Symbols	SI units		
		Unit	Symbol	Relation
m.m.f.	F	ampere = hour	At	$F = NI$
Permeability: Free space Abs. norm. permeability	 μ_o μ			 $\mu_0 = 4\pi 10^{-7}$ $\mu = 4\pi 10^{-7}\mu_r$
Length Area	l S	metre; metre^2	m m^2	
Reluctance	R_m			$R_m = \dfrac{l}{4\pi 10^{-7}\mu_r S}$
Flux	Φ	Weber	Wb	$\Phi = \dfrac{NI}{\dfrac{l}{\mu_r 4\pi 10^{-7} S}}$
Magnetic field intensity	H	$\dfrac{At}{metre}$	$\dfrac{At}{m}$	$H = \dfrac{NI}{l}$
Flux density	B	$\dfrac{webers}{metre^2} = \\ = tesla$	T	$B = \mu_r \mu_o H$

Table 9.3. Magnetic Units

Quantity	Symbols	Mixed English system	
		Unit	Relation
m.m.f.	F	At	$F = NI$
Permeability: Free space Abs. norm. perm.	μ_o μ		$\mu_o = 3.19$ $\mu = 3.19\mu_r$
Length Area	l S	inch inch^2	
Reluctance	R_m		$R_m = \dfrac{l}{3.19\mu_r S}$
Flux	Φ	Maxwell = = lines	$\Phi = \dfrac{NI}{\dfrac{l}{3.19\mu_r S}}$
Magnetic field intensity	H	$\dfrac{At}{inch}$	$H = \dfrac{NI}{l}$
Flux density	B	$lines/inch^2$	$B = 3.19\mu_r H$

9.11. Magnetic Circuit Computations

Basically magnetic circuit calculations involving ferromagnetic materials fall into two categories. In the first the value of the flux is known and it is required to find the magnetomotive force to produce it. This is the situation typical of the design of a-c and d-c electromechanical energy converters.

On the basis of the desired voltage rating of an electric generator or the torque rating of an electric motor information about the required magnetic flux is readily obtained. Then with this knowledge and the configuration of the magnetic circuit the total m.m.f. needed to establish the flux is determined straightforwardly.

In the second case it is the flux for which we must solve, knowing the geometry of the magnetic circuit and the applied m.m.f. An engineering application in which this situation prevails is the magnetic amplifier, where it is often necessary to find the resultant magnetic flux caused by one or more control windings.

Because the reluctance (or permeability) of the ferromagnetic material is not constant, the solution of this problem is considerably more involved than that of the first category as illustrated by the examples below.

Example 9.1. A toroid is composed of three ferromagnetic materials and is equipped with a coil having 100 turns as depicted in Fig. 9.13. Material *a* is a nickel-iron alloy having a mean arc length l_a of 0.3 m. Material *b* is medium silicon steel and has a mean arc length l_b of 0.2 m. Material *c* is of cast steel having a mean arc length equal to 0.1 m. Each material has a cross-sectional area of 0.001 m^2.

(a) Find the magnetomotive force needed to establish a magnetic flux of $\Phi = 6 \times 10^{-4}$ Wb = 60,000 lines.

(b) What current must be made to flow through the coil?

(c) Compute the relative permeability and reluctance of each ferromagnetic material.

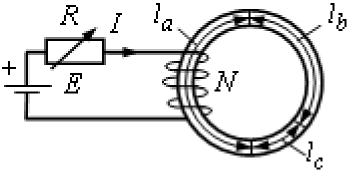


Fig. 9.13. Toroid composed of three different materials.

Solution: (a) To obtain the total m.m.f. of the coil all we need to do is to apply Ampere's circuital law. Thus

$$F = F_a + F_b + F_c = H_a l_a + H_b l_b + H_c l_c .$$

The unknown quantities here are H_a, H_b, H_c. These can readily be found from a knowledge of the flux density, which here is the same for each section because the flux is common and the cross-sectional areas are the same. Hence

$$B_a = B_b = B_c = \frac{\Phi}{S} = \frac{0{,}0006}{0.001} = 0.6\,\text{T}.$$

Now H_a is found by entering the *B*-*H* curve of the nickel-iron alloy of Fig. 9.10 corresponding to $B_a = 0.6$. This yields $H_a = 10 \; At/m$.

Similarly,

$$H_a = 10 \; At/m$$

Accordingly, the total required m.m.f. is

$$F = H_a l_a + H_b l_b + H_c l_c = 10 \times 0.3 + 77 \times 0.2 + 270 \times 0.1 =$$
$$= 3 + 15.4 + 27 = 45.4\ At.$$

Note that although the path length of cast steel is the smallest, it nonetheless requires the greatest portion of the m.m.f. to force the specified flux through. This happens because of its much lower permeability as shown in part (c).

(b) In the SI system the m.m.f. is equal to the number of ampere-turns. Hence

$$I = \frac{F}{N} = \frac{45.4}{100} = 0.454\,\text{A}.$$

From Eq. (9.10)

$$\mu_a = \frac{H_a}{B_a} = \frac{0.6}{10} = 0.06\ \text{H/m}.$$

Also,

$$\mu_{ra}\mu_0 = \mu_a$$
$$\mu_{ra} = \frac{\mu_a}{\mu_0} = \frac{0.06}{4\pi \times 10^{-7}} = 47.7$$

Furthermore, from Eq. (9.16) the reluctance is found to be

$$R_a = \frac{F_a}{\Phi} = \frac{3}{6 \times 10^{-4}} = 5000 \text{ rationalized mks units of reluctance.}$$

Proceeding in a similar fashion for materials *b* and *c* leads to the following results:

$$\mu_{rb} = 6207 \qquad R_b = 25{,}667$$
$$\mu_{rc} = 1768 \qquad R_c = 45{,}000$$

Next we consider the more difficult problem: that of finding the flux in a given magnetic circuit corresponding to a specified m.m.f. The solution cannot be arrived at directly because, as a result of the nonlinear relationship between *B* and *H*, there are too many unknowns. The easiest way of finding the solution is to employ a cut-and-try procedure guided by the knowledge of the permeability characteristics of the materials such as appears in their magnetization curves. The following example illustrates the technique involved.

Example 9.2 For the toroid of Example 9.1, shown in Fig. 9.13, find the magnetic flux produced by an applied magnetomotive force of $F = 35\ At$.

Solution: The solution cannot be determined directly because to do so we must know the reluctance of each part of the magnetic circuit, which can be known only if the flux density is known - which means that Φ must be known right at the start. This is clearly impossible.

To obtain the solution by the cut-and-try procedure, we begin by first assuming that all of the applied m.m.f. appears across the material having the highest reluctance. This yields an approximate value of Φ which can subsequently be refined.

A glance at Fig. 9.10 shows that the poorest magnetic "conductor" is cast steel. Hence by assuming the entire m.m.f. to appear across material c we can find H, from which B follows, which in turn yields.

Thus

$$H_c = \frac{F_c}{l_c} = \frac{F}{l_c} = \frac{35}{0.1} = 350\ At/m.$$

From Fig. 9.10

$$B_c = 0.65\ T$$

$$\Phi_1 = B_c S_c = 0.65 \times 0.001 = 0.00065\ Wb$$

This value represents the first approximation for the flux as indicated by the subscript. Also, since the cross-sectional area is the same for each material, it follows that

$$B_a = B_b = B_c = 0.65\ T.$$

Reference to the nickel-iron magnetization curve reveals that the value of H_a corresponding to B_a is negligibly small compared to H_c. Hence for all practical purposes its effect can be neglected.

However, note that for medium silicon steel the value of H_b is almost 90 At/meter. This, coupled with the fact that $l_b = 2l_c$, indicates that material b takes about half as much m.m.f. as material c in maintaining the flow of flux.

In other words, at this point in our analysis we can make a refinement on our original assumption of assigning the entire m.m.f. to material *c*. Now we see that about 50% of that assigned to *c* should be assigned to *b*. Thus

$$F_c + F_b = F \quad \text{(assumed)}.$$

But

$$F_b = 0.5F_c \quad \text{(assumed)}.$$

Hence

$$1.5F_c = F = 35, \quad F_c = 23.3\,\text{At}.$$

Accordingly, a second approximation for the solution can be obtained. Therefore,

$$H_c = \frac{23.3}{0.1} = 233 \text{ At/m}$$

which in turn yields

$$B_c = 0.4 \text{ T}$$

so that the value of the flux now becomes

$$\Phi_2 = B_c S_c = 0.0004\,\text{Wb}.$$

To determine whether or not this is the correct answer we must at this point compute the m.m.f. drops for each material and add to see whether they yield a value equal to the applied m.m.f.

If not, the foregoing procedure must be repeated until Ampere s circuital law is satisfied. Making this check for the second approximation we have

$H_b = 62\ At/m$ corresponding to $B_a = 0.4$.

And

$$H_a = 5.7\ At/m \qquad for\ B_a = 0.4.$$

Accordingly,

$$Hl = H_a l_a + H_b l_b + H_c l_c = 5.7 \times 0.3 + 62 \times 0.2 + 233 \times 0.1 =$$
$$= 1.7 + 12.4 + 23.3 = 37.4\ At.$$

Obviously this is too high by about 7%. Hence, as a third try, reduce the biggest contributor to the m.m.f. by a factor of 5%.

That is, assume that now

$$F_c = 22 \text{ At.}$$

Then

$$H_c = 220 \text{ At/m} \quad \text{and} \quad B_c = 0.375,$$

$$\Phi_3 = 0.000375 \text{ Wb.}$$

Corresponding to this flux we find

$$H_b = 59 \quad \text{and} \quad H_a = 5.$$

Hence

$$\text{m.m.f.} = (5)\,(0.3) + (59)(0.2) + 22 = 35.3 \text{ At}$$

Since this summation of m.m.f.'s agrees with the applied m.m.f. of 35 At, the correct solution for the flux is

$$\Phi = 0{,}000375 \text{ Wb} = 37{,}500 \text{ lines}$$

When making magnetic circuit computations of the kind just illustrated, it is common practice to accept as valid any solution that comes within ± 5% of the exact solution. The reason is that we are dealing with normal magnetization curves which neglect hysteresis and which are after all only *typical* of the material actually being used in the specified circuit. Deviations can and often do exist.

As a final example to illustrate magnetic circuit computations, we shall solve a problem involving parallel magnetic paths as well as the presence of an air gap. Moreover, we will consider only the first type where the m.m.f. needed to establish a specified flux is to be found. This is justified not only because the solution is straightforward but also because it is by far more representative of the kind of magnetic circuit problem the engineer is likely to be concerned with.

Example 9.3 A magnetic circuit having the configuration and dimensions shown in Fig. 9.14 is made of cast steel having a thickness of 0.05 m and an air gap of 0.002 m length appearing between points g and h. The problem is to find the m.m.f. to be produced by the coil in order to establish an air gap flux of 4×10^{-4} Wb (or 40,000 lines).

Solution: The method of solution can be readily ascertained by referring to the equivalent circuit of this magnetic circuit as shown in Fig. 9.15. Knowledge of Φ_g enables us to find the m.m.f. drop appearing across b and c.

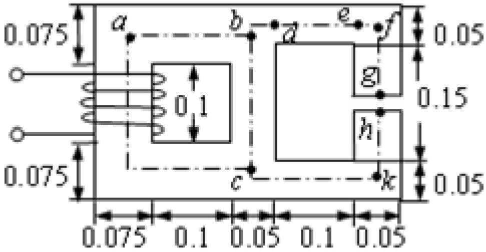


Fig. 9.14. Parallel magnetic circuit. All dimensions are in meters.

From this information the flux in leg *bc* can be determined and, upon adding it to we find the flux in leg *cab*. In turn, the m.m.f. needed to maintain the total flux in leg *cab* can be computed, and when we add it to the m.m.f. drop across *bc* we obtain the resultant magnetomotive force.

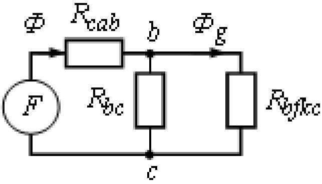


Fig. 9.15. Equivalent circuit of Fig. 9.14.

The computations involved for the various parts of the magnetic circuit are as follows.

Part gh. This is the air gap for which the flux is specified as 4×10^{-4} Wb. The cross-sectional area of the gap is (0.05)(0.05) = 0.0025 m$_2$.

Normally, however, this area is slightly higher because of the tendency of the flux to bulge outward along the edges of the air gap - which is often referred to as fringing. For convenience this effect is neglected.

Thus, the air gap flux density is found to be

$$B_g = \frac{\Phi_g}{S_g} = \frac{4\times10^{-4}}{0.0025} = 0.16\,\mathrm{T}.$$

Since the permeability of air is practically the same as that of free space, we have

$$H_g = \frac{B_g}{\mu_o} = \frac{0.16}{4\pi 10^{-7}} = 127{,}300 \text{ At/m}$$

and

$$F_g = H_g l_g = 127{,}300 \times 0.002 = 255 \text{ At}$$

Part bg and *hc*. The length of ferromagnetic material involved here is

$$l_{bg} + l_{hc} = 2\left(l_{bd} + l_{de} + l_{ef} + \frac{l_{fk}}{2} \right) - l_g =$$

$$= 2\big(0.025 + 0.1 + 0.025 + 0.1) - 0.002 = 0.498\big)\, m$$

Moreover, corresponding to a flux density in the cast steel of 0.16 T, the field intensity *H* is found to be 125 At/m. Hence

$$F_{bg+hc} = 125 \times 0.498 = 62.2 \text{ At.}$$

Part bc. Because path *bfkc* is in parallel with path *bc*, the total m.m.f. across path *bfkc* also appears across path *bc*. Hence

$$F_{bc} = 255 + 62.2 = 317.2 \text{ At.}$$

Also,

$$l_{bc} = 0.1 + 0.075 = 0.175\ m$$

$$H_{bc} = \frac{317.5}{0.175} = 1810\ At/m$$

and from Fig. 9.10 for cast steel the corresponding flux density is found to be

$$B_{bc} = 1.38 \text{ T.}$$

Hence

$$\Phi_{bc} = 1.38 \times 0.0025 = 0.00345 \text{ Wb.}$$

Part cab. Accordingly, the total flux existing in leg *cab* is

$$\Phi_{cab} = \Phi_{bc} + \Phi_g = 0.00345 + 0.0004 = 0.00385 \text{ Wb.}$$

Knowledge of this flux then leads to determination of the m.m.f. needed in; leg *cab* to sustain it. Thus

$$B_{cab} = \frac{0.00385}{0.00375} = 1.026\,\mathrm{T}$$

from which

$$H_{cab} = 690\ \mathrm{At/m}$$

Hence

$$F_{cab} = H_{cab} l_{cab} = 690 \times 0.5 = 345\ \mathrm{At}$$

Therefore the total m.m.f. required to produce the desired air gap flux is

$$F = F_{cab} + F_{bc} = 345 + 317.2 = 662.2\ \mathrm{At}$$

9.12. Hysteresis and Eddy-Current Losses in Ferromagnetic Materials

The process of magnetization and demagnetization of a ferromagnetic material in a symmetrical cyclic condition involves a storage and release of energy which is not completely reversible.

As the material is magnetized during each half-cycle, it is found that the amount of energy stored in the magnetic field exceeds that which is released upon demagnetization.

The background for understanding this behaviour was provided in Sec. 9.3. There the hysteresis loop was identified as the variation of flux density as a function of the magnetic field intensity for a ferromagnetic material in a cyclic condition. The salient feature of the hysteresis loop is the delayed reorientation of the domains in response to a cyclically varying magnetizing force.

A single hysteresis loop is depicted in Fig. 9.16. The direction of the arrows on this curve indicates the manner in which *B* changes as *H* varies from zero to a positive maximum through zero to a negative maximum and back to zero again, thus completing the loop.

To appreciate the meaning of the various shaded areas shown in Fig. 9.16, let us look at the units associated with the product of *B* and *H*. Thus

$$\text{units of } (HB) = \frac{amperes}{metre} \times \frac{newtons}{ampere - metre} = \frac{newtons}{metre^2} = \frac{N}{m^2}\,.$$

But

$$\text{Newton-metre} = \text{joule}$$

Hence

$$\text{Units of } (NB) = \frac{joule}{metre^3} = \frac{J}{m^3}$$

which is clearly recognized as an energy density. Therefore, in dealing with areas involving B and H in connection with a hysteresis loop we are really dealing with energy densities expressed on a per cycle basis because the hysteresis loop is repeatable for each cyclic variation of H.

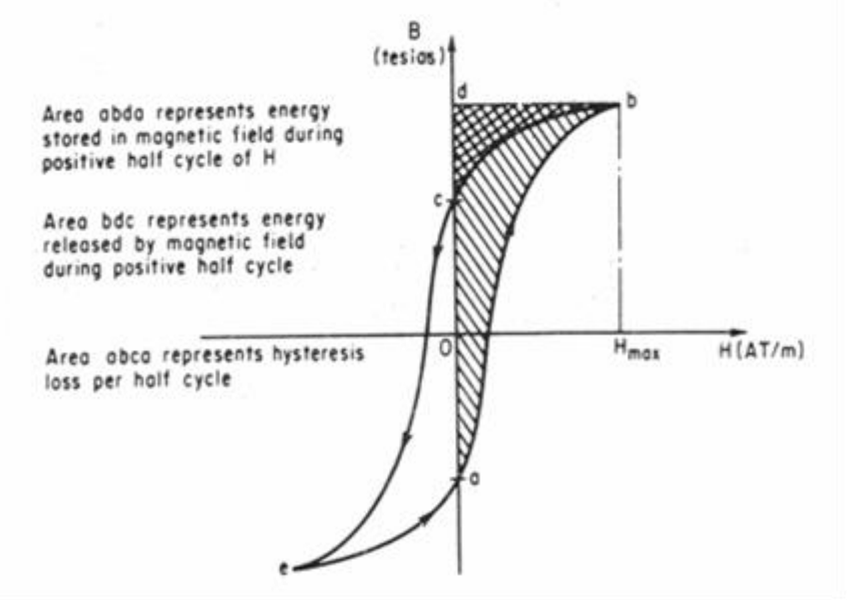


Fig.9.16. Hysteresis loop and energy relationship per half-cycle.

The energy stored in the magnetic field during that portion of the cyclic variation of H when it increases from zero to its positive maximum value (assuming the material is already in a cyclic state) is given by

$$W_1 = \int_{B_a}^{B_b} HdB \ \text{J/m}^3 \qquad (9.21)$$

when mks units are used; i.e., H must be expressed in At/m and B in teslas. Note that the axes of Fig. 9.16 are so labeled. Moreover, during that portion of its cyclic variation when H decreases from its positive maximum value to zero (as it follows along curve be of the hysteresis loop) energy is being released by the magnetic field and returned to the source, and this quantity can be represented as

$$W_2 = \int_{B_b}^{B_c} HdB \text{ J/m}^3 \tag{9.22}$$

In this equation, since $B_b > B_c$, the quantity W_2 will be negative, indicating that the energy is being released rather than stored by the magnetic field.

A graphical interpretation of Eq. (9.21) leads to the result that the energy absorbed by the field, when H is increasing in the positive direction, can be represented by the area *abdca*. Similarly, the energy released by the field as H varies from $H_{\max}$ to zero can be represented by area *bdcb*. The difference between these two energy densities represents the amount of energy which is not returned to the source but rather is dissipated as heat as the domains are realigned in response to the changing magnetic field intensity. This dissipation of energy is called *hysteresis loss*. Keep in mind that Fig. 9.16 depicts this energy density loss for a one-half cycle variation of H. Hence area *abca* represents the hysteresis loss per half-cycle. It certainly follows from symmetry that upon completion of the negative half-cycle variation of H an equal energy loss occurs. Therefore, as H varies over the complete cycle, the total energy loss per cubic meter is represented by the area of the hysteresis loop. More specifically, this energy loss per cycle can be expressed mathematically as

$$W_h = \text{(area of hysteresis loop)} \; \frac{J}{m^3 \times cycle} \tag{9.23}$$

where rationalized mks units are used for H and B.

It is frequently desirable to express the hysteresis loss of ferromagnetic materials in watts - the unit of power. A little thought about the units of W_h in Eq. (9.23) shows how this can be directly accomplished. Thus

$$W_h = \frac{energy}{vol \times cycles} = \frac{power \times \sec onds}{vol \times cycles} = \frac{power}{vol \times cycles\ / \sec onds} \qquad (9.24)$$

Now let $P_h =$ power loss in watts

V = volume of ferromagnetic material

f = cycles/seconds = frequency of variation of H

Then Eq. (9.24) becomes

$$W_h = \frac{P_h}{Vf} \qquad (9.25)$$

or

$$P_h = W_h V f \qquad (9.26)$$

where W_h - the energy density loss - is determined from Eq. (9.23).

To obviate the need of finding the area of the hysteresis loop in order to compute the hysteresis loss in watts from Eq. (9.26) Steinmetz obtained an empirical formula for W_h based on a large number of measurements for various ferromagnetic materials. He expressed the hysteresis power loss as

$$P_h = VfK_h B_m^n \qquad (9.27)$$

where B_m is the maximum value of the flux density and n lies in the range $1.5 \le n \le 2.5$ depending upon the material used. The parameter K_h also depends upon the material. Some typical values are: cast steel 0.025, silicon sheet steel 0.001, and permalloy 0.0001.

In addition to the hysteresis, another important loss occurs in ferromagnetic materials that are subjected to time-varying magnetic fluxes - the eddy-current loss. This term is used to describe the power loss associated with the circulating currents that are found to exist in closed paths within the body of a ferromagnetic material, causing an undesirable heat loss. These circulating currents are created by the differences in potential existing throughout the body of the material owing to the action of the changing flux. If the magnetic circuit is composed of solid iron, the ensuring power loss is appreciable because the circulating currents encounter relatively little resistance.

To increase significantly the resistance encountered by these eddy currents, the magnetic circuit is invariably composed of very thin *laminations* (usually 14 to 25 mils thick) whenever the electromagnetic device is such that a varying flux permeates it in normal operation. This is the case with transformers and all ac electric motors and generators. An empirical equation for the eddy-current loss is

$$P_e = K_e f^2 B_m^2 \tau^2 V \text{ W} \tag{9.28}$$

where K_e = a constant dependent upon the material
f = frequency of variation of flux in Hz
B_m = maximum flux density
τ = lamination thickness
V = total volume of the material

A comparison of this equation with Eq. (9.27) reveals that eddy-current losses vary as the square of the frequency, whereas the hysteresis loss varies directly with the frequency.

Taken together the hysteresis and eddy-current losses constitute what is frequently called the *core losses* of electromagnetic devices that involve time-varying fluxes for their operation. More than just passing attention is devoted to these losses here because, as will be seen, core losses have an important bearing on temperature rise, efficiency and rating of electromagnetic devices.

9.13. Relays - an Application of Magnetic force

A *relay* is an electromagnetic device which can often activated by relatively little energy, causing a movable ferromagnetic apmature to open or close one or several pairs of electrical contact points located in another control circuit or in a main circuit handling large amounts of energy. Ac and dc motor starters are equipped with relays designed to insure proper operation of motors during starting and running conditions. These devices are found in many applications in all fields of engineering, especially in situations where control of a process or machine is involved. Our objective in this section is to describe the principles that underlie the operation of these electromagnetic devices. Besides the knowledge it offers, this treatment gives the motivation for studying the theory of magnetic fields and circuits.

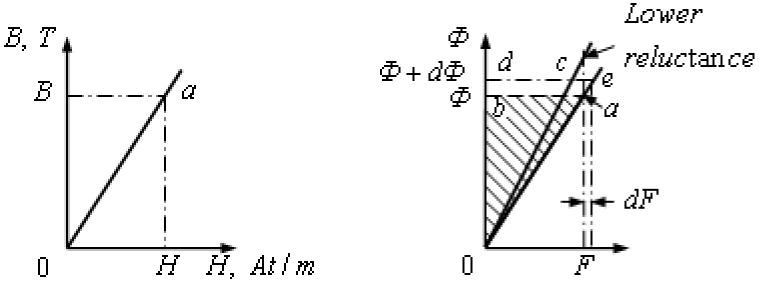
The derivation of a magnetic force equation is our primary interest because is shows how it is possible to do mechanical work - moving a relay armature - by abstracting energy from that stored in the magnetic field. Moreover, since the emphasis is on the principles involved, the simplifying assumptions of no saturation or no losses are imposed; i.e., linear analysis is used throughout. Accordingly, the magnetization curve of the ferromagnetic material material is assumed to be a straight line as depicted in Fig. 9.17, *a*. (Only the curve corresponding to positive values of *H* is shown). Now, by appropriately modifying the axes of the curve plotted in Fig. 9.17, *a*, a significant and useful thing happens. First recall that the area between the *B*-axis and the magnetization curve represents the energy absorbed from the source and stored in the magnetic field on a unit volume basis. Equation (9.21) states this result mathematically. Repeating we have

$$W_f = \int_{B_a}^{B_b} H dB \text{ J/m}^3 \tag{9.29}$$

In the simplified situation of Fig. 9.17, *a*, Eq. (9.29) becomes merely the area of triangle *0aB*. Thus

$$W_f = \frac{1}{2} BH \text{ J/m}^3 \tag{9.30}$$

where *H* is assumed fixed at the value shown.



a Area 0*aB* gives the energy density corresponding to fixed *H*

b Area 0*ab* gives the total stored magnetic field energy corresponding to fixed *F*

Fig. 9.17. Linear magnetization curve of a relay.

Moreover, since W_f is an energy density, the total energy stored in the magnetic field is found by multiplying Eq. (9.21) by the volume. Thus

$$W = W_f V = W_f S l \text{ J} \tag{9.31}$$

where l is the length and S the cross sectional area of the magnetic circuit. Inserting Eq. (9.30) into Eq. (9.31) yields

$$W = \frac{1}{2}(BS)(Hl) = \frac{1}{2}\Phi F \text{ J} \tag{9.32}$$

Note that in this equation S is combined with B to identify the flux Φ and H is combined with l to identify the magnetomotive force F. A graphical representation of Eq. (9.32) appears in Fig. 9.17, *b*. It should be apparent that Fig. 9.17, *b* derives from Fig. 9.17, *a* by multiplying the ordinate axis by S and the abscissa axis by l, thus plotting Φ versus F. Then for a fixed H (or F) area *0aB* of Fig. 9.17, *a* gives the energy density whereas the corresponding area *0ab* of Fig. 9.17, *b* gives the total energy stored in the magnetic field.

To understand how mechanical work can be done by the abstraction of energy stored in the magnetic field, consider the circuitry appearing in Fig. 9.18, which depicts the basic composition of an electromagnetic relay. It consists of an exciting coil placed on a fixed ferromagnetic core equipped with a movable element called the *relay armature*. The relay is energized from a constant voltage source through an adjustable resistor R.

To begin with, consider that R is fixed at that value which makes the coil m.m.f. equal to F and producing the flux Φ as shown in Fig. 9.17, *b*. Then adjust R to increase the m.m.f. by dF and thereby the flux by $d\Phi$, assuming the relay armature is held fast to keep the reluctance invariant. Figure 9.17, *b* shows that an additional amount of energy is absorbed from the source and stored in the magnetic field virtually equal to the area of rectangle *abdc*. Next readjust R to make dF zero, and then, keeping the m.m.f. fixed at F, release the armature, thus allowing it to move in the direction to decrease the air gap. Hold the armature fast again when the decreased reluctance causes the flux to increase by the amount $d\Phi$ obtained previously.

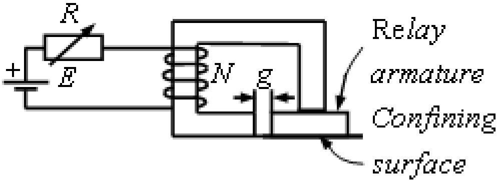


Fig. 9.18. Basic composition of an electromagnetic relay.

It is important to note here that, neglecting second-order effects, the new (lower-reluctance) position of the armature is also responsible for permitting the source to supply an additional amount of energy virtually equal to area *abdc* as before but with one significant difference. Whereas when the armature is held fixed, the additional energy supplied by the source is converted entirely to stored magnetic energy, on the other hand, when the armature is allowed to move towards a lower-reluctance position, only half of the same amount of additionally supplied energy is stored in the magnetic field.

The other half is consumed in doing the mechanical work involved in moving the relay armature from the higher- to the lower-reluctance position. That the amount is one-half, readily follows from a glance at Fig. 9.17, *b* if we note that the area of triangle *0ac* is one - half the area of rectangle *abdc*. The slope of the magnetization curve is the permeability and so can be used as a measure of the reluctance. Hence the higher the slope of the magnetization curve, the higher μ and the lower the reluctance.

For the magnetization curve in Fig. 9.17, *b* to go from position *0a* to *0c,* it is necessary for the relay armature to be moved to a position corresponding to a smaller air gap. The mechanical work involved in accomplishing this is represented by area *0ac* in Fig. 9.17, *b*.

The conclusions described in the foregoing can now be expressed mathematically. It is important to keep in mind, however, that the interchange of energy between the magnetic field and the mechanical system (the relay armature) necessarily involves a change in reluctance. In other words the change in energy to do mechanical work, dW_m, is equal to the change in magnetic field energy associated with a change in reluctance *dR*l. Thus by Eqs. (9.18) and (9.31)

$$dW_m = -\frac{1}{2}\Phi^2 dR \tag{9.33}$$

where the negative sign emphasizes that mechanical work is done through a decrease in reluctance.

An expression for the magnetic force developed on the relay armature is readily obtained by recalling that

$$Fdx = dW_m \tag{9.34}$$

where F is the force in newtons. Inserting this expression into Eq. (9.33) then yields

$$F = -\frac{1}{2}\Phi^2 \frac{dR}{dx}. \tag{9.35}$$

Hence the magnitude of the instantaneous magnetic force is dependent upon the value of the flux as well as the rate of change of reluctance. Moreover, the direction of this force is always such as to bring about a decrease in reluctance as indicated by the minus sign.

Example 9.4 In the relay circuit of Fig. 9.18 assume the cross-sectional area of the fixed core and the relay armature to be S, and the air gap flux to be Φ. Neglecting the reluctance of the iron, find the expression for the magnetic force existing on the relay armature.

Solution: We must find the rate of change of reluctance with distance along the sliding surface. Thus

$$R = \frac{g}{\mu_o S} \tag{9.36}$$

where g = air gap length.

$$\frac{dR}{dx} = \frac{1}{\mu_o S}\frac{dg}{dx}.$$

But

$$dg = -dx$$

Hence

$$\frac{dR}{dx} = -\frac{1}{\mu_o S}$$

Inserting this last expression into Eq. (9.35) yields the desired result.

Summary review questions

1. State Ampere's Law and comment about the motion of lines of force.

2. Describe how the magnetic quantity - flux density - is defined from Ampere's Law. Demonstrate why this quantity is aptly named.

3. Define permeability and show how this quantity can be experimentally determined for a particular medium. What is relative permeability?

4. Explain the meaning of magnetic flux and show how it is related to magnetic flux density.

5. What is magnetic field intensity? How is it different from magnetic flux density?

6. Describe Ampere's circuital law and illustrate its usefulness in magnetic circuit computations.

7. What is magnetomotive force? How does it differ from electromotive force? How is it similar?

8. How does the notion of reluctance arise in dealing with magnetic circuits? Why is this property useful? Name the physical parameters that influence this quantity.

9. Ampere's law deals with the force that exist between two current-carrying conductors. What influence, if any, does the orientation of one conductor relative to the other have on this force? Illustrate.

10. Write Ampere's law expressed in terms of the notation of vector analysis.

11. Describe how the direction of the force between two currents-carrying conductors is determined.

12. Explain the following terms: diamagnetic, paramagnetic, ferromagnetic, magnetic moment.

13. Describe briefly the domain theory of magnetism.

14. What is saturation as it applies to ferromagnetism?

15. Magnetic circuits are basically nonlinear. Explain what this statement means and why it is so.

16. Define the following: hysteresis, retentivity, coercivity, residual flux density, coercive force, normal magnetization curve.

17. Describe the premise on the basis of which it is possible to represent the three-dimensional field problems of magnetism by a magnetic circuit.

18. Describe the analogies that can be made between electric and magnetic circuits regarding the following items: driving force, field intensity, impedance drops and equivalent circuits.

19. Describe in some detail the two types of magnetic circuit problems that confront the designer of such circuits.

20. Demonstrate through dimensional analysis why the hysteresis loop represents an energy loss per cycle. What can be done to diminish this loss?

21. Cite an empirical formula for the evaluation of hysteresis loss expressed in watts. Repeat for eddy-current loss.

22. Describe how mechanical work can be done through the extraction of energy that is stored in a magnetic field.

23. Cite the formula for the force produced in a relay and explain the meaning of each term.

Problems

9.1. A long straight wire located in air carries a current of 4 A. Assume the relative permeability of air is unity.

(a) Find the value of the magnetic field intensity at a distance 0.5 m from the centre of the wire.

(b) A second long straight wire carrying a current of 2 A is placed parallel to the first one at a distance of 0.5 m with the current flowing in the same direction. Find the direction and magnitude of the force per meter existing between the wires.

(c) Repeat part (b) for the case where the wires are imbedded in iron having a relative permeability of 10,000 and a spacing of 0.05 m.

9.2. The wires shown in Fig. P9.2 are long, straight, and parallel and are completely embedded in iron having a relative permeability of 1000. Each wire carries a current of 10 A.

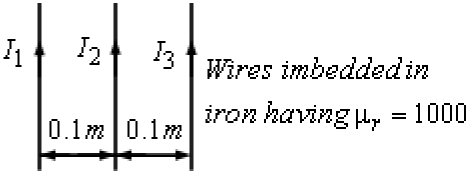


Fig. P9.2

(a) Compute the magnitude and direction of the resultant force per meter on the wire in which I_2 flows.

(b) Compute the magnitude and direction of the force per meter on the wire in which I_1 flows.

(c) Repeat part (b) for the third wire.

9.3. Repeat Problem 9.2 for the case where the current I_2 flows opposite to I_1 and I_3.

9.4. A uniform magnetic field of 0.7 T in the iron is applied to the configuration of Fig. P9.2.

(a) Compute the magnitude and direction of the resultant force per meter when the magnetic field is directed perpendicularly into the plane of the paper.

(b) What is the value of this resultant force per meter when the magnetic field is applied in the plane of the wires and directed from right to left?

(c) Compute the resultant force per meter when the magnetic field is applied at the angle of 45° relative to the plane of the paper and directed into it from right to left

(d) What the resultant force per meter when the magnetic field is applied at the angle of 60° relative to the plane of the paper and directed into it from top to bottom.

9.5. A circular loop of wire of radius r meters and consisting of a single turn carries the current I as shown in Fig. P9.4. Derive the expression for the magnetic field intensity at the centre.

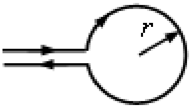


Fig. P9.5

9.6. A magnetic circuit composed of silicon sheet steel has the square construction shown in Fig. P9.6.

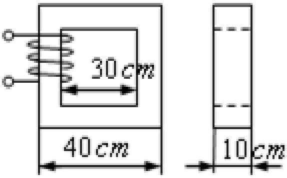


Fig. P9.6

(a) Find the magnetomotive force required to produce a core flux of 25×10^{-4} Wb.

(b) If the coil has 80 turns, how much current must be made to flow through the coil?

9.7. The magnetic circuit of Prob. 9.6 has an air gap of 0.1 cm cut in the right leg. For a coil having 100 turns find the current that must be allowed to flow in order that the core flux be 0.0025 Wb.

9.8. The core shown in Fig. P9.8 has a uniform cross-sectional area of $2\ in.^2$ and a mean length of 12 in. Also, coil A has 200 turns and carries 0.5 A, coil B has 400 turns and carries 0.75 A, and coil C carries 1.0 A.

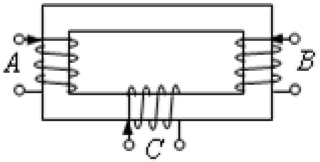


Fig. P9.8

How many turns must coil C have in order that the core flux be 120,000 lines? The coil currents have the directions indicated in the figure. The core is made of silicon sheet steel.

9.9. The magnetization curve of a relay is shown in Fig. P9.9.

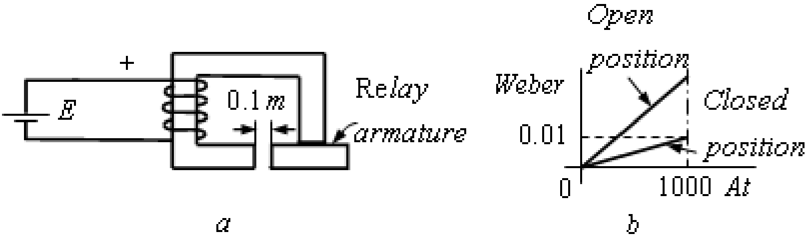


Fig. P9.9

(a) Find the energy stored in the field with the relay in the open position.

(b) Assume the relay armature moves rapidly under conditions of constant flux. Compute the work done in going from the open to the closed position.

(c) Calculate the force on newtons exerted on the armature in part (b).

(d) Assume the relay armature moves slowly at constant magnetomotive force. Compute the work done in going from the open to the closed position.

(e) Does the energy of part (d) come from the original stored field energy? Explain.

9.10. A ring of ferromagnetic material has a rectangular section. The inner diameter is 7.4 in., the outer diameter is 9 in., and the thickness is 0.8 in. There is a coil of 600 turns wound on the ring. When the coil carries a current of 2.5 A, the flux produced in the ring is 1.2×10^{-3} Wb.

Find the following quantities expressed in mks units: (a) magnetomotive force; (b) magnetic field intensity; (c) flux density; (d) reluctance; (e) permeability; (f) relative permeability.

9.11. The total core loss (hysteresis plus eddy current) for a specimen of magnetic sheet steel is found to be 1800 W at 60 Hz. If the flux density is kept constant and the frequency of the supply increased 50%, the total core loss is found to be 3000 W. Compute the separate hysteresis and eddy-current losses at both frequencies.

9.12. In the magnetic circuit shown in Fig. P9.12 the coil F_1 is supplied with 350 *At* in the direction indicated. Find the direction and magnitude of magnetomotive force required in coil F_2 in order that air-gap flux be 180,000 lines. The core has an effective cross-sectional area of $9\,in.^2$ and is made of silicon sheet steel. The length of the air gap is $0.05\,in$.

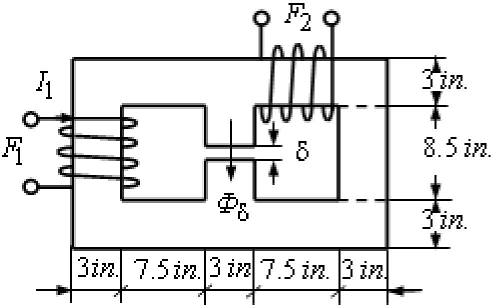


Fig. P9.12

9.13. In plotting a hysteresis loop the following scales are used: $1\,cm = 10\,At/in.$, and $1\,\tilde{n}m = 20\,kilolines/in.^2$. The area of the loop for a certain material is found to be $6.2\,cm^2$. Calculate the hysteresis loss in joules per cycle for the specimen tested if the volume is $400\,cm^3$.

9.14. The flux in a magnetic core is alternating sinusoidally at a frequency of 500 *Hz*. The maximum flux density is 50 $kilolines/in.^2$. The eddy-current loss then amounts to 14 *W*. Find the eddy-current loss in this core when the frequency is 750 *Hz* and the flux density 40 *kilolines per square inch.*

Chapter 10

FOUR-TERMINAL NETWORKS

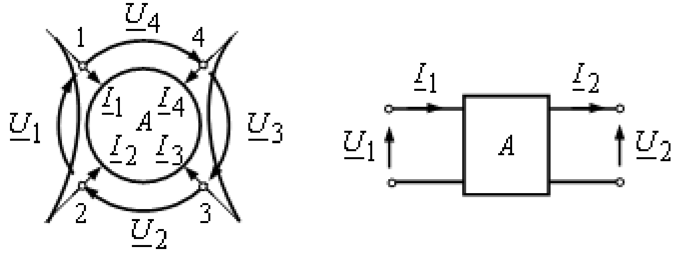
10.1. A Four-terminal Network and Network Equations

At the analysis of electric circuits in interconnection research problems between variables (currents, voltages, powers, etc.) of any two branches of electric circuit it very often happens convenient to gate out some circuit sections which have two pairs of terminals. Such sections are named *as four-terminal networks*.

As processes in electric circuits are usually linked to an energy transfer, one pair of terminals which is affiliated to an energy source, it is accepted to name *input*, and the second, which is affiliated to the receiver of electric energy is named *output*.

A four-terminal (or two-port) network is an electric circuit with two input and two output terminals. Practical examples of four-terminal networks are transformers, power-transmission lines, bridge circuits, electric filters, etc.

The four-terminal networks joined in pairs with their terminals to two external electric circuits are termed as *feed-through* ones. The rated circuit shown in Fig. 10.1, *a* which, obviously, is equivalent to the circuit in Fig. 10.1, *b* provided that $\underline{U}_3 = -\underline{U}_2$, is used in the classical theory of feed-through four-terminal networks.



a *b*

Fig. 10.1

In the given chapter the short theory of the feed-through four-terminal networks is considered. (The "feed-through" term will not be mentioned further).

It is customary to symbolize a four-terminal network by a box with two pairs of terminals *a-b* (or 1-1') and *c-d* (or 2-2') (Fig. 10.2), each pair making up a port.

If a four-terminal network contains a voltage or a current source, the convention is to put the letter *A* (for active) in the box. If there is no *A* in the box, the network is a passive one (marked with *P*), that is, contains no voltage source. We shall limit ourselves to passive four-terminal networks (except Sec.10.10).

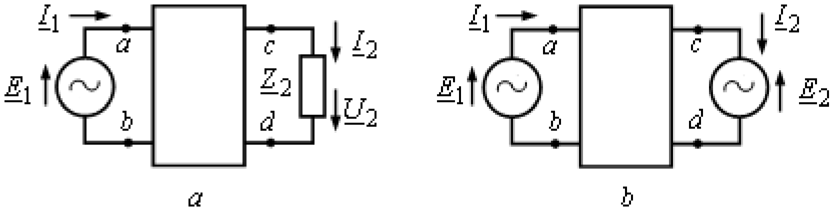


Fig.10.2. Four-terminal networks

A four-terminal network usually is an intermediate link between a source of supply and a load. The terminals to which the supply source is connected are called the input (*a-b* in Fig. 10.2), and the terminals to which the load is connected are referred to as the output (*c-d* in Fig. 10.2).

Four-terminal networks are divided into *linear* and *nonlinear*. The four-terminal network is linear if voltage and a current on its output (outlet) terminals linearly depend on voltage and a current on its input (incoming) terminals. The instances of the linear four-terminal networks are communication circuits, filters, and the instances of nonlinear ones are rectifiers, inverted rectifiers, frequency converters.

Four-terminal networks happen to be *symmetrical* and *asymmetrical*. The four-terminal network is symmetrical, if the change by places of input and output terminals does not change currents and voltages in the electric circuit to which the four-terminal network is connected. Otherwise the four-terminal network is asymmetrical.

Also four-terminal networks may be *reversible* and *non-reversible*. In *reversible one* the input voltage ratio to the output current (*a transmitting resistance*) does not depend on what pair of terminals is input, and what pair is output. Otherwise the four-terminal network is non-reversible. Passive linear four-terminal networks and the symmetrical four-terminal networks are always reversible. At the input terminals the current is $\underline{I}_1$ and the input voltage is $\underline{U}_1$. At the output terminals the respective values are $\underline{I}_2$ and $\underline{U}_2$.

If a four-terminal network is an intermediary link between a supply source and a load, it is assumed that both the load and the input voltage may vary, while the network configuration remains unchanged and the impedances in the network remain fixed (linear).

In any passive linear four-terminal network the input voltage and current are related to the output voltage and current by two equations which are basic to the theory of four-terminal networks:

$$\underline{U}_1 = \underline{A}\underline{U}_2 + \underline{B}\underline{I}_2, \tag{10.1}$$

$$\underline{I}_1 = \underline{C}\underline{U}_2 + \underline{D}\underline{I}_2. \tag{10.2}$$

The constants *A*, *B*, *C* and *D* are *linear parameters* of the network, dependent on the internal connections and the impedances of the network. For any four-terminal network they can be either computed or determined experimentally.

Different authors mark out these coefficients either *A*, *B*, *C* and *D* or A_{11}, A_{12}, A_{21} and A_{22}. It is possible to use any label from these two but it is necessary to remember that

$$\underline{A}_{11} = \underline{A}; \qquad \underline{A}_{12} = \underline{B}; \qquad \underline{A}_{21} = \underline{C}; \qquad \underline{A}_{22} = \underline{D}\,.$$

Then the linear parameters are related thus

$$\underline{A}\underline{D} - \underline{B}\underline{C} = 1 \quad \text{or} \quad \underline{A}_{11}\underline{A}_{22} - \underline{A}_{12}\underline{A}_{21} = 1. \tag{10.3}$$

To derive Eqs. (10.1) and (10.2), let there be a source of electromotive force $\underline{E}_1 = \underline{U}_{ab} = \underline{U}_1$ connected to the terminals *ab*, and the load Z_2 connected to the terminals *cd* (Fig. 10.2, *a*).

The complex voltage across the load is $\underline{U}_2 = \underline{I}_2\underline{Z}_2 = \underline{U}_{cd}$. By the compensation theorem (Sec.3.17), the complex load impedance $\underline{Z}_2$ may be replaced by an e. m. f. $\underline{E}_2$ directed against $\underline{I}_2$ and numerically equal to $\underline{U}_2$ (Fig. 10.1, *b*). Now we shall write the expressions for $\underline{I}_1$ and $\underline{I}_2$ in terms of $\underline{E}_1$ and $\underline{E}_2$, self and mutual admittance. If $\underline{I}_1$ and $\underline{I}_2$ be regarded as mesh currents, the mesh electromotive forces which are in the same direction as the mesh currents will enter equations similar to Eq. (3.32) with a "plus" sign, and the e.m.f.s opposing the respective mesh currents will be negative. The e. m.f. $\underline{E}_1$ is in the same direction as $\underline{I}_1$. Therefore it is positive in Eqs. (10.4) and (10.5). $\underline{E}_2$ is directed against $\underline{I}_2$, and will be negative in Eqs. (10.4) and (10.5):

$$\underline{I}_1 = \underline{Y}_{11}\underline{E}_1 + \underline{Y}_{12}\underline{E}_2\,, \tag{10.4}$$

$$\underline{I}_2 = \underline{Y}_{12}\underline{E}_1 + \underline{Y}_{22}\underline{E}_2 \tag{10.5}$$

From Eq. (10.5)

$$\underline{E}_1 = \underline{E}_2\frac{\underline{Y}_{22}}{\underline{Y}_{12}} + \underline{I}_2\frac{1}{\underline{Y}_{12}} \tag{10.6}$$

Substituting Eq. (10.6) in Eq. (10.4) gives

$$\underline{I}_1 = \underline{E}_2\frac{\underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}^2}{\underline{Y}_{12}} + \underline{I}_2\frac{\underline{Y}_{11}}{\underline{Y}_{12}}\,. \tag{10.7}$$

Putting

$$\left.\begin{aligned} &\underline{A} = \frac{\underline{Y}_{22}}{\underline{Y}_{12}}; \quad \underline{B} = \frac{1}{\underline{Y}_{12}};\\ &\underline{C} = \frac{\underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}^2}{\underline{Y}_{22}};\\ &\underline{D} = \frac{\underline{Y}_{11}}{\underline{Y}_{12}} \end{aligned}\right\} \tag{10.8}$$

and replacing $\underline{E}_1$ in Eqs. (10.6) and (10.7) by $\underline{U}_1$ and $\underline{E}_2$ by $\underline{U}_2$, and noting Eq. (10.8), we obtain the basic equations of four-terminal (two-port) networks (10.1) and (10.2)

$$\underline{U}_1 = \underline{A}\underline{U}_2 + \underline{B}\underline{I}_2; \quad \underline{I}_1 = \underline{C}\underline{U}_2 + \underline{D}\underline{I}_2\,.$$

We check that Eq. (10.2) holds:

$$\underline{A}\underline{D} - \underline{B}\underline{C} = \frac{\underline{Y}_{11}\underline{Y}_{22}}{\underline{Y}_{12}^2} - \frac{\underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}^2}{\underline{Y}_{12}^2} = 1$$

Now we shall consider the relationship between $\underline{U}_1$ and $\underline{I}_1$, $\underline{I}_2$ and $\underline{U}_2$, if the voltage source is connected to the terminals 2-2', and the load to the terminals 1-1' (Fig. 10.3).

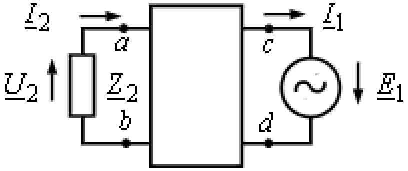


Fig.10.3. A four-terminal network with the source and the load interchanged.

As in the previous case, we replace Z_2 by the e. m. f. $\underline{E}_2$, directed against $\underline{I}_2$, and write expressions for $\underline{I}_1$ and $\underline{I}_2$:

$$\underline{I}_2 = -\underline{Y}_{11}\underline{E}_2 + \underline{Y}_{12}\underline{E}_1, \tag{10.9}$$

$$\underline{I}_1 = -\underline{Y}_{12}\underline{E}_2 + \underline{Y}_{22}\underline{E}_1. \tag{10.10}$$

From Eq. (10.9)

$$\underline{E}_1 = \underline{E}_2\frac{\underline{Y}_{11}}{\underline{Y}_{12}} + \underline{I}_2\frac{1}{\underline{Y}_{12}}. \tag{10.11}$$

Substituting Eq. (10.11) in Eq. (10.10) gives

$$\underline{I}_1 = \underline{E}_2\frac{\underline{Y}_{11}\underline{Y}_{12} - \underline{Y}_{12}^2}{\underline{Y}_{12}} + \underline{I}_2\frac{\underline{Y}_{22}}{\underline{Y}_{12}}. \tag{10.12}$$

On replacing $\underline{E}_1$ by $\underline{U}_1$ and $\underline{E}_2$ by $\underline{U}_2$, and using Eq. (10.7), we re-write Eqs. (10.11) and (10.12) thus

$$\underline{U}_1 = \underline{D}\underline{U}_2 + \underline{B}\underline{I}_2, \tag{10.13}$$

$$\underline{I}_1 = \underline{C}\underline{U}_2 + \underline{A}\underline{I}_2. \tag{10.14}$$

Eqs. (10.1) and (10.2) describe the operation of a four-terminal network with the source of supply connected to the terminals 1-1' and the load connected to the terminals 2-2'. Equations (10.13) and (10.14) do the same with the source and the load interchanged.

If, with the source and the load interchanged, the respective currents remain the same, such a network is termed symmetrical. In a symmetrical four-terminal network, $A = D$.

Equations (10.1) and (10.2) are often written thus

$$\underline{U}_1 = \underline{A}_{11}\underline{U}_2 + \underline{A}_{12}\underline{I}_2 \tag{10-1'}$$

$$\underline{I}_1 = \underline{A}_{21}\underline{U}_2 + \underline{A}_{22}\underline{I}_2 \tag{10-2'}$$

where $\underline{A}_{11} = \underline{A}$; $\underline{A}_{12} = \underline{B}$; $\underline{A}_{21} = C$; and $A_{22} = D$.

This constitutes what is known as the *A*-form of equations for four-terminal networks. For the *A*-form, the positive directions of $\underline{I}_1$ and $\underline{I}_2$ are as shown in Fig. 10.2, *a*. Of the four quantities ($\underline{U}_1$, $\underline{U}_2$, $\underline{I}_1$ and $\underline{I}_2$), any two may be known, and the remaining two determined from them.

Accordingly, there are, in addition to the *A*-form parameters, five more forms of the parameters for four-terminal networks, namely the *Y*, *Z*, *H*, *G*, and *B*-forms. For the *Y*, *Z*, *H* and *G*-parameters, the positive direction of $\underline{I}_1$ is as shown in Fig. 10.2, *a*, and for $\underline{I}_2$ is the opposite of that shown in Fig. 10.2, *a*.

For the *B*-parameters, the positive direction of $\underline{I}_1$ and $\underline{I}_2$ is the opposite of that shown in the figure.

In all the forms of parameters

$$\underline{U}_1 = \underline{U}_{ab} = \underline{E}_1 \text{ and } \underline{U}_2 = \underline{U}_{cd} = \underline{E}_2$$

The *Y*-form:

$$\underline{I}_1 = \underline{Y}_{11}\underline{U}_1 + \underline{Y}_{12}\underline{U}_2; \quad \underline{I}_2 = \underline{Y}_{21}\underline{U}_1 + \underline{Y}_{22}\underline{U}_2,$$

where Y_{12} and Y_{21} are opposite in sign to the same terms in Eqs. (10.4) and (10.5).

The *Z*-form:

$$\underline{U}_1 = \underline{Z}_{11}\underline{I}_1 + \underline{Z}_{12}\underline{I}_2; \quad \underline{U}_2 = \underline{Z}_{21}\underline{I}_1 + \underline{Z}_{22}\underline{I}_2.$$

The *H*-form:

$$\underline{U}_1 = \underline{H}_{11}\underline{I}_1 + \underline{H}_{12}\underline{U}_2; \quad \underline{I}_2 = \underline{H}_{21}\underline{I}_1 + \underline{H}_{22}\underline{U}_2.$$

The *G*-form:

$$\underline{I}_1 = \underline{G}_{11}\underline{U}_1 + \underline{G}_{12}\underline{I}_2; \quad \underline{U}_2 = \underline{G}_{21}\underline{U}_1 + \underline{G}_{22}\underline{I}_2.$$

The *B*-form:

$$\underline{U}_2 = \underline{B}_{11}\underline{U}_1 + \underline{B}_{12}\underline{I}_1; \quad \underline{I}_2 = \underline{B}_{21}\underline{U}_1 + \underline{B}_{22}\underline{I}_1 .$$

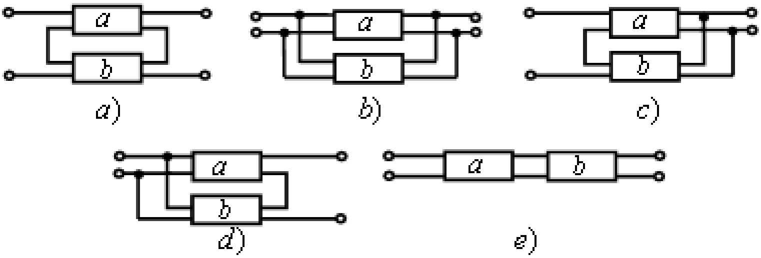


Fig. 10.4. Different connections of four-terminal networks.

When investigating the properties of four-terminal networks in combinations, one uses Z-parameters for series-connected four-terminal networks (Fig. 10.4, a), Y-parameters are used for parallel connection (Fig. 10.4, b), H-parameters for series-parallel connection (Fig. 10.4, c), and *G*-parameters for parallel-series connection (Fig. 10.4, *d*). With four-terminal networks in a cascade, *A*-parameters will do. The *Y*- and *Z*-forms are used most in circuit synthesis. The parameters of small-signal equivalent circuits for transistors are usually given in *H*- or *Z*-parameters.

10.2. Determination of the ABCD Parameters

The complex coefficients in Equations (10.1), (10.2), (10.13) and (10.14) can be found either by Eq. (10.8), if the internal connections and the circuit variables are known, or by the method of driving-point impedances, found by measurement or by calculation.

Experimentally, driving-point impedances are determined with the aid of a wattmeter, an ammeter and a voltmeter, using the circuit of Fig. 10.5 where the four-terminal network under examination is connected to the electromotive force source on the *ab* or the *cd* side (depending on which of the impedances is to be found).

A capacitor connected in parallel to the input terminals of four-terminal network serves for definition of a sign of the angle between voltage and current. The capacitor with a small capacitance is turned on to feeding terminals of the four-terminal network and one must trace a modification of an input current. If the current increases, the input impedance of a four-terminal network has capacitive character and an angle is negative. If the current decreases, the input impedance has the inductive character, i.e. an angle is positive.

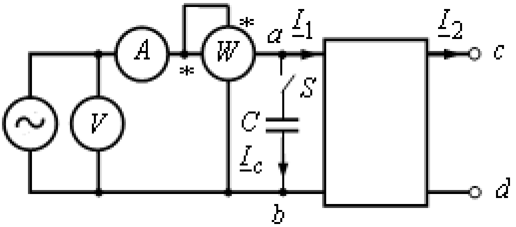


Fig. 10.5. The circuit with the ammeter, the voltmeter and the wattmeter

Consider the procedure for determining the driving-point impedances of a four-terminal network under three sets of operating conditions:

1. Measure the impedance Z_{10} looking into the generator terminals *ab* with the load terminals *cd* open-circuited (the load is disconnected, and the respective subscript is zero):

$$\underline{Z}_{10} = z_{10} e^{j\varphi_{10}}.$$

2. Measure the impedance Z_{1k} looking into the terminals *ab* with the terminals *cd* short-circuited (the load end is short-circuited, and the respective subscript is *k*):

$$\underline{Z}_{1k} = z_{1k} e^{j\varphi_{1k}}.$$

3. Measure the complex impedance Z_{2k} looking into the terminals *cd*, with the terminals *ab* short-circuited:

$$\underline{Z}_{2k} = z_{2k} e^{j\varphi_{2k}}.$$

Now we write $\underline{Z}_{10}$, $\underline{Z}_{1k}$ and $\underline{Z}_{2k}$ in terms of the *ABCD* parameters. To this end, we find $\underline{Z}_{10}$ and $\underline{Z}_{1k}$ by Eqs. (10.1) and (10.2) in terms of *A*, *B*, *C* and *D*, and Z_{2k} by Eqs. (10.13) and (10.14) in terms of *B* and *A*. In the measurement of Z_{10}, the load end (terminals *cd*) was open-circuited. So, $I_2 = 0$, and Eqs. (10.1) and (10.2) give

$$\underline{U}_{10} = \underline{A}\underline{U}_{20}, \quad \underline{I}_{10} = \underline{C}\underline{U}_{20}.$$

Hence

$$\underline{Z}_{10} = \frac{\underline{U}_{10}}{\underline{I}_{10}} = \frac{\underline{A}}{\underline{C}}.$$

In the measurement of Z_{1k}, the load end was short-circuited, and so $U_2 = 0$. Eqs. (10.1) and (10.2) give:

$$\underline{U}_{1k} = \underline{B}\underline{I}_{2k}, \quad \underline{I}_{1k} = \underline{D}\underline{I}_{2k};$$

$$\underline{Z}_{1k} = \frac{\underline{U}_{1k}}{\underline{I}_{1k}} = \frac{\underline{B}}{\underline{D}}.$$

In the measurement of Z_{2k}, the source was connected to the terminals *cd* and the terminals *ab* were short-circuited, so $U_2 = 0$, and Eqs. (10.13) and (10.14) give:

$$\underline{Z}_{2k} = \frac{B\underline{I}_{2k}}{A\underline{I}_{2k}} = \frac{\underline{B}}{\underline{A}}.$$

To sum up, we have got four equations for the four unknown constants *A*, *B*, *C* and *D*:

$$\underline{A}\underline{D} - \underline{B}\underline{C} = 1; \tag{10.3}$$

$$\underline{Z}_{10} = \frac{\underline{A}}{\underline{C}}; \tag{10.15}$$

$$\underline{Z}_{1k} = \frac{\underline{B}}{\underline{D}}; \tag{10.16}$$

$$\underline{Z}_{2k} = \frac{\underline{B}}{\underline{A}}. \tag{10.17}$$

Writing the difference, we obtain

$$1 - \frac{\underline{Z}_{1k}}{\underline{Z}_{2k}} = 1 - \frac{\underline{B}\underline{C}}{\underline{A}\underline{D}} = \frac{1}{\underline{A}\underline{D}}$$

or

$$\frac{\underline{Z}_{10} - \underline{Z}_{1k}}{\underline{Z}_{10}} = \frac{1}{\underline{A}\underline{D}}. \tag{10.18}$$

Dividing Eq. (10.17) by Eq. (10.16) gives

$$\frac{\underline{Z}_{2k}}{\underline{Z}_{1k}} = \frac{\underline{D}}{\underline{A}}\,. \tag{10.19}$$

Multiplying Eq. (10.18) by Eq. (10.19) gives

$$\frac{(\underline{Z}_{10} - \underline{Z}_{1k})\,\underline{Z}_{2k}}{\underline{Z}_{10}\,\underline{Z}_{1k}} = \frac{1}{\underline{A}^2}$$

Hence

$$\underline{A} = \sqrt{\frac{\underline{Z}_{10}\,\underline{Z}_{1k}}{\underline{Z}_{2k}\,(\underline{Z}_{10} - \underline{Z}_{1k})}}\,. \tag{10.20}$$

Equation (10.20) gives *A* in terms of $\underline{Z}_{10}$, $\underline{Z}_{1k}$ and $\underline{Z}_{2k}$. After *A* is found, *C* can be found by Eq. (10.15), *B* by Eq. (10.17) and *D* by Eq. (10.16). The constants *A* and *D* are dimensionless, the constant *B* is expressed in ohms, and the constant *C* in Siemens (1 siemens = 1 ampere/1 volt).

Example 10.1. By measurement it has been found that $\underline{Z}_{10} = 8.68e^{-j50^o}$ Ω; $\underline{Z}_{1k} = 10.25e^{j70^o}$ Ω; and $\underline{Z}_{2k} = 3.33e^{j27^o}$ Ω. Find the constants, *A*, *B*, *C* and *D* of the four-terminal network.

Solution:

$$\underline{Z}_{10} - \underline{Z}_{1k} = 5.58 - j6{,}65 - 3.5 - j9.63 = 2.08 - j16.28 = 16.68e^{-j83^o}\,.$$

By Eq. (10.20)

$$\underline{A} = \pm\sqrt{\frac{8.86e^{-j50^o} \times 10.25e^{j70^o}}{3.33e^{j27^o} \times 16.68e^{-j83^o}}} = \pm\sqrt{\frac{90.815e^{j20^o}}{55.544e^{-j56^o}}} = 1.28e^{j38^o}\,.$$

The change of sign does not change the module of the given magnitude, but changes the exponent of a complex quantity by 180°. Only the first case (with a "plus" sign) is considered in further calculation.

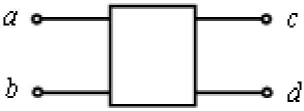
Then, we can find the unknown constants *B*, *C*, *D*:

$$\underline{C} = \frac{\underline{A}}{\underline{Z}_{10}} = \frac{1.28e^{j38^o}}{8.86e^{-j50^o}} = 0.144e^{j88^o} \text{ siemens (approx.);}$$

$$\underline{B} = \underline{A}\underline{Z}_{2k} = 1.28e^{j38^o} \cdot 3.33e^{j27^o} = 4.26e^{j65^o}\ \Omega \text{ (approx.);}$$

$$\underline{D} = \frac{\underline{B}}{\underline{Z}_{1k}} = \frac{4.33e^{j37^o}}{10.25e^{j70^o}} = 0.42e^{-j33^o} \text{ (approx.).}$$

Example 10.2. The terminals *cd* of the four-terminal network in Example 10.1 receive a load $\underline{Z}_2 = 6 + j6\ \Omega$. The voltage source is connected to the terminals *ab*. Find $\underline{U}_1$ and $\underline{I}_1$ if $\underline{I}_2 = 1$ ampere.



Solution: By Eq. (10.1)

$$\underline{U}_1 = \underline{A}\underline{U}_2 + \underline{B}\underline{I}_2 == \underline{A}\underline{I}_2\underline{Z}_2 + \underline{B}\underline{I}_2 = \underline{I}_2(\underline{A}\underline{Z}_2 + \underline{B});$$

After substituting known values

$$\underline{U}_1 = 1 \times (1.28e^{j40^o} \times 6\sqrt{2}e^{j45^o} + 4.26e^{j67^o}) = 14.85e^{j80^o}\ V.$$

By Eq. (10.2)

$$\underline{I}_1 = \underline{C}\underline{U}_2 + \underline{D}\underline{I}_2 = \underline{I}_2(\underline{C}\underline{Z}_2 + \underline{D}) =$$

$$1 \times (0.167e^{j90^o} \times 6\sqrt{2}e^{j45^o} + 0.34) = 1.2e^{j123^o}\ A.$$

10.3. Equivalent Circuits of a Passive Four-terminal Network

In depending on the circuit design of internal connections (or a substitution equivalent circuit) one can discriminate: *T*-network (Fig. 10.6, *a),* *P*-network (or *pi*-network) (Fig. 10.6, *b*). *G*-network (Fig. 10.6, *c*), *bridge* network (Fig. 10.6, *d*) and other types of four-terminal networks.

In fact, some of the physical components are *T*–and *P*–sections. In an equivalent *T*– or *P*–network the three impedances should be such that the equivalent network has the same general circuit parameters *A*, *B*, *C* and *D* as the original four-terminal network. These two forms are used more often. This is a single-valued problem, because the equivalent network contains three impedances, and the original network is also determined uniquely by three parameters (one of the relations between *A*, *B*, *C* and *D* is given by the equation *AD* – *BC* = l).

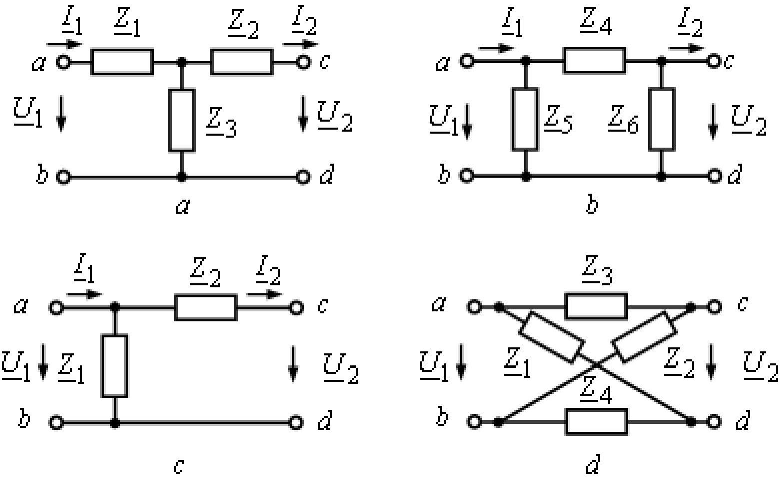


Fig.10.6. Equivalent *T*–network and *P*–network.

Expressing $\underline{U}_1$ and $\underline{I}_1$, at the input of a *T*–network (Fig. 10.6, *a*) in terms of the same values at its output, we obtain

$$\underline{I}_1=\underline{I}_2+\frac{\underline{U}_2+\underline{I}_2\underline{Z}_2}{\underline{Z}_3}=\underline{U}_2\frac{1}{\underline{Z}_3}+\underline{I}_2\left(1+\frac{\underline{Z}_2}{\underline{Z}_3}\right); \tag{10.21}$$

$$\begin{aligned}\underline{U}_1&=\underline{U}_2+\underline{I}_2\underline{Z}_2+\underline{I}_1\underline{Z}_1=\\&=\underline{U}_2\left(1+\frac{\underline{Z}_1}{\underline{Z}_3}\right)+\underline{I}_2\left(\underline{Z}_1+\underline{Z}_2+\frac{\underline{Z}_1\underline{Z}_2}{\underline{Z}_3}\right).\end{aligned} \tag{10.22}$$

Comparing Eq. (10.22) with Eq. (10.1), and Eq. (10.21) with Eq. (10.2), we find that

$$\underline{A}=1+\frac{\underline{Z}_1}{\underline{Z}_3};\quad \underline{B}=\underline{Z}_1+\underline{Z}_2+\frac{\underline{Z}_1\underline{Z}_2}{\underline{Z}_3};\quad \underline{C}=\frac{1}{\underline{Z}_3};\quad \underline{D}=1+\frac{\underline{Z}_2}{\underline{Z}_3}.$$

Consequently,

$$\underline{Z}_3=\frac{1}{\underline{C}}; \tag{10.23}$$

$$\underline{Z}_1 = \frac{\underline{A}-1}{\underline{C}}; \tag{10.24}$$

$$\underline{Z}_2 = \frac{\underline{D}-1}{\underline{C}}. \tag{10.25}$$

Equations (10.23) through (10.25) give the impedances in the equivalent *T*–network of Fig. 10.6, *a* in terms of the constants *A*, *B*, *C* and *D* of the original four-terminal network.

Similarly for the *P*–network of Fig. 10.6, *b* we get

$$\underline{A} = 1 + \frac{\underline{Z}_4}{\underline{Z}_6}; \quad \underline{B} = \underline{Z}_4; \quad \underline{C} = \frac{\underline{Z}_4 + \underline{Z}_5 + \underline{Z}_6}{\underline{Z}_5 \underline{Z}_6}; \quad \underline{D} = 1 + \frac{\underline{Z}_4}{\underline{Z}_5}.$$

Consequently,

$$\underline{Z}_4 = \underline{B}; \tag{10.26}$$

$$\underline{Z}_5 = \frac{\underline{B}}{\underline{D}-1}; \tag{10.27}$$

$$\underline{Z}_6 = \frac{\underline{B}}{\underline{A}-1}. \tag{10.28}$$

If the original four-terminal network is a symmetrical one, then *A* = *D*, and so $\underline{Z}_1 = \underline{Z}_2$ in the equivalent *T*–network, and $\underline{Z}_5 = \underline{Z}_6$ in the equivalent *P*–network.

Example 10.3. Find the impedances of a *T*–equivalent for the four-terminal network of Example 10.1.

Solution: By Equations (10.23) through (10.25)

$$\underline{Z}_3 = \frac{1}{\underline{C}} = \frac{1}{0.147e^{j60^o}} = 6.8e^{-j60^o}\ \Omega \text{ (approx.)};$$

$$\underline{Z}_2 = \frac{\underline{D}-1}{\underline{C}} = \frac{0.42e^{-j33^o} - 1}{0.147e^{j60^o}} = 4.76e^{j100^o}\ \Omega;$$

$$\underline{Z}_1 = \frac{\underline{A}-1}{\underline{C}} = \frac{1.3e^{j10^o} - 1}{0.147e^{j60^o}} = 2.45e^{-j21^o}\ \Omega.$$

10.4. The systems of equations of four-terminals networks

The complex effective values of input and output currents and voltages of linear four-terminal network are linked among themselves by the system of two linear equations, and any two of these four magnitudes can be considered in the capacity of initial, and remaining two – in the capacity of unknown values (in the case of nonlinear four-terminal networks these dependences have more complicated character).

The coefficients at currents and voltages in equations are represented by complex quantities which depend on internal structure of the four-terminal network and the frequency of the energy source connected to the four-terminal unit.

Let's consider in detail the determination both input and output currents and voltages of any four-terminal network, using its parameters. There are six different systems of equations, each of which expresses various pairs of magnitudes through the remaining ones. The coefficients of these six equations at constant frequency of the energy source are invariable and consequently are four-terminal network parameters. Traditionally they are called *A* - *B* - *G* - *H* - *Y* - and *Z*-parameters. The basic sense of the theory of four-terminal networks consists in using four-terminal network parameters to find all currents and voltages of four-terminal network without researching the processes inside it. The variant with the currents $\underline{I}_1$, $\underline{I}_2$ is called a *direct transmission*, and with the currents $\underline{I}'_1$, $\underline{I}'_2$ is termed an *inverse transmission*.

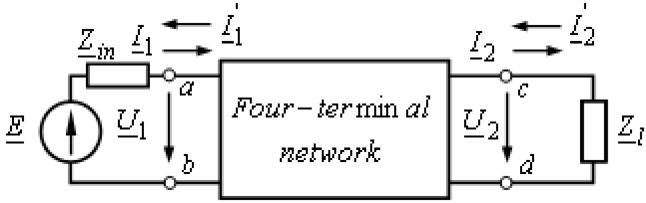


Fig. 10. 7. A four-terminal network.

As we can see from the circuit (Fig. 10. 7), $\underline{I}_1 = -\underline{I}'_1, \underline{I}_2 = -\underline{I}'_2$. In a case when the four-terminal network appears as an intermediate link between the energy source and a load resistance, the output voltage $\underline{U}_2$ and current $\underline{I}_2$ are the given magnitudes, and input voltage $\underline{U}_1$ and current $\underline{I}_1$ are the unknown quantities characterizing the working rate of a four-terminal network.

The connection between them is installed by the system of direct transmission parameters:

$$\begin{cases} \underline{U}_1 = \underline{A}_{11}\underline{U}_2 + \underline{A}_{12}\underline{I}_2; \\ \underline{I}_1 = \underline{A}_{21}\underline{U}_2 + \underline{A}_{22}\underline{I}_2, \end{cases} \tag{10.21}$$

where $\underline{A}_{11}$, $\underline{A}_{12}$, $\underline{A}_{21}$, $\underline{A}_{22}$ are the *A*-form parameters for four-terminal network.

When the energy source is connected to the terminals (*c*–*d*), and a load – to the terminals (*a*–*b*), one can use the system of equations of *inverse transmission*:

$$\begin{cases} \underline{U}_2 = \underline{B}_{11}\underline{U}_1 + \underline{B}_{12}\underline{I}'_1; \\ \underline{I}'_2 = \underline{B}_{21}\underline{U}_1 + \underline{B}_{22}\underline{I}'_1, \end{cases} \tag{10.22}$$

where $\underline{B}_{11}$, $\underline{B}_{12}$, $\underline{B}_{21}$, $\underline{B}_{22}$ are the *B*-form parameters for four-terminal network.

In engineering literature there are other symbols of parameters *A*- and *B*-forms for four-terminal network:

$$\begin{gathered} \underline{A} = \underline{A}_{11} = \underline{B}_{22}; \quad \underline{B} = \underline{A}_{12} = \underline{B}_{12}; \\ \underline{C} = \underline{A}_{21} = \underline{B}_{21}; \quad \underline{D} = \underline{A}_{22} = \underline{B}_{11}. \end{gathered} \tag{10.23}$$

Parameters $\underline{A}$ and $\underline{D}$ are measured in *the relative units* (*r.un.*), parameter $\underline{B}$ is measured in *Ohms* (Ω), and parameter $\underline{C}$ – in S*iemens* (*Sm*).

If an input voltage and an output current are given magnitudes, and an output voltage and an input current are the unknown quantities one must use the system of equations with *G*-parameters:

$$\begin{cases} \underline{I}_1 = \underline{G}_{11}\underline{U}_1 + \underline{G}_{12}\underline{I}'_2; \\ \underline{U}_2 = \underline{G}_{21}\underline{U}_1 + \underline{G}_{22}\underline{I}'_2, \end{cases} \tag{10.24}$$

where $\underline{G}_{11}$, $\underline{G}_{12}$, $\underline{G}_{21}$, $\underline{G}_{22}$ are the *G*-form parameters for four-terminal network.

If an output voltage and an input current are given magnitudes, and an input voltage and an output current are the unknown quantities one must use the system of equations with *H*-parameters:

$$\begin{cases} \underline{U}_1 = \underline{\underline{H}}_{11}\underline{I}_1 + \underline{\underline{H}}_{12}\underline{U}_2; \\ \underline{I}_2 = \underline{\underline{H}}_{21}\underline{I}_1 + \underline{\underline{H}}_{22}\underline{U}_2, \end{cases} \tag{10.25}$$

where $\underline{H}_{11}$, $\underline{H}_{12}$, $\underline{H}_{21}$, $\underline{H}_{22}$ are the *H*-form parameters for four-terminal network.

In a case if the four-terminal network is reversible the equalities $\underline{G}_{12} = \underline{G}_{21}$ and $\underline{H}_{12} = \underline{H}_{21}$ are fulfilled. If the four-terminal network is symmetrical the equalities $\underline{G}_{11} = \underline{G}_{22}$ and $\underline{H}_{11} = \underline{H}_{22}$ are fulfilled in addition. In this case the properties of the four-terminal network are determined only by two independent parameters (for example, $\underline{Z}_{11}$ and $\underline{Z}_{12}$ or $\underline{H}_{11}$ and $\underline{H}_{12}$). The unit of parameters $\underline{G}_{12}$, $\underline{G}_{21}$, $\underline{H}_{12}$ and $\underline{H}_{21}$ is *the relative unit* (*r.un.*), the unit of parameters $\underline{G}_{22}$ and $\underline{H}_{11}$ is Ohm (Ω), the unit of parameters $\underline{G}_{11}$ и $\underline{H}_{22}$ is *Siemens* (*Sm*).

If input and output voltages are given magnitudes, and input and output currents are unknown quantities one must use the system of equations with *Y*-parameters:

$$\begin{cases} \underline{I}_1 = \underline{Y}_{11}\underline{U}_1 + \underline{Y}_{12}\underline{U}_2; \\ \underline{I}'_2 = \underline{Y}_{21}\underline{U}_1 + \underline{Y}_{22}\underline{U}_2, \end{cases} \qquad (10.26)$$

where $\underline{Y}_{11}$, $\underline{Y}_{12}$, $\underline{Y}_{21}$, $\underline{Y}_{22}$ - are the *Y*-form parameters for four-terminal network.

The unit of parameters *Y*-form is *Siemens* (*Sm*), as these parameters, in the physical main, are complex admittances.

Really, $\underline{Y}_{11} = \left.\dfrac{\underline{I}_1}{\underline{U}_1}\right|_{\underline{U}_2=0}$ is the *complex input admittance* under short circuit of output terminals (*c*–*d*), $\underline{Y}_{22} = \left.\dfrac{\underline{I}'_2}{\underline{U}_2}\right|_{\underline{U}_1=0}$ is the *complex input admittance* looking at terminals (*c*–*d*), when there is a short circuit of input terminals (*a*–*b*), $\underline{Y}_{12} = \left.\dfrac{\underline{I}_1}{\underline{U}_2}\right|_{\underline{U}_1=0}$ is the *complex transmitting* (*mutual*) *admittance* under short circuit of input terminals (*a*–*b*), $\underline{Y}_{21} = \left.\dfrac{\underline{I}'_2}{\underline{U}_1}\right|_{\underline{U}_2=0}$ is the *complex transmitting* (*mutual*) *admittance* under short circuit of output terminals (*c*–*d*).

In a case if the four-terminal network is reversible the equality $\underline{Y}_{12} = \underline{Y}_{21}$ is fulfilled. If the four-terminal network is symmetrical the equality $\underline{Y}_{11} = \underline{Y}_{22}$ is in addition fulfilled and its properties are determined only by two independent parameters (for example, $\underline{Y}_{11}$ and $\underline{Y}_{12}$). If input and output currents are given magnitudes, and input and output voltages are unknown quantities one must use the system of equations with *Z*-parameters:

$$\begin{cases} \underline{U}_1 = \underline{Z}_{11}\underline{I}_1 + \underline{Z}_{12}\underline{I}'_2; \\ \underline{U}_2 = \underline{Z}_{21}\underline{I}_1 + \underline{Z}_{22}\underline{I}'_2, \end{cases} \tag{10.27}$$

The unit of parameters *Z*-form is *Ohm* (*Ω*), as these parameters, in the physical main, are complex impedances. Really, $\underline{Y}_{11} = \left.\dfrac{\underline{I}_1}{\underline{U}_1}\right|_{\underline{U}_2=0}$ is the *complex input admittance* under short circuit of output terminals (*c*-*d*), $\underline{Y}_{22} = \left.\dfrac{\underline{I}'_2}{\underline{U}_2}\right|_{\underline{U}_1=0}$ is the *complex input admittance* looking at terminals (*c*–*d*), when there is a short circuit of input terminals (*a*–*b*), $\underline{Y}_{12} = \left.\dfrac{\underline{I}_1}{\underline{U}_2}\right|_{\underline{U}_1=0}$ is the *complex transmitting* (*mutual*) *admittance* under short circuit of input terminals (*a*–*b*). Really, $\underline{Z}_{11} = \left.\dfrac{\underline{U}_1}{\underline{I}_1}\right|_{\underline{I}'_2=0}$ is the *complex input impedance* looking at terminals (*a*–*b*), when there is an open circuit of input terminals (*a*–*b*) under open circuit of output terminals (*c*–*d*), $\underline{Z}_{12} = \left.\dfrac{\underline{U}_1}{\underline{I}'_2}\right|_{\underline{I}_1=0}$ is the *complex transmitting* (*mutual*) *impedance* under open circuit of input terminals (*a*–*b*), $\underline{Z}_{21} = \left.\dfrac{\underline{U}_2}{\underline{I}_1}\right|_{\underline{I}'_2=0}$ is the *complex transmitting* (*mutual*) *impedance* under open circuit of output terminals (*c*–*d*), $\underline{Z}_{22} = \left.\dfrac{\underline{U}_2}{\underline{I}'_2}\right|_{\underline{I}_1=0}$ is the *complex input impedance* looking at terminals (*c*–*d*), when there is an open circuit of input terminals (*a*–*b*).

In a case if the four-terminal network is reversible the equality $\underline{Z}_{12} = \underline{Z}_{21}$ is fulfilled. If the network is symmetrical the equality $\underline{Z}_{11} = \underline{Z}_{22}$ is in addition fulfilled and its properties are determined only by two independent parameters (for example, $\underline{Z}_{11}$ and $\underline{Z}_{12}$).

10.5. The Parameter Interconnection of Various Notations

If the parameters in one of six systems are known it is possible to transfer to any another system. The transition is carried out by the solution of equation set of four-terminal network concerning the unknown quantities.

As example we will find the interconnection between *A* - and Y-parameters. Taking into account the relationship $\underline{I}_2 = -\underline{I}'_2$ we will express voltage $\underline{U}_1$ from the second equation of system (10.26), and substitute it in the first equation of system.

$$\begin{cases} \underline{U}_1 = -\dfrac{\underline{Y}_{22}}{\underline{Y}_{21}}\underline{U}_2 - \dfrac{1}{\underline{Y}_{21}}\underline{I}_2; \\ \underline{I}_1 = \underline{Y}_{11}\left(-\dfrac{\underline{Y}_{22}}{\underline{Y}_{21}}\underline{U}_2 - \dfrac{1}{\underline{Y}_{21}}\underline{I}_2 \right) + \underline{Y}_{12}\underline{U}_2 \end{cases} \tag{10.28}$$

or after transformation

$$\begin{cases} \underline{U}_1 = -\dfrac{\underline{Y}_{22}}{\underline{Y}_{21}}\underline{U}_2 - \dfrac{1}{\underline{Y}_{21}}\underline{I}_2; \\ \underline{I}_1 = \dfrac{\underline{Y}_{12}\underline{Y}_{21} - \underline{Y}_{11}\underline{Y}_{22}}{\underline{Y}_{21}}\underline{U}_2 - \dfrac{\underline{Y}_{11}}{\underline{Y}_{21}}\underline{I}_2. \end{cases} \tag{10.29}$$

Comparing (10.21) to (10.29), we will get the result as

$$\begin{aligned} &\underline{A} = \underline{A}_{11} = -\frac{\underline{Y}_{22}}{\underline{Y}_{21}}; \qquad && \underline{B} = \underline{A}_{12} = -\frac{1}{\underline{Y}_{21}}; \\ &\underline{C} = \underline{A}_{21} = -\frac{|\underline{Y}|}{\underline{Y}_{21}}; \qquad && \underline{D} = \underline{A}_{22} = -\frac{\underline{Y}_{11}}{\underline{Y}_{21}}, \end{aligned} \tag{10.30}$$

where $|\underline{Y}| = \underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}\underline{Y}_{21}$ is the determinant made up of *Y*-parameters.

As in accordance with a reciprocity theorem $\underline{Y}_{12}=\underline{Y}_{21}$, it is visible, that parameters of the *A*-form (*B*-forms) of the four-terminal network are linked among themselves by a relationship

$$\underline{A}\underline{D}-\underline{B}\underline{C}=\underline{A}_{11}\underline{A}_{22}-\underline{A}_{12}\underline{A}_{21}=\frac{\underline{Y}_{22}\underline{Y}_{11}}{\underline{Y}_{21}\underline{Y}_{21}}-\frac{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}}{\underline{Y}_{21}\underline{Y}_{21}}=1. \qquad (10.31)$$

By analogy there is a transition from the arbitrary parameters of one form to the parameters of other form. As all six systems of parameters present one four-terminal network they are linked among themselves by transition formulas. The transitions from some form of parameters of four-terminal networks to another one are presented in the matrix form in Table 10.1. Taking into account relationships (10.30) *B*-parameters are not given in the Table.

Table 10.1

		From which form				
		A	*H*	*G*	*Y*	*Z*
To which form	*A*	$\begin{matrix}\underline{A}_{11} & \underline{A}_{12}\\ \underline{A}_{21} & \underline{A}_{22}\end{matrix}$	$\begin{matrix}\dfrac{-\Delta_H}{\underline{H}_{21}} & \dfrac{\underline{H}_{11}}{\underline{H}_{21}}\\ \dfrac{-\underline{H}_{22}}{\underline{H}_{21}} & \dfrac{-1}{\underline{H}_{21}}\end{matrix}$	$\begin{matrix}\dfrac{1}{\underline{G}_{21}} & \dfrac{\underline{G}_{22}}{\underline{G}_{21}}\\ \dfrac{\underline{G}_{11}}{\underline{G}_{21}} & \dfrac{\Delta_G}{\underline{G}_{21}}\end{matrix}$	$\begin{matrix}\dfrac{-\underline{Y}_{22}}{\underline{Y}_{21}} & \dfrac{-1}{\underline{Y}_{21}}\\ \dfrac{-\Delta_Y}{\underline{Y}_{21}} & \dfrac{-\underline{Y}_{11}}{\underline{Y}_{21}}\end{matrix}$	$\begin{matrix}\dfrac{\underline{Z}_{11}}{\underline{Z}_{21}} & \dfrac{\Delta_Z}{\underline{Z}_{21}}\\ \dfrac{1}{\underline{Z}_{21}} & \dfrac{\underline{Z}_{22}}{\underline{Z}_{21}}\end{matrix}$
	H	$\begin{matrix}\dfrac{\underline{A}_{12}}{\underline{A}_{22}} & \dfrac{1}{\underline{A}_{22}}\\ \dfrac{-1}{\underline{A}_{22}} & \dfrac{\underline{A}_{21}}{\underline{A}_{22}}\end{matrix}$	$\begin{matrix}\underline{H}_{11} & \underline{H}_{12}\\ \underline{H}_{21} & \underline{H}_{22}\end{matrix}$	$\begin{matrix}\dfrac{\underline{G}_{22}}{\Delta_G} & \dfrac{-\underline{G}_{12}}{\Delta_G}\\ \dfrac{-\underline{G}_{21}}{\Delta_G} & \dfrac{\underline{G}_{11}}{\Delta_G}\end{matrix}$	$\begin{matrix}\dfrac{1}{\underline{Y}_{11}} & \dfrac{-\underline{Y}_{12}}{\underline{Y}_{11}}\\ \dfrac{\underline{Y}_{21}}{\underline{Y}_{11}} & \dfrac{\Delta_Y}{\underline{Y}_{11}}\end{matrix}$	$\begin{matrix}\dfrac{\Delta_Z}{\underline{Z}_{22}} & \dfrac{\underline{Z}_{12}}{\underline{Z}_{22}}\\ \dfrac{-\underline{Z}_{21}}{\underline{Z}_{22}} & \dfrac{1}{\underline{Z}_{22}}\end{matrix}$
	G	$\begin{matrix}\dfrac{\underline{A}_{21}}{\underline{A}_{11}} & \dfrac{-1}{\underline{A}_{11}}\\ \dfrac{1}{\underline{A}_{11}} & \dfrac{\underline{A}_{12}}{\underline{A}_{11}}\end{matrix}$	$\begin{matrix}\dfrac{\underline{H}_{22}}{\Delta_H} & \dfrac{-\underline{H}_{12}}{\Delta_H}\\ \dfrac{-\underline{H}_{21}}{\Delta_H} & \dfrac{\underline{H}_{11}}{\Delta_H}\end{matrix}$	$\begin{matrix}\underline{G}_{11} & \underline{G}_{12}\\ \underline{G}_{21} & \underline{G}_{22}\end{matrix}$	$\begin{matrix}\dfrac{\Delta_Y}{\underline{Y}_{22}} & \dfrac{\underline{Y}_{12}}{\underline{Y}_{22}}\\ \dfrac{-\underline{Y}_{21}}{\underline{Y}_{22}} & \dfrac{1}{\underline{Y}_{22}}\end{matrix}$	$\begin{matrix}\dfrac{1}{\underline{Z}_{11}} & \dfrac{-\underline{Z}_{12}}{\underline{Z}_{11}}\\ \dfrac{\underline{Z}_{21}}{\underline{Z}_{11}} & \dfrac{\Delta_Z}{\underline{Z}_{11}}\end{matrix}$
	Y	$\begin{matrix}\dfrac{\underline{A}_{22}}{\underline{A}_{12}} & \dfrac{-1}{\underline{A}_{12}}\\ \dfrac{-1}{\underline{A}_{12}} & \dfrac{\underline{A}_{11}}{\underline{A}_{12}}\end{matrix}$	$\begin{matrix}\dfrac{1}{\underline{H}_{11}} & \dfrac{-\underline{H}_{12}}{\underline{H}_{11}}\\ \dfrac{\underline{H}_{21}}{\underline{H}_{11}} & \dfrac{\Delta_H}{\underline{H}_{11}}\end{matrix}$	$\begin{matrix}\dfrac{\Delta_G}{\underline{G}_{22}} & \dfrac{\underline{G}_{12}}{\underline{G}_{22}}\\ \dfrac{-\underline{G}_{21}}{\underline{G}_{22}} & \dfrac{1}{\underline{G}_{22}}\end{matrix}$	$\begin{matrix}\underline{Y}_{11} & \underline{Y}_{12}\\ \underline{Y}_{21} & \underline{Y}_{22}\end{matrix}$	$\begin{matrix}\dfrac{\underline{Z}_{22}}{\Delta_Z} & \dfrac{-\underline{Z}_{12}}{\Delta_Z}\\ \dfrac{-\underline{Z}_{21}}{\Delta_Z} & \dfrac{\underline{Z}_{11}}{\Delta_Z}\end{matrix}$
	Z	$\begin{matrix}\dfrac{\underline{A}_{11}}{\underline{A}_{21}} & \dfrac{1}{\underline{A}_{21}}\\ \dfrac{1}{\underline{A}_{21}} & \dfrac{\underline{A}_{22}}{\underline{A}_{21}}\end{matrix}$	$\begin{matrix}\dfrac{\Delta_H}{\underline{H}_{22}} & \dfrac{\underline{H}_{12}}{\underline{H}_{22}}\\ \dfrac{-\underline{H}_{21}}{\underline{H}_{22}} & \dfrac{1}{\underline{H}_{22}}\end{matrix}$	$\begin{matrix}\dfrac{1}{\underline{G}_{11}} & \dfrac{-\underline{G}_{12}}{\underline{G}_{11}}\\ \dfrac{\underline{G}_{21}}{\underline{G}_{11}} & \dfrac{\Delta_G}{\underline{G}_{11}}\end{matrix}$	$\begin{matrix}\dfrac{\underline{Y}_{22}}{\Delta_Y} & \dfrac{-\underline{Y}_{12}}{\Delta_Y}\\ \dfrac{-\underline{Y}_{21}}{\Delta_Y} & \dfrac{\underline{Y}_{11}}{\Delta_Y}\end{matrix}$	$\begin{matrix}\underline{Z}_{11} & \underline{Z}_{12}\\ \underline{Z}_{21} & \underline{Z}_{22}\end{matrix}$

The following denotations are used in the Table 10.1:

$$\Delta_H = \underline{H}_{11}\underline{H}_{22} - \underline{H}_{12}\underline{H}_{21}; \qquad \Delta_Y = \underline{Y}_{11}\underline{Y}_{22} - \underline{Y}_{12}\underline{Y}_{21};$$
$$\Delta_Z = \underline{Z}_{11}\underline{Z}_{22} - \underline{Z}_{12}\underline{Z}_{21}; \qquad \Delta_G = \underline{G}_{11}\underline{G}_{22} - \underline{G}_{12}\underline{G}_{21}.$$

The analysis of relationships between parameters of four-terminal networks which are resulted in Table 10.1, shows that

$$\underline{H}_{12} = -\underline{H}_{21};\ \underline{G}_{12} = -\underline{G}_{21};\ \underline{Y}_{12} = \underline{Y}_{21};\ \underline{Z}_{12} = \underline{Z}_{21}. \tag{10.32}$$

From the formula (10.31) and expressions (10.32) follows, that only three of four parameters are independent under any notation of the equations of four-terminal networks. Thus, any passive asymmetric four-terminal network is set by three any parameters. The definition of parameters is carried out either by a calculation way if the internal structure of the four-terminal network and the parameters of its units are known, or by the experimental way under three conditions.

Example 10.4. The parameters of the asymmetrical four-terminal network are

$$\underline{A}_{11} = -j10\ r.un.,\ \underline{A}_{12} = 20\ \ \Omega,\ \underline{A}_{21} = j0{,}05\ \ Sm.$$

Determine the parameter $\underline{A}_{22}$.

Solution. From the formula (10.31) it follows

$$\underline{A}_{22} = \frac{1+\underline{A}_{12}\cdot\underline{A}_{21}}{\underline{A}_{11}} = \frac{1+20\cdot j0{,}05}{-j10} = \frac{1+j1}{-j10} =$$
$$= -0.1 + j0.1 = 0.141e^{j135^o}\ r.un.$$

10.6. The transfer functions of a four-terminal network.

As transfer function of the loaded four-terminal network is called the ratio of complex magnitudes (or complex effective values) of output current or voltage to complex magnitudes (or complex effective values) of input current or voltage.

If the transfer function represents the ratio of two electric magnitudes of the same name it is termed as a *transfer ratio*. If load impedance and the parameters of a four-terminal network are known then transfer ratios can be presented as follows

$$\begin{aligned} \underline{k}_U &= \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{U}_2}{\underline{A}_{11}\underline{U}_2 + \underline{A}_{12}\underline{I}_2} = \frac{\underline{Z}_l}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}}; \\ \underline{k}_I &= \frac{\underline{I}_2}{\underline{I}_1} = \frac{\underline{I}_2}{\underline{A}_{21}\underline{U}_2 + \underline{A}_{22}\underline{I}_2} = \frac{1}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{22}} \end{aligned} \tag{10.43}$$

where $\underline{Z}_l$ is a load impedance.

If the transfer function represents the ratio of two opposite electric magnitudes it is either *transmitting impedance* or *transmitting conductivity*:

$$\begin{aligned} \underline{Z}_{tr} &= \frac{\underline{U}_2}{\underline{I}_1} = \frac{\underline{U}_2}{\underline{A}_{21}\underline{U}_2 + \underline{A}_{22}\underline{I}_2} = \frac{\underline{Z}_l}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{22}}; \\ \underline{Y}_{tr} &= \frac{\underline{I}_2}{\underline{U}_1} = \frac{\underline{I}_2}{\underline{A}_{11}\underline{U}_2 + \underline{A}_{12}\underline{I}_2} = \frac{1}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}}. \end{aligned} \tag{10.44}$$

The unit of measure for the transfer ratios $\underline{k}_U$ and $\underline{k}_I$ are relative unit (*r.un.*), for transmitting impedance $\underline{Z}_{tr}$ is *Ohm* (*Ω*), and for transmitting conductivity $\underline{Y}_{tr}$ is *Siemens* (*Sm*).

10.7. The ways of joint of passive four–terminal networks

Let us consider different connections of passive four–terminal networks more in detail. *A compound* four-terminal network may be organized as a result of the joint of two (or several) four-terminal networks. The parameters of such a composite four-terminal network can be determined if the parameters of each making four-terminal network are known. Depending on the circuit joint the calculation of parameters of equivalent four-terminal network is made using corresponding equations in a matrix form. In general, it is made out cascade, parallel, series, series-parallel and parallel-serial joints.

The cascade connection of four-terminal networks is the most important in practice. It is the connection when the output terminals of the first four-terminal network are joined to the matching input terminals of the second one (Fig. 10.7).

Such an equivalent joint of two simple four-terminal networks can be observed as the new compound four-terminal network which parameters are necessary to determine.

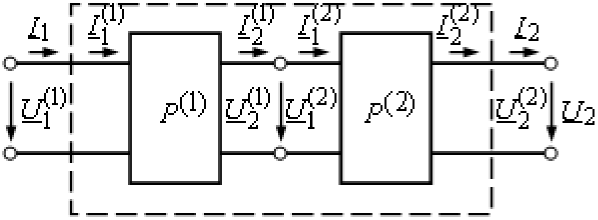


Fig.10.7. The cascade connection of four-terminal networks.

As we can see from the circuit of this cascade connection $\underline{U}_2^{(1)} = \underline{U}_1^{(2)}$; $\underline{I}_2^{(1)} = \underline{I}_1^{(2)}$, that is why it is convenient to use *A*-parameters in a matrix form:

$$\begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{I}_1^{(1)} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11}^{(1)} & \underline{A}_{12}^{(1)} \\ \underline{A}_{21}^{(1)} & \underline{A}_{22}^{(1)} \end{vmatrix} \cdot \begin{vmatrix} \underline{U}_2^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix} = \left|\underline{A}\right|^{(1)} \cdot \begin{vmatrix} \underline{U}_2^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix};$$

$$\begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{I}_1^{(2)} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11}^{(2)} & \underline{A}_{12}^{(2)} \\ \underline{A}_{21}^{(2)} & \underline{A}_{22}^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \underline{U}_2^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix} = \left|\underline{A}\right|^{(2)} \cdot \begin{vmatrix} \underline{U}_2^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix}. \tag{10.45}$$

As for the cascade connection $\underline{U}_1 = \underline{U}_1^{(1)}$; $\underline{I}_1 = \underline{I}_1^{(1)}$ and $\underline{U}_2^{(2)} = \underline{U}_2$; $\underline{I}_2^{(2)} = \underline{I}_2$, then

$$\begin{vmatrix} \underline{U}_1 \\ \underline{I}_1 \end{vmatrix} = \left|\underline{A}\right|^{(1)} \cdot \left|\underline{A}\right|^{(2)} \cdot \begin{vmatrix} \underline{U}_2 \\ \underline{I}_2 \end{vmatrix} = \left|\underline{A}\right| \cdot \begin{vmatrix} \underline{U}_2 \\ \underline{I}_2 \end{vmatrix} \tag{10.46}$$

Hence, the matrix of *A*-parameters of the equations of the resulting equivalent four-terminal network is equal to the product of the matrixes of *A*-parameters of component four-terminal networks:

$$\left|\underline{A}\right| = \left|\underline{A}\right|^{(1)} \cdot \left|\underline{A}\right|^{(2)} = \begin{vmatrix} \underline{A}_{11}^{(1)} \cdot \underline{A}_{11}^{(2)} + \underline{A}_{12}^{(1)} \cdot \underline{A}_{21}^{(2)} & \underline{A}_{11}^{(1)} \cdot \underline{A}_{12}^{(2)} + \underline{A}_{12}^{(1)} \cdot \underline{A}_{22}^{(2)} \\ \underline{A}_{21}^{(1)} \cdot \underline{A}_{11}^{(2)} + \underline{A}_{22}^{(1)} \cdot \underline{A}_{21}^{(2)} & \underline{A}_{21}^{(1)} \cdot \underline{A}_{12}^{(2)} + \underline{A}_{22}^{(1)} \cdot \underline{A}_{22}^{(2)} \end{vmatrix} \tag{10.47}$$

In case of cascade connection of *N* four-terminal networks, the matrix of parameters of *A*-form of the equivalent four-terminal network is obtained by consecutive multiplication of *A*-parameter matrixes of the cascades in their turn:

$$|\underline{A}| = |\underline{A}|^{(1)} \cdot |\underline{A}|^{(2)} \cdot \ldots \cdot |\underline{A}|^{(N)} = \prod_{n=1}^{N} |\underline{A}|^{(n)} \qquad (10.48)$$

We have the *parallel* connection when both input and output terminals of four-terminal networks are joined in parallel (Fig.10.8).

To determine the parameters of the equivalent (concerning terminals 1-1', 2-2') four-terminal network it is convenient to use *Y*-parameters as input and output voltages of four-terminal networks are equal: $\underline{U}_1 = \underline{U}_1^{(1)} = \underline{U}_1^{(2)}$ и $\underline{U}_2 = \underline{U}_2^{(1)} = \underline{U}_2^{(2)}$.

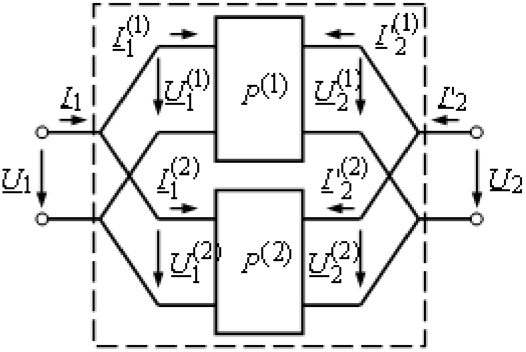


Fig.10.8. The parallel connection of four-terminal networks.

Y-parameters in a matrix form:

$$\begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix} = \begin{vmatrix} \underline{Y}_{11}^{(1)} & \underline{Y}_{12}^{(1)} \\ \underline{Y}_{21}^{(1)} & \underline{Y}_{22}^{(1)} \end{vmatrix} \cdot \begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix} = |\underline{Y}|^{(1)} \cdot \begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix};$$

$$\begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix} = \begin{vmatrix} \underline{Y}_{11}^{(2)} & \underline{Y}_{12}^{(2)} \\ \underline{Y}_{21}^{(2)} & \underline{Y}_{22}^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix} = |\underline{Y}|^{(2)} \cdot \begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix}. \qquad (10.49)$$

By the first Kirchhoff's law for terminals 1 and 2 we can write $\underline{I}_1 = \underline{I}_1^{(1)} + \underline{I}_1^{(2)}$ и $\underline{I}_2 = \underline{I}_2^{(1)} + \underline{I}_2^{(2)}$. Then in a matrix form:

$$\begin{vmatrix} \underline{I}_1 \\ \underline{I}_2 \end{vmatrix} = \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix} = \left(\begin{vmatrix} \underline{Y}_{11}^{(1)} & \underline{Y}_{12}^{(1)} \\ \underline{Y}_{21}^{(1)} & \underline{Y}_{22}^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{Y}_{11}^{(2)} & \underline{Y}_{12}^{(2)} \\ \underline{Y}_{21}^{(2)} & \underline{Y}_{22}^{(2)} \end{vmatrix} \right) \cdot \begin{vmatrix} \underline{U}_1 \\ \underline{U}_2 \end{vmatrix} = |\underline{Y}| \cdot \begin{vmatrix} \underline{U}_1 \\ \underline{U}_2 \end{vmatrix}. \qquad (10.50)$$

Hence, the matrix of *Y*-parameters of the equations of the resulting equivalent four-terminal network is equal to the sum of the matrixes of *Y*-parameters of component four-terminal networks:

$$|\underline{Y}| = |\underline{Y}|^{(1)} + |\underline{Y}|^{(2)} = \begin{vmatrix} \underline{Y}_{11}^{(1)} + \underline{Y}_{11}^{(2)} & \underline{Y}_{12}^{(1)} + \underline{Y}_{12}^{(2)} \\ \underline{Y}_{21}^{(1)} + \underline{Y}_{21}^{(2)} & \underline{Y}_{22}^{(1)} + \underline{Y}_{22}^{(2)} \end{vmatrix}. \tag{10.51}$$

We have the *series* connection when both input and output terminals of four-terminal networks are joined in series (Fig.10.9). To determine the parameters of the equivalent (concerning terminals 1-1', 2-2') four-terminal network it is convenient to use *Z*-parameters as input and output currents of four-terminal networks are equal: $\underline{I}_1 = \underline{I}_1^{(1)} = \underline{I}_1^{(2)}$ и $\underline{I}_2 = \underline{I}_2^{(1)} = \underline{I}_2^{(2)}$.

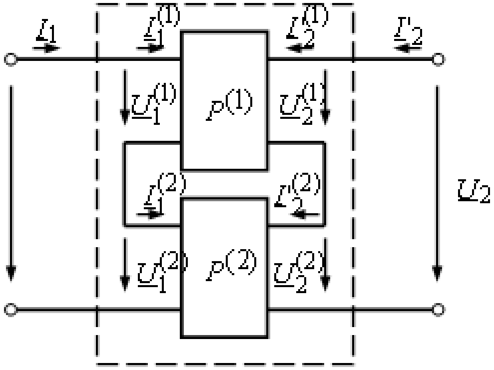


Fig.10.9. The series connection of four-terminal networks.

Z-parameters in a matrix form:

$$\begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix} = \begin{vmatrix} \underline{Z}_{11}^{(1)} & \underline{Z}_{12}^{(1)} \\ \underline{Z}_{21}^{(1)} & \underline{Z}_{22}^{(1)} \end{vmatrix} \cdot \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix} = |\underline{Z}|^{(1)} \cdot \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix};$$
$$\begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix} = \begin{vmatrix} \underline{Z}_{11}^{(2)} & \underline{Z}_{12}^{(2)} \\ \underline{Z}_{21}^{(2)} & \underline{Z}_{22}^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix} = |\underline{Z}|^{(2)} \cdot \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix}. \tag{10.52}$$

By the second Kirchhoff's law we can write. $\underline{U}_1 = \underline{U}_1^{(1)} + \underline{U}_1^{(2)}$ and $\underline{U}_2 = \underline{U}_2^{(1)} + \underline{U}_2^{(2)}$. Then in a matrix form:

$$\begin{vmatrix} \underline{U}_1 \\ \underline{U}_2 \end{vmatrix} = \begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix} = \left(\begin{vmatrix} \underline{Z}_{11}^{(1)} & \underline{Z}_{12}^{(1)} \\ \underline{Z}_{21}^{(1)} & \underline{Z}_{22}^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{Z}_{11}^{(2)} & \underline{Z}_{12}^{(2)} \\ \underline{Z}_{21}^{(2)} & \underline{Z}_{22}^{(2)} \end{vmatrix} \right) \cdot \begin{vmatrix} \underline{I}_1 \\ \underline{I}_2 \end{vmatrix} = |\underline{Z}| \cdot \begin{vmatrix} \underline{I}_1 \\ \underline{I}_2 \end{vmatrix}. \tag{10.53}$$

Hence, the matrix of *Z*-parameters of the equations of the resulting equivalent four-terminal network is equal to the sum of the matrixes of *Z*-parameters of component four-terminal networks:

$$\left|\underline{Z}\right| = \left|\underline{Z}\right|^{(1)} + \left|\underline{Z}\right|^{(2)} = \begin{vmatrix} \underline{Z}_{11}^{(1)} + \underline{Z}_{11}^{(2)} & \underline{Z}_{12}^{(1)} + \underline{Z}_{12}^{(2)} \\ \underline{Z}_{21}^{(1)} + \underline{Z}_{21}^{(2)} & \underline{Z}_{22}^{(1)} + \underline{Z}_{22}^{(2)} \end{vmatrix}. \qquad (10.54)$$

We have the *series-parallel* connection when input terminals of four-terminal networks are joined in series and output terminals are joined in parallel (Fig.10.10).

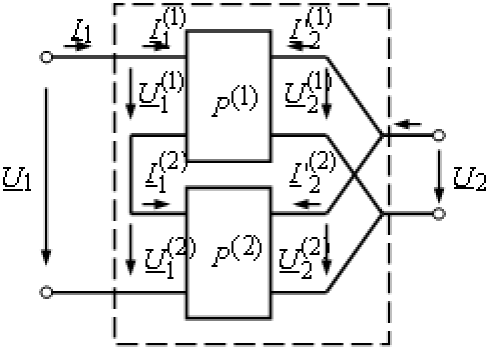


Fig.10.10. The series-parallel connection of four-terminal networks.

To determine the parameters of the equivalent (concerning terminals 1-1', 2-2') four-terminal network it is convenient to use *H*-parameters as input currents and output voltages are equal: $\underline{I}_1 = \underline{I}_1^{(1)} = \underline{I}_1^{(2)}$ и $\underline{U}_2 = \underline{U}_2^{(1)} = \underline{U}_2^{(2)}$.

H-parameters in a matrix form:

$$\begin{vmatrix} \underline{U}_1^{(1)} \\ \underline{I}_2^{(1)} \end{vmatrix} = \begin{vmatrix} \underline{H}_{11}^{(1)} & \underline{H}_{12}^{(1)} \\ \underline{H}_{21}^{(1)} & \underline{H}_{22}^{(1)} \end{vmatrix} \cdot \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix} = \left|\underline{H}\right|^{(1)} \cdot \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix};$$
$$\begin{vmatrix} \underline{U}_1^{(2)} \\ \underline{I}_2^{(2)} \end{vmatrix} = \begin{vmatrix} \underline{H}_{11}^{(2)} & \underline{H}_{12}^{(2)} \\ \underline{H}_{21}^{(2)} & \underline{H}_{22}^{(2)} \end{vmatrix} \cdot \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix} = \left|\underline{H}\right|^{(2)} \cdot \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix}. \qquad (10.55)$$

By the first and the second Kirchhoff's laws we can write $\underline{U}_1 = \underline{U}_1^{(1)} + \underline{U}_1^{(2)}$ и $\underline{I}_2 = \underline{I}_2^{(1)} + \underline{I}_2^{(2)}$.

Then in a matrix form

$$\begin{vmatrix}\underline{U}_1\\ \underline{I}_2\end{vmatrix}=\begin{vmatrix}\underline{U}_1^{(1)}\\ \underline{I}_2^{(1)}\end{vmatrix}+\begin{vmatrix}\underline{U}_1^{(2)}\\ \underline{I}_2^{(2)}\end{vmatrix}=\left(\begin{vmatrix}\underline{H}_{11}^{(1)} & \underline{H}_{12}^{(1)}\\ \underline{H}_{21}^{(1)} & \underline{H}_{22}^{(1)}\end{vmatrix}+\begin{vmatrix}\underline{H}_{11}^{(2)} & \underline{H}_{12}^{(2)}\\ \underline{H}_{21}^{(2)} & \underline{H}_{22}^{(2)}\end{vmatrix}\right)\cdot\begin{vmatrix}\underline{I}_1\\ \underline{U}_2\end{vmatrix}=|\underline{H}|\cdot\begin{vmatrix}\underline{I}_1\\ \underline{U}_2\end{vmatrix} \qquad (10.56)$$

Hence, the matrix of *H*-parameters of the equations of the resulting equivalent four-terminal network is equal to the sum of the matrixes of *H*-parameters of component four-terminal networks:

$$|\underline{H}|=|\underline{H}|^{(1)}+|\underline{H}|^{(2)}=\begin{vmatrix}\underline{H}_{11}^{(1)}+\underline{H}_{11}^{(2)} & \underline{H}_{12}^{(1)}+\underline{H}_{12}^{(2)}\\ \underline{H}_{21}^{(1)}+\underline{H}_{21}^{(2)} & \underline{H}_{22}^{(1)}+\underline{H}_{22}^{(2)}\end{vmatrix}. \qquad (10.57)$$

We have the *parallel-series* connection when input terminals of four-terminal networks are joined in parallel and output terminals are joined in series (Fig.10.11).

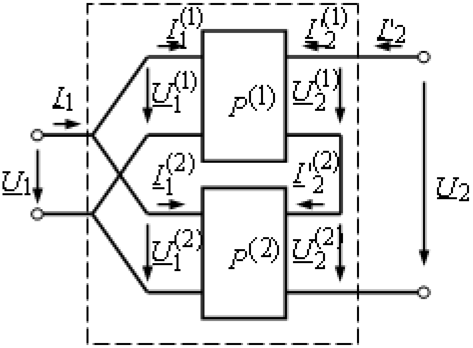


Fig.10.11. The parallel- series connection of four-terminal networks.

To determine the parameters of the equivalent (concerning terminals 1-1', 2-2') four-terminal network it is convenient to use *G*-parameters as input voltages and input currents are equal: $\underline{U}_1=\underline{U}_1^{(1)}=\underline{U}_1^{(2)}$ и $\underline{I}_2=\underline{I}_2^{(1)}=\underline{I}_2^{(2)}$.

G-parameters in a matrix form:

$$\begin{vmatrix}\underline{I}_1^{(1)}\\ \underline{U}_2^{(1)}\end{vmatrix}=\begin{vmatrix}\underline{G}_{11}^{(1)} & \underline{G}_{12}^{(1)}\\ \underline{G}_{21}^{(1)} & \underline{G}_{22}^{(1)}\end{vmatrix}\cdot\begin{vmatrix}\underline{U}_1^{(1)}\\ \underline{I}_2^{(1)}\end{vmatrix}=|\underline{G}|^{(1)}\cdot\begin{vmatrix}\underline{U}_1^{(1)}\\ \underline{I}_2^{(1)}\end{vmatrix};$$

$$\begin{vmatrix}\underline{I}_1^{(2)}\\ \underline{U}_2^{(2)}\end{vmatrix}=\begin{vmatrix}\underline{G}_{11}^{(2)} & \underline{G}_{12}^{(2)}\\ \underline{G}_{21}^{(2)} & \underline{G}_{22}^{(2)}\end{vmatrix}\cdot\begin{vmatrix}\underline{U}_1^{(2)}\\ \underline{I}_2^{(2)}\end{vmatrix}=|\underline{G}|^{(2)}\cdot\begin{vmatrix}\underline{U}_1^{(2)}\\ \underline{I}_2^{(2)}\end{vmatrix}. \qquad (10.58)$$

By the first and the second Kirchhoff's laws we can write $\underline{I}_1 = \underline{I}_1^{(1)} + \underline{I}_1^{(2)}$ and $\underline{U}_2 = \underline{U}_2^{(1)} + \underline{U}_2^{(2)}$. Then in a matrix form:

$$\begin{vmatrix} \underline{I}_1 \\ \underline{U}_2 \end{vmatrix} = \begin{vmatrix} \underline{I}_1^{(1)} \\ \underline{U}_2^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{I}_1^{(2)} \\ \underline{U}_2^{(2)} \end{vmatrix} = \left(\begin{vmatrix} \underline{G}_{11}^{(1)} & \underline{G}_{12}^{(1)} \\ \underline{G}_{21}^{(1)} & \underline{G}_{22}^{(1)} \end{vmatrix} + \begin{vmatrix} \underline{G}_{11}^{(2)} & \underline{G}_{12}^{(2)} \\ \underline{G}_{21}^{(2)} & \underline{G}_{22}^{(2)} \end{vmatrix} \right) \cdot \begin{vmatrix} \underline{U}_1 \\ \underline{I}'_2 \end{vmatrix} = \left|\underline{G}\right| \cdot \begin{vmatrix} \underline{U}_1 \\ \underline{I}'_2 \end{vmatrix} \qquad (10.59)$$

Hence, the matrix of *G*-parameters of the equations of the resulting equivalent four-terminal network is equal to the sum of the matrixes of *G*-parameters of component four-terminal networks:

$$\left|\underline{G}\right| = \left|\underline{G}\right|^{(1)} + \left|\underline{G}\right|^{(2)} = \begin{vmatrix} \underline{G}_{11}^{(1)} + \underline{G}_{11}^{(2)} & \underline{G}_{12}^{(1)} + \underline{G}_{12}^{(2)} \\ \underline{G}_{21}^{(1)} + \underline{G}_{21}^{(2)} & \underline{G}_{22}^{(1)} + \underline{G}_{22}^{(2)} \end{vmatrix}. \qquad (10.60)$$

It is necessary to mean that the specified formulas of matrix determination of the complicated four-terminal networks are correct only when the *regularity condition* of their connection is fulfilled The four-terminal network connection is regularly in the case when the currents which are flowing past through both primary and secondary terminals of each of four-terminal networks, are equal on magnitude and are opposite in direction.

Example 10.5. Two equal four-terminal networks with the parameters $\underline{A}_{11} = j4$ *r.un.*; $\underline{A}_{12} = 100 - j20\,\Omega$; $\underline{A}_{21} = j0{,}05\,Sm$ and $\underline{A}_{22} = 5$ *r.un.* are connected in cascade so that the output terminals of the first four-terminal network are joined with the input terminals of the second four-terminal network (Fig. 10.7). Determine *A*-parameters of equivalent four-terminal network.

Solution: Use Eq. (10.47) taking into account that we have two equal four-terminal networks:

$$\left|\underline{A}\right|_{eq} = \begin{vmatrix} \underline{A}_{11} \cdot \underline{\underline{A}}_{11} + \underline{A}_{12} \cdot \underline{\underline{A}}_{21} & \underline{A}_{11} \cdot \underline{\underline{A}}_{12} + \underline{A}_{12} \cdot \underline{\underline{A}}_{22} \\ \underline{A}_{21} \cdot \underline{\underline{A}}_{11} + \underline{A}_{22} \cdot \underline{\underline{A}}_{21} & \underline{A}_{21} \cdot \underline{\underline{A}}_{12} + \underline{A}_{22} \cdot \underline{\underline{A}}_{22} \end{vmatrix} = \begin{vmatrix} \underline{A}_{11eq} & \underline{A}_{12eq} \\ \underline{A}_{21eq} & \underline{A}_{22eq} \end{vmatrix}$$

Then *A*-parameters of equivalent four-terminal network:

$$\underline{A}_{11eq} = \underline{A}_{11} \cdot \underline{\underline{A}}_{11} + \underline{A}_{12} \cdot \underline{\underline{A}}_{21} = j1 \cdot j1 + (100 - j20) \cdot j0{,}05 =$$

$$= j5 = 5e^{j90^o} \ \ r.un.;$$

$$\underline{A}_{12_{eq}} = \underline{A}_{11}\cdot\underline{\underline{A}}_{12} + \underline{A}_{12}\cdot\underline{\underline{A}}_{22} = j1\cdot(100 - j20) + (100 - j20)\cdot 5 =$$

$$= 520 = 520e^{j0^{o}} \quad \Omega;$$

$$\underline{A}_{21_{eq}} = \underline{A}_{21}\cdot\underline{\underline{A}}_{11} + \underline{A}_{22}\cdot\underline{\underline{A}}_{21} = j0{,}05\cdot j1 + 5\cdot j0{,}05 =$$

$$= -0{,}05 + j0{,}25 = 0{,}255e^{j101^{o}} \quad Sm;$$

$$\underline{A}_{22_{eq}} = \underline{A}_{21}\cdot\underline{\underline{A}}_{12} + \underline{A}_{22}\cdot\underline{\underline{A}}_{22} = j0{,}05\cdot(100 - j20) + 5\cdot 5 =$$

$$= 26 + j5 = 26{,}48e^{11^{o}} \quad r.un.$$

We will check up the correctness of the calculation by Eq. (10.31):

$$\underline{A}_{11_{eq}}\underline{A}_{22_{eq}} - \underline{A}_{12_{eq}}\underline{A}_{21_{eq}} = j5\cdot(26 + j5) - 520\cdot(-0{,}05 + 0{,}25) = 1.$$

As we can see, the calculation is correct.

10.8. Symmetrical four-terminal networks

As we have already known, a four-terminal network is symmetrical if the change by places of input and output terminals does not change currents and voltages in the electric circuit to which the four-terminal network is connected.

There is a condition for symmetrical four-terminal network:

$$\underline{A}_{11} = \underline{A}_{22} = \underline{B}_{11} = \underline{B}_{22} = \underline{A} = \underline{D}\,. \qquad (10.61)$$

That is why Eqs. (10.1) и (10.2) are identical in this case. For such a four-terminal network the number of independent parameters is two. All the relationships which characterize a four-terminal network earlier, remain just to a symmetrical four-terminal network, taking into account Eq.(10.61).

The relationship between *A*-form parameters

$$\underline{A}_{11}^{2} - \underline{A}_{12}\underline{A}_{21} = 1\,. \qquad (10.62)$$

The system of equations of direct and inverse transmission

$$\begin{cases} \underline{U}_1 = \underline{A}_{11}\underline{U}_2 + \underline{A}_{12}\underline{I}_2; \\ \underline{I}_1 = \underline{A}_{21}\underline{U}_2 + \underline{A}_{11}\underline{I}_2; \end{cases} \qquad \begin{cases} \underline{U}_2 = \underline{A}_{11}\underline{U}_1 + \underline{A}_{12}\underline{I}'_1; \\ \underline{I}'_2 = \underline{A}_{21}\underline{U}_1 + \underline{A}_{11}\underline{I}'_1. \end{cases} \qquad (10.63)$$

Input impedances

$$\underline{Z}_{in1} = \underline{Z}_{in2} = \underline{Z}_{in} = \frac{\underline{U}_1}{\underline{I}_1} = \frac{\underline{U}_2}{\underline{I}'_2} = \frac{\underline{A}_{11}\underline{Z}_{l2} + \underline{A}_{12}}{\underline{A}_{21}\underline{Z}_{l2} + \underline{A}_{11}} = \frac{\underline{A}\underline{Z}_{l2} + \underline{B}}{\underline{C}\underline{Z}_{l2} + \underline{D}}. \qquad (10.64)$$

The characteristic equations

$$\underline{Z}_{C1} = \underline{Z}_{C2} = \underline{Z}_C = \sqrt{\frac{\underline{A}_{12}}{\underline{A}_{21}}}. \qquad (10.65)$$

The coefficient of transformation of a symmetrical four-terminal network taking into account Eq (10.65) $k_{tr} = 1$.

The transfer functions

$$\begin{aligned} \underline{k}_U &= \frac{\underline{U}_2}{\underline{U}_1} = \frac{\underline{Z}_l}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}}; \\ \underline{k}_I &= \frac{\underline{I}_2}{\underline{I}_1} = \frac{1}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{11}} \end{aligned} \qquad (10.66)$$

The transmitting impedance

$$\underline{Z}_{tr} = \frac{\underline{U}_2}{\underline{I}_1} = \frac{\underline{Z}_l}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{11}}. \qquad (10.67)$$

The transmitting admittance

$$\underline{Y}_{tr} = \frac{\underline{I}_2}{\underline{U}_1} = \frac{1}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}}. \qquad (10.68)$$

Example 10.5. The symmetrical four-terminal network has the following parameters: $\underline{A}_{11} = 2$ r.un. and $A_{12} = j5\ \Omega$ for the.

Determine a characteristic impedance and transition functions at a coordinated load.

Solution. Using Eq. (10.62) we will find the parameter $\underline{A}_{21}$

$$\underline{A}_{21} = \frac{\underline{A}_{11}^2 + 1}{\underline{A}_{12}} = \frac{2^2 + 1}{j5} = -j1\ Sm.$$

Using Eq. (10.65) we will calculate the characteristic impedance

$$\underline{Z}_C = \sqrt{\frac{\underline{A}_{12}}{\underline{A}_{21}}} = \sqrt{\frac{j5}{-j1}} = 2{,}24j\ \Omega.$$

The impedance of coordinated load

$$\underline{Z}_l = \underline{Z}_C = 2{,}24j\ \Omega.$$

The transfer functions

$$\underline{k}_U = \frac{\underline{Z}_l}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}} = \frac{2{,}24j}{2\cdot 2{,}24j + j5} = 0{,}236 \ \ r.un.$$

$$\underline{k}_I = \frac{1}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{11}} = \frac{1}{-j1\cdot 2{,}24j + 2} = 0{,}236 \ \ r.un.$$

The transmitting impedance

$$\underline{Z}_{tr} = \frac{\underline{Z}_l}{\underline{A}_{21}\underline{Z}_l + \underline{A}_{11}} = \frac{2{,}24j}{-j1\cdot 2{,}24j + 2} = 0{,}528j = 0{,}528e^{j90^o}\,\Omega\,.$$

The transmitting admittance

$$\underline{Y}_{tr} = \frac{1}{\underline{A}_{11}\underline{Z}_l + \underline{A}_{12}} = \frac{1}{2\cdot 2{,}24j + j5} = -j0{,}105 = 0{,}105e^{-j90^o} \ \ Sm.$$

10.9. Bridge four-terminal networks

Bridge four-terminal networks (Fig. 10.6, *d)* are widely used at the analysis of passive symmetrical four-terminal networks.

It proved, that for any passive symmetrical four-terminal network it is possible to find equivalent symmetrical bridge (Fig. 10.12, *a)*.

The bridge four-terminal network may be presented as a parallel connection of two simple four-terminal networks (Fig. 10.12, *b)*. We will make on the basis of Kirchhoff's laws of simultaneous equations of the A-form of each of these four-terminal networks

Compose the system of *A*-form equations for each of these four-terminal networks by Kirchhoff's laws:

$$\begin{cases}\underline{U}_1^{(1)} = \underline{U}_2^{(1)} - 2\underline{Z}_1\underline{I}'^{(1)}_2; \\ \underline{I}_1^{(1)} = 0\cdot\underline{U}_2^{(1)} - \underline{I}'^{(1)}_2;\end{cases} \quad \begin{cases}\underline{U}_1^{(2)} = -\underline{U}_2^{(2)} + 2\underline{Z}_2\underline{I}'^{(2)}_2; \\ \underline{I}'^{(2)}_1 = 0\cdot\underline{U}_2^{(2)} + \underline{I}'^{(2)}_2.\end{cases} \qquad (10.69)$$

Taking into account Eqs. (10.69), we will transfer to the system of *Y*-form equations for each of the elementary four-terminal networks:

$$\begin{cases}\underline{I}_1^{(1)} = \frac{1}{2\underline{Z}_1}\underline{U}_1^{(1)} - \frac{1}{2\underline{Z}_1}\underline{U}_2^{(1)}; \\ \underline{I}'^{(1)}_2 = -\frac{1}{2\underline{Z}_1}\underline{U}_1^{(1)} + \frac{1}{2\underline{Z}_1}\underline{U}_2^{(1)};\end{cases} \quad \begin{cases}\underline{I}_1^{(2)} = \frac{1}{2\underline{Z}_2}\underline{U}_1^{(2)} + \frac{1}{2\underline{Z}_2}\underline{U}_2^{(2)}; \\ \underline{I}'^{(2)}_2 = \frac{1}{2\underline{Z}_2}\underline{U}_1^{(2)} + \frac{1}{2\underline{Z}_2}\underline{U}_2^{(2)}.\end{cases} \qquad (10.70)$$

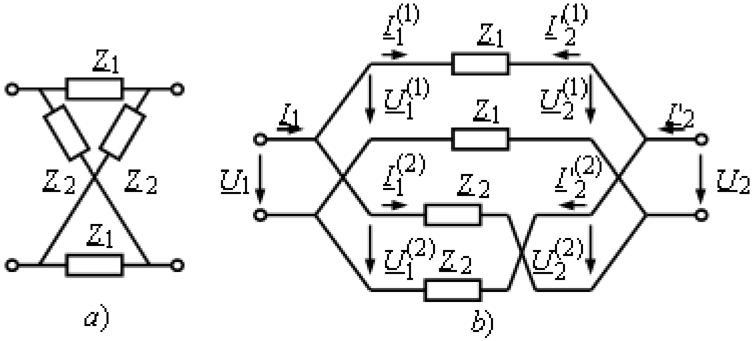


Fig.10.12. The bridge four-terminal network.

The matrix of *Y*-parameters of the equations of the equivalent bridge four-terminal network according to (10.51) is

$$|\underline{Y}| = \begin{vmatrix} \frac{1}{2\underline{Z}_1}+\frac{1}{2\underline{Z}_2} & -\frac{1}{2\underline{Z}_1}+\frac{1}{2\underline{Z}_2} \\ -\frac{1}{2\underline{Z}_1}+\frac{1}{2\underline{Z}_2} & \frac{1}{2\underline{Z}_1}+\frac{1}{2\underline{Z}_2} \end{vmatrix} = \begin{vmatrix} \frac{\underline{Z}_1+\underline{Z}_2}{2\underline{Z}_1\underline{Z}_2} & \frac{\underline{Z}_1-\underline{Z}_2}{2\underline{Z}_1\underline{Z}_2} \\ \frac{\underline{Z}_1-\underline{Z}_2}{2\underline{Z}_1\underline{Z}_2} & \frac{\underline{Z}_1+\underline{Z}_2}{2\underline{Z}_1\underline{Z}_2} \end{vmatrix}. \qquad (10.71)$$

Using Eqs. (10.30) we will transfer to a matrix of *A*-parameters

$$|\underline{A}| = \begin{vmatrix} \frac{\underline{Z}_1+\underline{Z}_2}{\underline{Z}_2-\underline{Z}_1} & \frac{2\underline{Z}_1\underline{Z}_2}{\underline{Z}_2-\underline{Z}_1} \\ \frac{2}{\underline{Z}_2-\underline{Z}_1} & \frac{\underline{Z}_1+\underline{Z}_2}{\underline{Z}_2-\underline{Z}_1} \end{vmatrix}. \qquad (10.72)$$

The characteristic equations is found by using Eq. (10.65)

$$\underline{Z}_C = \sqrt{\frac{\underline{A}_{12}}{\underline{A}_{21}}} = \sqrt{\underline{Z}_1\underline{Z}_2}\ . \qquad (10.73)$$

The transfer functions by voltage at the coordinated load

$$\underline{k}_U = \frac{\sqrt{\underline{Z}_2}-\sqrt{\underline{Z}_1}}{\sqrt{\underline{Z}_2}+\sqrt{\underline{Z}_1}} = \frac{\underline{Z}_C-\underline{Z}_1}{\underline{Z}_C+\underline{Z}_1}\ . \qquad (10.74)$$

When $\underline{Z}_1$ and $\underline{Z}_2$ are ideal reactive but heterogeneous (i.e. have different signs) the bridge four-terminal network possesses interesting properties. In this case the characteristic impedance is active:

$$\underline{Z}_C = \sqrt{\underline{Z}_1\underline{Z}_2} = \sqrt{-jX_1\cdot jX_2} = \sqrt{X_1X_2} = R_C \qquad (10.75)$$

Then the transfer functions by voltage at the coordinated load:

$$\underline{k}_U = \frac{\underline{Z}_C - \underline{Z}_1}{\underline{Z}_C + \underline{Z}_1} = \frac{R_C \mp jX_1}{R_C \pm jX_1}\,. \qquad (10.76)$$

From (10.76) it follows, that the module of the coefficient of transformation by voltage does not depend on frequency

$$\left|\underline{k}_U\right| = \sqrt{\frac{R_C^2 + X_1^2}{R_C^2 + X_1^2}} = 1$$

and, it means, that such a four-terminal network passes all frequencies without changing of their amplitudes.

10.10. The active four-terminal network

An active four-terminal network is called such a unit which contains some energy sources in itself, and the action of these sources is not compensated inside the four-terminal network.

The examples of active four-terminal networks are amplifiers and transmission lines with additional energy sources which are connected between its input and output terminals.

Input non-compensated sources excite voltages across one or both pairs of open terminals (or currents through one or both pairs of open terminals), in a case if the four-terminal network is disconnected from exterior electric circuits.

Any active four-terminal network can be replaced by equivalent passive four-terminal network, to input and output terminals of which either some additional equivalent sources of the e.m.f. are connected in series (Fig. 10.13), or some additional equivalent current sources are connected in parallel (Fig. 10.14).

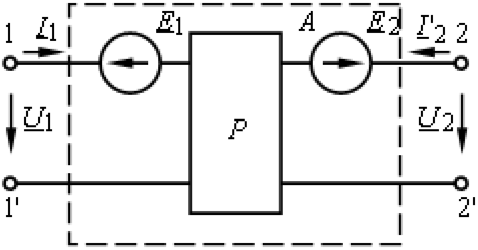


Fig. 10.13. Equivalent passive four-terminal network with e.m.f. sources.

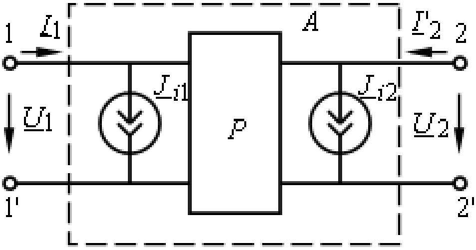


Fig. 10.14. Equivalent passive four-terminal networks with current sources.

For describing of an active network, the equivalent circuit of which is presented in Fig. 10.13, it is convenient to use the system of equations with *Z*-parameters

$$\begin{cases}\underline{U}_1 = \underline{\underline{Z}}_{11}\underline{I}_1 + \underline{\underline{Z}}_{12}\underline{I}'_2 + \underline{E}_{i1};\\ \underline{U}_2 = \underline{\underline{Z}}_{21}\underline{I}_1 + \underline{\underline{Z}}_{22}\underline{I}'_2 + \underline{E}_{i2},\end{cases} \tag{10.77}$$

and for describing of an active network, the equivalent circuit of which is presented in Fig. 10.14, it is convenient to use the system of equations with *Y*-parameters:

$$\begin{array}{l}\underline{I}_1 = \underline{\underline{Y}}_{11}\underline{U}_1 + \underline{\underline{Y}}_{12}\underline{U}_2 + \underline{J}_{i1};\\ \underline{I}'_2 = \underline{\underline{Y}}_{21}\underline{U}_1 + \underline{\underline{Y}}_{22}\underline{U}_2 + \underline{J}_{i2}.\end{array} \tag{10.78}$$

The asymmetric active four-terminal network is characterized by five independent parameters.

The parameters $\underline{Z}_{11}, \underline{Z}_{12}, \underline{Z}_{21}, \underline{Z}_{22}$ (or $\underline{Y}_{11}, \underline{Y}_{12}, \underline{Y}_{21}, \underline{Y}_{22}$) of its passive part correspond to the parameters of the passive four-terminal network and do not depend on internal energy sources. The parameters of internal energy sources $\underline{E}_{i1}$ and $\underline{E}_{i2}$ (or $\underline{J}_{i1}$ and $\underline{J}_{i2}$) depend both on energy sources inside a four-terminal network and on internal circuit design of the joint and the parameters of passive elements. Their values are determined by the calculation method.

Summary review questions

10.1 What is called a four-terminal network?

10.2. Distinguish between the active and the passive two-port networks.

10.3. Distinguish between the input and output terminals. How are they marked?

10.4. How are input voltage and current related to the output voltage and current?

10.5. What do linear parameters of the network depend on?

10.6 How many forms of the parameters for four-terminal network are there? Name these forms.

10.7. In which cases does one use different kinds of parameters for four-terminal networks?

10.8. What equations are used for determination of the *ABCD* parameters?

10.9. What is the procedure for determining the driving-point impedances of a four-terminal network?

10.10. What equivalent circuits of a passive four-terminal network do you know? How many impedances does the equivalent network contain?

10.11. Write six notations of the four-terminal network equations, specify the positive directions of currents and voltages for them.

10.12. Describe the advantages of each notation of equations before the remaining?

10.13. How many coefficients of the asymmetrical four-terminal network are independent?

10.14. How can one determine *A*- parameters by practical way?

10.15. If we know the coefficients of one notation, how can we define the coefficients of other form?

10.16. What is the characteristic parameter of the four-terminal network?

10.17. How can one determine the characteristic impedances of a four-terminal network?

10.18. What is termed as an input resistance of the four-terminal network?

10.19. What is called a transfer function of the loaded four-terminal network?

10.20. How may a compound four-terminal network be organized?

10.21. For what purpose are equivalent circuits of four-terminal networks used?

10.22. What four-terminal network is called a symmetrical one?

10.23. How can an active four-terminal network replaced?

10.24. What is the asymmetric active four-terminal network characterized by?

10.25. What do the parameters of internal energy sources depend on in the asymmetric active four-terminal network?

Problems

10.1. Define *A*-parameters of the four-terminal network (see Fig. P10.1): *a*) using Kirchhoff's laws; *b*) when there is an idling condition and short circuit in the four-terminal network.

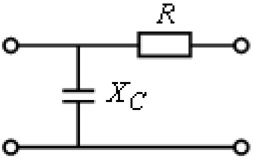


Fig. P10.1

10.2. Determine *A*-, *Z*-, *H*-parameters of symmetrical four-terminal networks in Fig. P10.2, *a*, *b*.

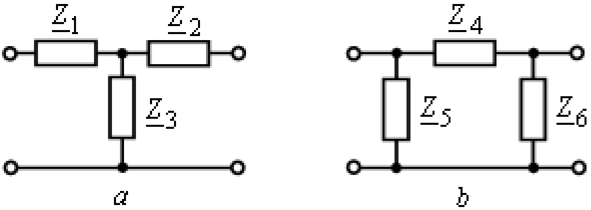


Fig. P10.2

10.3. Express impedances $\underline{Z}_1$, $\underline{Z}_2$, $\underline{Z}_3$ of the equivalent *T*-network and the impedances $\underline{Z}_4$, $\underline{Z}_5$, $\underline{Z}_6$ of the equivalent *P*-network of the four-terminal network (see Fig. P10.2) through *A*-parameters.

10.4. Express impedances of the equivalent *T*- and *P*-network of four-terminal network through *Z*-parameters.

10.5. Express *Z*-, *Y*-, *H*-, *B*-parameters through *A*-parameters.

10.6. Express *A*-, *Z*-, *H*-parameters of a four-terminal network through the impedances of direct open circuit, reverse open circuit and short circuit rate.

10.7. Calculate *Y*-parameters of the four-terminal network in Fig. P10.3 if $\underline{Z}_1=\underline{Z}_3=j20\ \Omega$; $\underline{Z}_2=j40\ \Omega$; $\underline{Z}_4=\underline{Z}_5=-j40\ \Omega$.

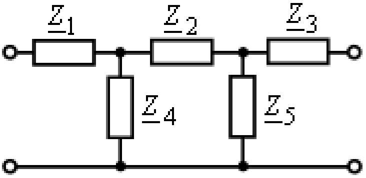


Fig. P10.3

10.8. The load connected to the terminals *ab* of the four-terminal network (Fig. P10.4) is $\underline{Z}_2$ = (6+j6) Ω. The voltage source is connected to the terminals *cd*. What should be the e.m.f. of the source to produce a load current (I_2) of 1 ampere?.

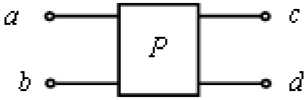


Fig. P10.4

10.9. The *A*-parameters of the passive four-terminal network are known: $\underline{A}_{11}=1-j$ *r.un*; $\underline{A}_{12}=-j10\ \Omega$; $A_{21}=-j0{,}1$ *Sm*. Determine the impedances of equivalent *T*-network.

10.10. There are the impedances of equivalent *T*-network: $\underline{Z}_1=\underline{Z}_2=10\ \Omega$; $\underline{Z}_3=-j10\ \Omega$. Determine, at which load impedance the input impedance of the four-terminal network is equal to the load impedance.

10.11. Substitute the four-terminal network resulted in Fig. P10.3 by the equivalent *P*-network.

10.12. 10.4. By measurement it has been found that $\underline{Z}_{10}=7.815e^{-j50^o}$ Ω; $\underline{Z}_{1k}=12.5e^{j66^o}$ Ω; and $\underline{Z}_{2k}=3.33e^{j27^o}$ Ω. Find the constants, *A*, *B*, *C* and *D* of the four-terminal network.

Appendix A

Answers to selected problems

Chapter 3

3.4. (*a*) 70 μF; (*b*) $14 \cdot 10^{-3}\ C$; (*c*) $Q_1 = 2 \cdot 10^{-3}$, $Q_2 = 4 \cdot 10^{-3}$, $Q_3 = 8 \cdot 10^{-3} C$. **3.5**. (*a*) 40/7 μF; (*b*) $2 \cdot 10^{-3}$, (*c*) U_1=200 V, U_2=100 V, U_3=50 V. **3.10**. 165V. **3.11**. 4.78 Ω. **3.13**. 18.75Ω. **3.14**. (a) 150 V; (b) 40 V. **3.15**. 15.44 Ω. **3.17**. 1.67 A from bottom to top. **3.18**. 0.448 A. **3.21**. 2.86 A. **3.23**. (*a*) $Q_1 = 240 \mu C$, $Q_2 = 180\ \mu C$, $Q_3 = 60\ \mu C$; (*c*) $W_1 = 4.8 \cdot 10^{-3} J$, $W_2 = 5.4 \cdot 10^{-3} J$, $W_3 = 1.8 \cdot 10^{-3} J$. **3.26**. 4.5 and 14.5 H. **3.29**. $U_{O.C.} = 12$ V, $R_i = 48$ Ω.

Chapter 4

4.2. 862 W. **4.3.** *a*) $8.63 e^{-j5^{o}}\ A$; *b*) $12.2 \sin(377t - 5^{o})\ A$; *c*) 863 *VA*; *d*) 0.966. **4.7**. *a*) $15 e^{j53^{o}} \Omega$; *b*) $300\sqrt{2} \sin(377t + 53^{o})$; *c*) 0.0317 H. **4.11**. *a*) $39.4 \sin(377t + 15^{o}) A$; **4.16.** $\underline{U}_{ab} = 84^{ej28^{o}} V$; $\underline{U}_{bc} = 47.3 e^{-j57^{o}} V$. **4.17**. The reactive component is 33.3 Ω. **4.18.** $X_1 = 4\ \Omega$; $X_2 = 3\ \Omega$; $\underline{I} = 39.6 e^{-j45^{o}} A$ **4.19**. $11.8 e^{-j80^{o}}$. **4.20**. *a*) 11.72+j47.1 Ω; *b*) the reactance $X = -$ j47.1 Ω; *c*) 53.2 μ*F*; *d*) $12.5 e^{-j90^{o}} A$. **4.28**. *a*) $6 e^{j37^{o}} A$; *b*) $6\sqrt{2} \sin(\omega t + 37^{o})\ A$; *c*) $\underline{U}_{ab} = 24 e^{-j53^{o}} V$; $\underline{U}_{bc} = 107.4 e^{j10^{o}} V$. **4.29**. $\underline{I}_1 = 17.7 e^{-j63^{o}} A$; $\underline{I}_2 = 14.6 e^{-j114^{o}} A$; $\underline{I}_R = 14.12 e^{-j10^{o}} A$. **4.30**. $\underline{I}_1 = 16 e^{-j60^{o}} A$; $\underline{I}_2 = 14.3 e^{-j86^{o}} A$; $\underline{I} = 29.52 e^{-j73^{o}} A$.

Chapter 5

5.1. $0{,}5 U_{ph}$. **5.2**. The electromotor can work with line voltage $U_l = 380\ V$ at a delta-connection, and with line voltage $U_l = 660\ V$ at a wye-connection. **5.4**, *a*) The line current is $17.32e^{-j60^o}$; *b*) 5710 Wt. **5.5**, *a*) $15.21e^{-j83^o}$; $15.21e^{-j203^o}$; $15.21e^{j37^o}$; *b*) 1160 W, *c*) 0.

5.9 *a*) $I_p = 10\,A;\ I_l = 17{,}3\,A;\ U_p = 220\,V;\ P_1 = 1900\,Wt;\ P_2 = 3800\,Wt;$ *b*) $U_{AB} = 220\ V;\ U_{BC} = U_{CA} = 110\ V;\ I_{AB} = 10\ A;\ I_{BC} = I_{CA} = 5\ A;$ $I_A = I_B = 15\ A;\ I_C = 0;\ P_1 = 2850\ Wt;\ P2 = 0;$ *c*) $U_{AB} = U_{BC} = U_{CA} =$ $= 220\,V;\ I_{AB} = I_{CA} = I_B = I_C = 10\,A;\ I_{BC} = 0;\ I_A = 17{,}3\,A;\ P = 1.9\,kWt.$ **5.10**. $I_p = 2.2\,A;\ I_l = 3.81\ A; P = 878\,Wt$. **5.12**. $I_A = I_B = I_C = 3.8\ A$; $I_{AB} = I_{BC} = I_{CA} = 2.2\ A$. **5.13**. Line currents $I_A = I_B = I_C = 8.47\,A$; phase currents $I_{AB} = I_{BC} = I_{CA} = 4.89\ A$. **5.14**. $P = 777\ Wt$; $Q = 70.5\,VAr$; $S = 779\,VA$. **5.15**. $R = 20\ \Omega;\ X_L = 5.5\ \Omega;\ X_C = 22.7\ \Omega$.

Chapter 6

6.1. $i(t) = 2 - e^{-25t}$ A. **6.2**. (a) $i(t) = 2(1 - e^{-25/3\,t})$ A; (b) 2 A; (c) 9/25 s. **6.3**. $i(t) = 5(1 - e^{-200t})$ A. **6.5**. $i(t) = 10 - 6e^{-10t}$ A. **6.6**. $i(t) = 15 - 3e^{-5.4t}$ A; t = 0.185 s. **6.8**. $i(t) = {}^3\!/_2(1 - e^{-32t})$ A. **6.9**. $i_2(t) = 4\times10^{-3}e^{-t}$ A. **6.10**. (a) $p^2 + 1.25p + 0.25 = 0$; (b) 1s, 4s. **6.11**. (a) 20 V, (b) 20 V, (c) $u(t) = 7.5 + 12.5e^{-4/15\,t}$ V. **6.12**. (a) 0, (b) $2\times10^{-3}e^{-t}$ A, (c) 5s, (d) $i(t) = q_0\omega\sin\omega t$. **6.13**. (a) e^{-30t}, (b) 1000 V, (c) tenfold increase. **6.16**. $f(t) = 5e^{-2t}\sin 2t$.

6.19. $f(t) = 3e^{-3t}\cos 4t - 0.25e^{-3t}\sin 4t$. **6.20**. *a*) $Z(p) = R + \dfrac{1}{pC}$; *b*) $Z(p) = R + \ + pL$; *c*) $Z(p) = \dfrac{1}{pC} + \dfrac{RLp}{R + pL}$; *d*) $pL_2 + \dfrac{1}{pC} + \dfrac{RL_1p}{R + pL_1}$.

6.21. *a*) $I(p) = \dfrac{100}{p(p + 100)}$; $i(t) = 1 - e^{-100t}A$;

b) $I(p) = \dfrac{100}{(p + 200)(p + 100)}$; $i(t) = e^{-200t} + e^{-100t}A$.

6.22. $u_C(t) = 50 + 50e^{-t}V$; $i_2(t) = 50 + \ + 50e^{-t}A$.

Chapter 7

7.1. $0.2\ A$. **7.2**. $R_{L\min} = 3\ k\Omega$; $R_{L\max} = 8\ k\Omega$. **7.4**. *a*) $R_3 = 18\ k\Omega$; $R_{st} = 6\ k\Omega$; $R_d = -6\ k\Omega$; *b*) $U_{out} = -15\ V$; $R_{st} = -11.5\ k\Omega$; $R_d = -7\ k\Omega$. **7.5**. $I_1 = 5.33\ A$; $I_2 = 1.33\ A$; the current through NR is $I_3 = 3\ A$, and the voltage across NR is $U_{ab} = U_{NR} = 2\ V$. **7.6**. $I_{NR} = 1\ A$; $R_{st} = 5\ \Omega$; $R_d = -1\ \Omega$; $I_{NR} = 4\ A$; $R_{st} = 2.5\ \Omega$; $R_d = 7\ \Omega$. **7.7**. *a*) $I_{NR} = 2\ A; I_{R1} = 4.5\ A; I_{R2} = 1.5\ A; I_{R3} = 1\ A; I_{R4} = 3\ A$; b) $I_{NR} = 0.8\ A$; $I_{R1} = 2\ A$; $I_{R2} = 0.7\ A$; $I_{R3} = 0.5\ A$; $I_{R4} = 1.3\ A$. **7.8**. $I_{NR1} = 3.1\ A$; $I_{R1} = I_{R3} = 4.3\ A$; $I_{R2} = 1.2\ A$; $I_{NR2} = 2.3\ A$.
7.9. $I_1 = 0.7\ A$; $I_2 = 0.46\ A$; $U_1 = U_2 = 10\ V$.

Chapter 8

8.1. *a*) 0, I_m; *b*) $0.5 I_m$, $0.707 I_m$. **8.2**. *a*) 0, $0.577 U_m$; *b*) $0.25 U_m$, $U_m / \sqrt{6}$. **8.3**. *a*) $0.5 I_p$, $0.577\, Ip$; *b*) 0, $0.577 I_p$; *c*) $0.25 I_p$, $I_p / \sqrt{6}$. **8.4**. $u(t) = \dfrac{8}{\pi^2}(\sin \omega t - \dfrac{1}{3^2}\sin 3\omega t + \dfrac{1}{5^2}\sin 5\omega t - \dfrac{1}{7^2}\sin 7\omega t + \ldots)\ V$. **8.6.** $i(t) = \dfrac{I_p}{2} + \dfrac{1}{\pi} I_p\,(\sin\ \omega t + \dfrac{1}{2}\sin\ 2\omega t + \dfrac{1}{3}\sin\ 3\omega t + \ldots)$. **8.11**. $i(t) = 14.1\sin(10^3 t + 45^o) + 9.7\ \sin\ (2\times 10^3 t + 166^o) + 4.7\ \sin\ (3\times 10^3 t - 45^o)\ \ A$; $I = 12.5\ A$; $U_R = 125 V$;
$u_R = 141\sin(10^3 t + 45^o) + 97\sin(2\times 10^3 t + 166^o) + 47\sin(3\times 10^3 t - 45^o)$
$u_L(t) = 71\ \sin\ (10^3 t + 135^o) + 97\ \sin\ (2\times 10^3 t - 104^o) + 71\ \sin\ (3\times 10^3 t + 45^o)\ V$; $U_L = 98.5\ V$; $u_C(t) = 212\ \sin\ (10^3 t - 45^o) + 75\ \sin\ (2\times 10^3 t + 76^o) + 23.5\sin(3\times 10^3 t - 135^o)\ \ V$; $U_C = 170\ V$;

$P = 1580\ W$; $Q = 1230\ VAr$; $S = 2060\ VA$.

8.12. $i(t) = 3 + 3.8\sin(314t + 162^o) + 1.41\sin\left(3\times 314t - 45^o\right)A$;

$U = 65{,}4\ V,\ I = 4.27\ A$. **8.14**. $I = 2.15\ A,\ U = 62.5\ V,$

$i = 2 + \sin(314t - 7^o) + 0.5\sin(628t - 33^o)$; $P = 133.2\ Wt$.

8.15. $i_L(t) = 10 + 10\sin(\omega t - 30^o) + 2.24\sin(2\omega t - 116^o)\ A$;

$I_L = 12.3\ A$.

Chapter 9

9.1. (*a*) 4/π; (b) 32×10^{-7} N/m; (c) 0.32 N/m. **9.6**. (*a*) 98 At; (*b*) 1.225 A. **9.7**. 4.96 A. **9.9**. (*a*) 5 J; (*b*) 2 W-s; (*c*) 20 H; (*d*) 3.35 W-s; (*e*) no, it comes from electrical source. **9.10**. (*a*) 1500 At; (b) 2300 At/m; (c) $2.9\ W/m^2$; (d) 1.25×10^6; (e) 1.26×10^{-3}; (f) 1010. **9.11**. At 60 Hz: $P_h = 1400\ W$, $P_e = 400\ W$; at 90 Hz: $P_h = 2100\ W$ and $P_e = 900\,W$. **9.12**. 367 *At* applied in direction to cause flux in right leg to flow CCW. **9.14**. 20.2 W.

Chapter 10

10.1. (*a*) $A_{11} = 1$; $A_{12} = R\,\Omega$; $A_{21} = {-1}/{-jX_C}$; $A_{22} = \dfrac{R - jX_C}{-jX_C}$;

(*b*) the idling condition: $A_{11} = 1$; $A_{21} = 1/-jX_C$; the short circuit: $A_{12} = R\,\Omega$; $A_{22} = (R - jX_C)/(-jX_C)$.

10.7. $Y_{11} = Y_{22} = j/60\,\Omega$; $Y_{12} = Y_{21} = j/30\ Sm$.

10.8 $\underline{U}_1 = 6.98\,e^{j58^o}\,V$. **10.9**. $\underline{Z}_1 = 10\ \Omega$; $\underline{Z}_2 = -j10\ \Omega$; $\underline{Z}_3 = j10\ \Omega$.

10.10. $\underline{Z}_l = 14.95e^{j32^o}\,\Omega$. **10.12**. $A = 1.28\,e^{j40^o}$; $B = 4.25\,e^{j67^o}$;

$C = 0.17e^{j90^o}$; $D = 0.34\,e^{j1^o}$.

Appendix B

VOCABULARY

A	
accept	приймати, допускати
access	доступ, підхід
accident	аварія, пошкодження
accordant connection	узгоджене з'єднання
accumulate	акумулювати, накопичувати
action	дія, хід
actuate	надавати дії, збуджувати
add	додавати, складати
adapt	пристосовувати, приганяти
addend	додаток
adding connection	згідне включення
addition	додаток, приєднання, додавання
additional	додатковий
adjacent	суміжний, який межує, граничить
adjust	регулювати, налагоджувати, настроювати
adjustment range	межі регулювання
admittance	повна електрична провідність
adoption	прийняття, засвоєння
advantage	перевага
affect	впливати
air gap	повітряний зазор
alarm signal	аварійний сигнал
algorithm	алгоритм
allowable voltage	допустиме напруження
alternating	змінний
alternating component	змінна складова
alternating current circuit	коло змінного струму
alternating-current measurement	вимір на змінному струмі
alternating current a.c. system	електрична мережа змінного струму
ammeter	амперметр

amount	кількість, сума, доза
ampere	ампер
ampere-hour	ампер-година
ampere-turns	ампер-витки
amplifier	підсилювач
amplify	розширювати
amplitude	амплітуда
amplitude modulation	амплітудна модуляція
amplitude-frequency characteristic	амплітудно-частотна характеристика
analogously	аналогічно
angular	кутовий
anode connection	анодний приєднання
anti-damping	анті-загасання, анті-затухання
apparatus	прибор, пристрій, прилад
apparent	очевидний, явний
apparent power	загальна потужність
application	застосування, включання
applied (impressed) voltage	прикладена напруга
apply	застосувати
approach	наближення, доступ
appropriate	відповідний
arc	електрична дуга
arcing earth	дугове замикання на землю
area	площа, поверхня
arrange	розташовувати, розкладати, розміщати
arrangement	розташування, розміщення, улаштування (*дія*)
array	ряд, черга, масив
arrest	гальмувати, зупиняти, вимикати
arrested	загальмований
assume	набувати (*форму, вид*)
asynchronous	асинхронный
asynchronous generator	асинхронный генератор
attach	приєднувати
attain	досягати, набувати

attitude	положення, розташування
average	середнє значення
axis (pl. axes)	вісь
axis of abscissas	вісь абсцис
axis of ordinates	вісь ординат
B	
back electromotive force	протидіюча ЕРС
backflow	зворотна течія, протиток
backing	обертання в зворотний бік
backward	зворотний
backwash	зворотний потік
balance	рівновага, зрівноважувати, балансувати
band	смуга, діапазон
band switch	перемикач
band width	ширина смуги
bare	неізольований (*ел.*)
battery	акумуляторна (гальванічна) батарея
bearing	підшипник, опора
beat	биття
beat frequency	частота биття
beating	биття
bias electrical restraint	електричне гальмування
bias voltage	напруга зсуву
bias neutral voltage	напруга зсуву нейтралі
binding	з'єднання (*проводів*)
bipolar	двополюсний
blanket	покривання, поверхневий шар
blocking	блокування, замикання
blowing out	видування, перегоряння
blow-out	іскрогасник, зрив дуги
bond	зв’язок, сполучення
branch	вітка (*ел. кола*)
bridge	перемичка, шунт
break	розмикання
bridge	міст
bridging	шунтований

burn-out	перегоріння
busbar sectionalising switch	секційний вимикач
bus coupler circuit breaker	шиноз'єднувальний вимикач
bushing of a transformer	уведення трансформатора
C	
cable	кабель, трос, линва
cable conductor	кабельна жила
cal	вольфраміт
call	викликати
calculate	обчислювати, рахувати
capability	здатність
capacitance	ємність, ємнісний опір
capacitive current	ємнісний струм
capacitive feedback	ємнісний зворотний зв'язок
capacitive load	ємнісне навантаження
capacitive potential divider	ємнісний дільник напруги
capacitive reactance	ємнісний опір
capacitive residual current	остатній ємнісний струм залишковий ємнісний струм
capacitive susceptance	ємнісна провідність
capacitor	конденсатор, ємнісний накопичувач
capacitor discharge	розряд конденсатора
capacity	ємність (*ел.*)
carriage	несуча конструкція, шасі
cascade connection	каскадне вмикання
catch	фіксатор, захват
causes	причини
cell	елемент (*ел.*)
charging current	зарядний струм
change	зміна, змінитися
character	характерна ознака, символ
characteristic equation	характеристичне рівняння
charge (of capacitors or batteries)	заряд, завантаження
chart	діаграма, схема, графік, таблиця
check	перевірка, перевіряти
circle	круг, коло
circuit	мережа, коло, контур, схема

circuit closer	замикатель кола
circuit breaker	вимикач
circuit breaker closing	вмикання вимикача
circuit breaker opening	відключення (вимикання) вимикача
circuit closed in working position	коло замкнуте в робочому положенні
circuit opening contact	розмикальний контакт
circulate	циркулювати, повторюватися, розповсюджуватися
clean-off	чистове оброблення
clean-up	очищення
clear the short	усунути коротке замикання
closing (manual)	вмикання (вручну)
closing time	час вмикання
clip	затискач, затискати
closed	замкнений, закритий
coefficient	коефіцієнт
coefficient of conductivity	коефіцієнт провідності
~ of correction	поправочний коефіцієнт
~ of coupling	коефіцієнт зв'язку
~ of expansion	коефіцієнт розширення
~ of leakage	коефіцієнт розсіяння
~ of mutual inductance	коефіцієнт взаємної індуктивності
~ of performance	коефіцієнт корисної дії
~ of permeability	коефіцієнт проникності
~ of reflection	коефіцієнт відбивання
~ of resistance	коефіцієнт опору
~ of self-inductance	коефіцієнт самоіндукції
~ of shear	коефіцієнт зсуву
~ of power	коефіцієнт потужності
coil	котушка, віток (ел.)
coincide	збігатися
column	колонка, стовпець
combination	з'єднання, сполучення
common	спільний
common fraction	звичайний дріб
commutated current	комутаційний струм
commutation	комутація

comparative	порівняльний
comparison	порівняння
compensated	компенсований, зрівноважений
compile	складати
complementary	додатковий
complex	комплексний, складний
complex impedance	комплексний опір
complex plane	комплексна площина
complex power	комплексна потужність
component	компонент, складова частина, складова
compose	складати
composition	додавання (векторів)
compound	з'єднання, з'єднати, складний, складений
comprehensive	повний
compressed	стиснений
compulsory	обов'язковий, змушений
computation	підрахування, обчислення
computing	обчислення
concentrated	зосереджений, концентрований
condition	умова, режим, стан, положення
conduct	проводити
conductor	провідник
conductance	активна провідність
conductive coupling	гальванічний зв'язок
conductivity	провідність, електропровідність
conduction current	струм провідності
connections	сполучення, з'єднання
connection diagram	схема з'єднання
connector	роз'їм
conjugate	спряжений
connect	з'єднати, вмикати
connected	з'єднаний, ввімкнений
consequent	послідовний
conserve	зберігати, запобігати
constant resistance	постійний опір

contact	контакт
contact voltage	напруга між контактами
control current	струм управління
controlled value	регульований параметр
control voltage	керуюча напруга
control winding	керуюча обмотка
construct	споруджувати
constructed	споруджений
consumer	споживач
contact breaking	вимикання, розмикання
contain	містити в собі
content	склад
continuous	постійної дії, безперервний
continuous current	постійний струм
conventional thermal power station	електростанція на органічному паливі
converter	перетворювач
converter substation	перетворювальна підстанція
conversion of electricity	перетворення електричної енергії
conversion	перетворення
convert back	перетворити зворотно
copper loss	втрати в міді
core	осердя, стрижень
core of a transformer	осердя (магнітопровід) трансформатора
corona discharge	коронування
correct	коригувати, правильний, безпомилковий
correlation	кореляція, корекція, зв'язок
corrupted data	перекручені дані
couple (to)	замикання (замикати)
coupling between different phases of two circuits of a high voltage link	взаємоіндукція між фазами двох кіл високої напруги
coulomb	кулон (*ел.*)
coupled	сполучений, спарений
cross connection	поперечне з'єднання

cross-over	перехід
cross-section	поперечний переріз
cumbersome	громіздкий
current	струм
current balance	баланс струмів
current carrying capacity	пропускна здатність по струму
current circuit	струмове коло
current-dependent	залежний від струму
current limiter	обмежувач струму
current resonance	резонанс струмів
current reversal	зміна напрямку струму
current rush	кидок струму
current transformer	трансформатор струму
~ charging current	струм заряджання
~ constant current	постійний струм
~ control current	керуючий струм
~ direct current	постійний струм
~ earth current	струм заземлення
~ eddy ~s current	вихрові струми (Фуко)
~ effective value of current	діюче значення струму
~ fault current	струм пошкодження
~ exciting current	струм збудження
~ idle current	реактивний струм
~ induced current	індукований струм
~ inverse current	зворотний струм
~ joint current	сумарний струм
~ lagging current	струм, що відстає за фазою
~ leading current	струм, що опереджає за фазою
~ leakage current	струм витоку
~ magnetization current	струм намагнічування
~ mesh current	контурний струм
~ operating current	робочий струм
~ primary current	струм у первинній обмотці
~ pulsating current	пульсуючий струм
~ reverse current	зворотний струм
~ sinusoidal current	синусоїдний струм
~ starting current	пусковий струм

~ transient current	перехідний струм
current-carrying	струмовидний
current-conducting	струмопровідний
curve	крива (*лінія*)
cushion	зм’якшувати, послабляти
cusp	вершина, точка перетину двох кривих
cut	канал, перетин, профіль
cut-off	відключення
cut-out	рубильник (*ел.*)
cut-off relay	реле відсічення
cycle	цикл, оборот, період (*синусоїдного струму*)
D	
damage	пошкодження
damped oscillations	загасаючі коливання
damped transient	загасаючий перехідний процес
damp	згасати (*про коливання*)
damping	загасання, демпфування, гальмування
damper winding	демпферна обмотка
damping circuit	коло, що демпфірує
damping magnet	магніт, що демпфірує
damping decrement	декремент загасання
dash	риска, штрих, тире, штрихувати
data	дані, відомості
date	дата, строк, період
decay	розпадання, загасання
decaying	загасання
deceleration	уповільнення (гальмування)
decoupling	розв'язка
deenergized line	лінія без напруги
defence	захист, оборона
define	визначати
definite	визначений, певний
deflection	відхилення
deform	деформувати, викривляти
degree	ступень, градус
delay	затримка, запізнювання

delete	викреслювати, стирати
delta connection	з'єднання у трикутник
delta	зірка (*ел.*)
delta-star connection	з'єднання зірка - трикутник
demand	попит, вимога
demount	розбирати, демонтувати
density	густина, густість
denominator	знаменник
denote	позначати
departure	відхилення
dependence	залежність
dependent	залежний
derivative	похідна (*функції*)
derive	брати похідну
derive a formula	виводити формулу
deserve	заслужити
design	план, креслення, проект
desire	вимагати
determine	визначати
determination	визначення, вирішування
develop	створювати, розвивати
device	обладнання, апарат, прилад, механізм
diagram	графік, діаграма
dielectric	діелектрик
differential	диференціал, диференційний
difference	різниця, відмінність
differentiate	диференціювати
digit	цифра, одиниця, символ
dimension	розмір, величина
diode	діод
dipole	диполь
direct	постійний (*про струм*)
direct current	постійний струм
direct-current amplifier -	підсилювач постійного струму
direct-current circuit	коло постійного струму
direct current system (d.c. system)	електрична мережа постійного струму

directed	направлений, спрямований
direction	направлення, спрямування
directional operation	спрямована дія
direct voltage	постійна напруга
disconnector	роз'єднувач
discharge	скидання, випуск, вихід, розряд
~ capacitor discharge	розряд конденсатору
~ corona discharge	коронний розряд
~ spark discharge	іскровий розряд
disconnect	розмикати, вимикати
disconnection	розмикання, вимикання
discontinuity	обрив, розрив
disengage	вимикати, розмикати
dislocate	зрушувати, переміщати
dismount	знімати, демонтувати, розбирати
dismountable	знімний, роз'ємний
dispense	розподіляти, роздавати
displacement	переміщення, зсув
displacement voltage of the neutral points voltage	напруга зсуву нейтралі
disposal	розміщення, видалення
disposition	розташування, розміщення
disruption	розрив, пробій
dissociate	роз'єднати, розділяти
distance	відстань, інтервал
distension	розширення, розтягання
distribution	розподіл
distortion	перекручування
distortion factor	коефіцієнт перекручування
distribution of electricity	розподіл електричної енергії
divide	поділяти, розділяти
division	поділ, розподіл
dot	точка
double earth fault	подвійне замикання на землю
double phase short circuit	двофазне коротке замикання
drive	привід, передача, рухати, обертати
duration	тривалість

duty	навантажування, режим, цикл
dynamo	генератор постійного струму
E	
earth	заземлення, замикання на землю
earthed	заземлений
earth connection	з'єднання з землею
earth current	струм витоку на землю
earthed neutral	заземлена нейтраль
earth fault	замикання на землю
earth fault current	струм замикання (КЗ) на землю
earthing resistance	опір заземлення
earth leakage current	струм витоку на землю
eddy	вихор, завихрення
eddy currents	вихрові струми
effect	дія, результат, наслідок
effective	діючий
effective value	ефективне (діюче) значення
efficiency	коефіцієнт корисної дії (к.к.д.), ефективність
effluent	потік, витікання
effort	зусилля, напруга
ejection	викид, викидання
electric current	електричний струм
electric arc	електрична дуга
electric charge	електричний заряд
electric circuit	електричне коло
electric field	електричну поле
electric line	електрична лінія
electrical circuit	електричне коло
electrical measurement	електричний вимір
electrical power system	система електропостачання
electrical power network	електрична мережа
electric losses	електричні втрати
electricity	електрика
electromotive force	електрорушійна сила
electrostatics	електростатика
element	елемент, частина, деталь

eliminate	виключати, вилучати
emergency	аварія, запасний, аварійний, критичний
emission	виділення, випромінювання
empty	вивантажувати, спорожняти
end	кінець, вивід (обмотки)
end-to-end	впритул, по всій довжині
end winding	вивід обмотки
endurance	витривалість, стійкість
energise	подача напруги
energised facility	установка під напругою
energy	потужність, енергія
engage	включати, зістиковувати
engaging	включення, вмикання
engine	машина, двигун
entry	вхід, введення
environment	навколишнє середовище
equal	рівний
equality	рівність
equation	рівняння (*мат.*)
equipment	обладнання
equipotential	еквіпотенційний
equivalent impedance	еквівалентний опір
erect	установлювати, збирати, монтувати
error	помилка, похибка
erase	видалення
error correction	усунення помилки (похибки)
error detection	виявлення помилки (похибки)
escape	випуск, витікання
establishment	установа, підприємство
estimate	розрахунок, оцінка
estimated	розрахований
evaluate	оцінювати, знаходити значення величини
examination	огляд, дослідження
example	приклад (*мат.*)
excitation	збудження

excite	збуджувати (*ел.*)
exciting current	струм збудження
excursion	зсув, відхилення
expand	розкладати формулу, розкладати у ряд (*мат.*)
expansion	розширення, розкладання
expend	витрачати
experiment	дослід, експеримент
exponential curve	експонентна крива
exposed	оголений, незахищений, відкритий (*про проводку*)
expression	вираз
external characteristic	зовнішня характеристика
external terminal	зовнішній затискач
extra high voltage network	мережа надвисокої напруги
extend	подовжувати, збільшувати
extreme	крайній, граничний
F	
face	лицьовий бік, вид спереду, фас
factor	множник, коефіцієнт, показник
~ common factor	загальний множник
factory tests	заводські випробування
fade	затухати
fail	пошкоджуватися, виходити з ладу
failure	пошкодження, відмова
falling	падання, зниження
false switching	помилкове включення
family of curves	сім'я кривих
fault	пошкодження
fault between turns	міжвіткове коротке замикання
fault between windings	коротке замикання між обмотками
fault clearance	відключення короткого замикання
fault situation	аварійний режим
feature	характерна риса, особливість
feed	подача, живлення
feedback	зворотний зв'язок
feedback amplifier	підсилювач зі зворотним зв'язком

feedback control	керування зі зворотним зв'язком
feedback ratio -	коефіцієнт зворотного зв'язку
feed-through	прохідний
feed-through capacitor	прохідний конденсатор
feed-through four-terminal network	прохідний чотириполюсник
feeder	живильна лінія
feeder circuit-breaker –	лінійний вимикач
feeder disconnector	лінійний роз'єднувач
field	поле, простір
~ of force	силове поле
~ of gravity	поле тяжіння
~ of vorticity	вихрове поле
~ rotating field	обертове поле
figure	рисунок, фігура, цифра
filter	фільтр, фільтрувати
finding	визначення, пошук
finger	покажчик, вказівник
finite	кінцевий, визначний
firm	густий, твердий
fit	установлювати, монтувати
fitting	монтаж, збирання
fix	укріплювати, установлювати
fixed	нерухомий, закріплений
flame	полум’я, палати
flash	спалах, дуговий розряд
flashover	іскріння, пробій, коротке замикання
flat	плоский, рівний
flaw	розрив, дефект
flexing	вигин, вигинання
flicker	мигання, мигати, мерехтіти
flow	потік, течія, теча
flowing	течія, текучий
fluctuation	качання
~ frequency fluctuation	коливання частоти
~ pressure fluctuation	коливання тиску
~ random fluctuation	випадкова флуктуація

fluent	текучий, рідкий
fluid	рідина, рідкий, газ
flush	струмінь, швидкий приплив
flux	потік, витікати
flux linkage	потокозчеплення
force	сила, заставляти, примушувати
forced	примушений
formula	формула, аналітичний вираз
formulated	сформульований
foundation	основа, фундамент
founding	лиття, плавка
fraction	дріб, частина, частка
fractional	дрібний, дробовий
fracture	розрив, злом
frame	корпус, каркас, конструкція, збирати
freezing	замерзання, застигання
freight	вантаж, фрахт, фрахтувати
frequency	частота
frequency changer	перетворювач частоти
frequency converter	перетворювач частоти
frequency response	частотна характеристика
function	функція, дія
fuse	топкий (плавкий) запобіжник
fusibility	топкість, плавкість
G	
gadget	пристосування, обладнання, технічна новинка
gang	набір, комплект, агрегат
gap	зазор, щілина
gas generator	газогенератор
gasket	прокладка
gate	прохід, вентиль
gauge	міра, масштаб, розмір
gear	механізм, пристрій, обладнання
generate	генерувати, утворювати
generation	генерація, утворювання
generator	генератор, джерело енергії

glance	блиск, блищати
glow	розпікання, розпікати
glissade	ковзання
go down	спускатися
go on	продовжувати
go out slowly	затухати
governing equipment	кероване обладнання
grid	енергетична система
grip	затискач, захват, лещата
ground	заземлювання, заземлювати
grounded	заземлений
grummet	прокладка
guide	напрямна
H	
habilitate	устатковувати, постачати
habit	особливість, властивість
hade	нахил, падіння
hairline	дуже тонка лінія
half-and-half	що складається з двох компонентів
half-axle	піввісь
half-cycle	півперіод
half-wave	однопівперіодний
half-wave rectifier	однопівперіодний випрямляч
hand	стрілка
handle	держак, керувати
hang	вішати, підвішувати
hardness	твердість, міцний
harmonic	гармоніка
harmonic component	гармонійна складова
harmonic content	зміст гармонік
harmonic function	гармонійна функція
harmonic oscillation	гармонійне коливання
headway	рух уперед
heat	теплота, топлення, розжарювання, нагрівання
heating	нагрівання, розжарювання
heating-up	прогрів

heat-proof	теплостійкий
heat-resistant	теплостійкий
heat-sensitive	теплочутливий
heavy	потужний, великий
heavy-duty	потужний
high-cycle	високочастотний
height	висота, максимум, межа
high-frequency	високочастотний
high-frequency cable	високочастотний кабель
high-frequency generator	високочастотний генератор
high frequency disturbance test	перевірка перешкодостійкості
higher harmonic	вища гармоніка
higher harmonic voltage	напруга вищих гармонік
high-level	інтенсивний
high-speed excitation system	швидкодіюча система збудження
high voltage	висока напруга
high-voltage installation	установка високої напруги
high-voltage network	мережа високої напруги
high-voltage side	сторона високої напруги
high-voltage switchgear	розподільний пристрій (електроустаткування) високої напруги
high-voltage winding	обмотка високої напруги
hit	зіткнення
holding winding	утримуюча обмотка
hole	отвір, дірка
hollow	порожнина, западина
hood	кожух
hydroelectric set	гідроагрегат
hydroelectric power station	гідроелектростанція (ГЕС)
hysteresis	гістерезис
hysteresis loop	петля гістерезису
hysteresis losses	втрати на гістерезис
hydrostable	водостійкий
hydrous	водний
hydroxide	гідроокис
hysteresis	гістерезис

I	
ideal rectifier	ідеальний випрямляч
identical	тотожний
ideal synchronizing	точна синхронізація
identify	пізнавати, визначати
idle	непрацюючий, неробочий, не навантажений
idling	неробочий хід, робота на неробочому ході
ignite	займатися, зайнятися
ignition	запалювання
illuminate	освітлювати
image	зображення
imaginary	уявний
immediately prior	безпосередньо до
immediately after	безпосередньо після
immobilization –	висновок з роботи
immovable	нерухомий
impact	удар, поштовх
impedance	загальний опір, імпеданс
impedance earthed (neutral) system	електрична мережа з заземленою через опір нейтраллю
impedance protection	дистанційний захист
impedance voltage (of a transformer)	напруга короткого замикання (трансформатору)
impermeability	непроникність
improvement	поліпшення
impulse counter	лічильник імпульсів
impulse voltage test	перевірка ізоляції
inadvertent operation	неправильне спрацьовування
incorrect operation of relay protection	неправильна дія захисту
independent time relay	реле з незалежною витримкою часу
indicating relay	реле вказівне
indicator lamp	сигнальна лампа
inadvertent operation	неправильне спрацьовування
incentive	стимул, спонука

inception	початок
incidental	випадковий
incline	нахил, нахилення
inclusion	вмикання
incorrect operation of relay protection	неправильна дія захисту
increase	збільшення, зріст
increment	збільшення, зріст, приріст
indefinite	невизначений
independent	незалежний
indeterminate	невизначний
index	показник степені, коефіцієнт
indicate	вказувати
indicating relay	реле вказівне
indicator lamp	сигнальна лампа
indicator of sense of rotation	вказівник напрямку обертання
indivisible	неподільний, неділимий
indoor substation	закрита підстанція
indicator of sense of rotation	покажчик напрямку обертання
indoor apparatus	апаратура внутрішньої установки
indoor substation	закрита підстанція
indoor switch-gear	закритий розподільний пристрій
induced voltage	наведена напруга
induce	викликати, індукувати, спонукати
induced	індукований
induced voltage	наведена напруга
inductance	індуктивність
inductance coil	котушка індуктивності
induction	індукція
infinity	нескінченність
inflow	впуск, втікання
inhibition	гальмування
initial	початковий

initiation	виникання
inject	вдувати, впорскувати
injection	підживлення (*ел.*)
injury	ушкодження, аварія
inlet	вхід, впуск
inner	внутрішній
in-parallel	паралельно ввімкнений
in-phase	збіжний за фазою
input	вхідний
input winding	вхідна обмотка
integral control	інтегральне регулювання
integrated circuit	інтегральна схема
interconnected systems	об'єднані енергосистеми
interconnection	міжсистемний зв'язок
insert	включати, запроваджувати
in service	у експлуатації
inspection	огляд
instability	нестабільність
install	розташовувати
installed	розташований
installation	установка, пристрій
instantaneous	миттєвий
instantaneous relay	швидкодіюче реле (миттєвої дії)
instantaneous value	миттєве значення
instrument	прилад, апарат, інструмент, пристрій
insulance	опір ізоляції
insulated	ізольований
insulation	ізоляція
insulation resistance	опір ізоляції
intact	неушкоджений, цілий
intake	підвід, споживана (*двигуном*) потужність
integrate	інтегрувати, об'єднувати

intensify	підсилювати
intensity	напруженість (*ел.*), сила, енергія
interact	взаємодіяти
interaction	взаємодія
intercept	перервати, вимикати
interchange	(*взаємний*) обмін, чергування, зміна
interface	поверхня поділу
interfere	заважати, перешкоджати
interference	взаємний вплив
interference effect	вплив перешкод
interference pulse	імпульс перешкоди
interflow	злиття
interfuse	змішуватися
interior	внутрішній
interlinkage of mutual inductance	потокозчеплення взаємної індукції
interlinked	зв'язаний, сполучений
interlock	взаємно з'єднувати
intermediate	проміжний, допоміжний
intermittent	стрибкоподібний, переривчастий, пульсуючий
intermittent contact	переривчастий контакт
intermittent earth	перемежоване замикання на землю
intermittent fault	нестійке ушкодження
intermix	змішуватися
internal	внутрішній
internal combustion set	електроапарат із двигуном внутрішнього згоряння
internal short circuit	коротке замикання у зоні дії захисту
interphase	поверхня поділу фаз
iterruption arc	дуга при розмиканні контактів
interruption of supply	порушення електропостачання
interturn fault	міжвиткове коротке замикання
introduce	вводити

inverse	поворотний, протилежний, обернений (*мат.*)
inverse characteristic relay	реле с обратнозависимой времятоковой характеристикой
inverse time; very inverse time; extremely inverse time current relay	реле струмового захисту, що мають велику та дуже велику залежність часу спрацьовування від струму
inversion	обертання, інверсія, інвертування (*ел.*)
invertible	оборотний
invertor	зворотний перетворювач, інвертор
investigate	досліджувати
investigation	дослідження
involution	піднесення до степеня
iron losses	втрати у сталі
isolated	окремий, ізольований, вимкнений
isolated neutral system	система з ізольованою нейтраллю
isolation	ізоляція, відділення, вимикання
isolation transformer	розділювальний трансформатор
issuing of permit to work	дозвіл на провадження робіт
in-rush, inrush	кидок струму
issuance	вихід, випуск
issue	вихід, випуск, результат, витікання
item	пункт, параграф, деталь
itemize	класифікувати, складати специфікацію
J	
jack	затискач, важіль, гніздо (*ел.*)
jacket	кожух, чохол, оболонка
jam, jamming	заїдання, защемлення
jar	поштовх, тремтіння, струшувати
jet	струмінь, реактивний (*про двигун*)
jobbing	дрібний ремонт
joggling	смикання, підштовхування
join, joining	з'єднання, зв’язок, складання, з'єднувати

joined	складовий, поєднаний
joint	з'єднання, зв’язок, стик, вузол, шарнір
jointing	з'єднання, спайка
joist	балка, брус, дошка
joule	джоуль
jump	стрибок, зміна
jumper	перемичка
jumper board	збирання затискачів
jumping	биття, пульсація
junction	вузол, з'єднання
juncture	шов, з'єднання, спай, спайка
K	
keep	тримати, зберігати
key	ключ, клин, шпонка
~ cut-off key	розмикаюча кнопка
~ magnetic key	магнітне реле
~ resetting key	розмикаюча кнопка
~ rocking key	комутаторний ключ
keyboard	клавіатура, комутатор
keying	з'єднання шипами
killed	вимкнений
kink	петля, вузол, перегин
knead	місити, змішувати
knife	ніж, різець, скребачка
knit	скріплювати, з'єднувати
knob	перемикач
knob with indicator	перемикач з сигналізацією
knock	удар, поштовх, детонація, битися
knot	вузол
knotty	вузлуватий, заплутаний
L	
label	маркірувати, мітити
lag	відставання, запізнення, зсув фаз (*ел.*)

lag behind	відставати від
lagging	зсув фаз (*ел.*)
lagging current	відстаючий струм
latitude	широта
lead	провідник (*ел.*), живлячий провід, випереджувати
leading	ведучий, направляючий
leakage	текти, протікання
leg	сторона, катет, опора
length	довжина, протяг
let	пускати, відпускати
~ to let in	впускати, вмикати
~ to let off (out)	спускати, випускати
letter	позначати літерами
levelling	вирівнювання
life	термін служби, довговічність
lightning discharge	розряд блискавки
limit	границя, межа
line	лінія, пряма, границя
linear system	лінійна система
line attenuation	лінійне загасання
line charging current	струм заряду лінії
line-drop compensation	компенсація спадання напруги в лінії
line fault	пошкодження на лінії
line voltage	лінійна напруга
link	ланка (*кола*), зчеплення, зв’язок, з'єднувати
linkage	зчеплення, зв’язок
linked	сполучений, з’єднаний, зчленований
list	список, перелік
live	що знаходиться під напругою, заряджений
live circuit breaker	включений вимикач
live line	лінія під напругою
load	навантаження, навантажувати
load in a system	навантаження енергосистеми
load curve	графік навантаження

load duration curve	графік тривалості навантаження
loaded	навантажений
loading	навантаження, заряджання
loading resistor	навантажувальний резистор
location	розташування, розміщення
lock	замок, замикач, з'єднувати, блокувати
locomotion	пересування
locus	місцеположення
logical multiplication	логічне множення
logic scheme	логічна схема
longitude	довгота
longitudinal	поздовжній
long-lived	з тривалим строком служби
long power transmission line	лінія довгих електропередач
long-term	тривалий, довготерміновий
loop	виток, петля, контур
loop around	обмотувати, обвивати
loop current	контурний струм
loose	вільний, з’єднаний нежорстке
loss	втрата, збиток
loosen	ослабляти
loss of excitation	втрата порушення
loss of load	втрата навантаження
loss of stability	втрата стійкості
loss of voltage	втрата напруги
loss of voltage relay	реле втрати напруги
low	низький, недостатній
lower frequency (to)	зниження частоти
low frequency	низька частота
low frequency amplifier	низькочастотний підсилювач
low frequency band	низькочастотний діапазон
low operating	повільно діючий
low voltage	низька напруга
low-voltage apparatus	апаратура низької напруги
low-voltage side	сторона нижчої напруги
luminous	ясний, світлий, блискучий
M	

machine	машина, станок, механізм, обробляти
magnet	магніт
magnetic circuit	магнітопровід
magnetizing current	струм намагнічування
magnetic field	магнітне поле
magnetize	намагнічувати
magnetizing inrush	кидок струму намагнічування
magnetizing inrush restrain	гальмування при кидку струму намагнічування
magneto-electric relay	магнітоелектричне реле
magnetomotive force	магніторушійна сила
magnify	збільшувати, підсилювати
magnitude	величина
main protection	основний захист
mains voltage	напруга мережі
maintenance	експлуатація (технічне обслуговування)
maintenance tests	експлуатаційні випробування
maintenance work	поточний ремонт
make and break	переключення, перемикання
maloperation	помилкова дія
manage	керувати, провадити
manifold	колектор, патрубок
manual closing	включення вручну
manual opening	відключення вручну
manual regulation	ручне регулювання
mark	зазначка, позначка, позначати
master controller	центральний регулятор
matched	погоджений
matching	погодження, припасування
matching transformer	узгоджений трансформатор
mated	парний, спряжений
matter	речовина, матерія, справа, питання, предмет
maximum voltage relay	реле максимальної напруги
mean	середня величина, середній; спосіб, засіб

mean deviation	середнє відхилення
mean square error	середньоквадратична помилка
mean value (of a periodic quantity)	середнє значення (періодичної складової)
measured	виміряний
measuring relay	вимірювальне реле
measuring winding	вимірювальна обмотка
melt	топлення, топитися
melting	топлення
merge	зливати, поєднувати(ся)
mesh	замкнений контур (*ел.*)
mesh current	контурний струм
meshed network	замкнуте коло
metering winding	вимірювальна обмотка
method	метод, спосіб
millivolt	мілівольт
mirror	відбивач, відбивати
miscalculation	помилка в розрахунках, неслушний розрахунок
miss	пропуск, перебій, пропускати
mistake	помилка
mixed	змішаний
mobile	рухомий, мобільний
mobility	рухомість
mode	спосіб, метод, режим
moderate	помірний, середній
modify	змінювати
modulate	модулювати
modulated	модульований
module, modulus	модуль
molten	розтоплений
momentary	миттєвий, миттьовий
monitoring	контроль, керування, спостереження
monofrequent	одночастотний
monophase	однофазний
mortise	паз, канавка, гніздо
motion	рух, переміщення

motor	двигун
~alternating-current motor	двигун змінного струму
~ asynchronous motor	асинхронний двигун
~ direct-current motor	двигун постійного струму
~ single-phase motor	однофазний двигун змінного струму
motor protection	захист електродвигуна
motoring	режим двигуна
mount	кріплення, опора, збирати, установлювати
mounting	кріплення, монтаж, цоколь (*ел.*), основа
mounting	монтаж
mouth	вхідний отвір
movable	рухомий, рухливий
movable contact	рухливий контакт
moving coil	рухлива котушка
moving point	рухома точка
muff	муфта
muffle	шумоглушник
multiengined	з кількома двигунами
multiform	різноманітний
multiloop	багато контурний, багатовитковий
multinomial	багаточлен, поліном
multipartite	поділений на кілька частин
multiphase	багатофазний
multiple	кратний, складовий, паралельний (*ел.*)
multiple connection	паралельне з'єднання
multipled	паралельно з'єднаний
multiple earth fault	багаторазове замикання на землю
multi-terminal line	розгалужена лінія
multiwinding transformer	багатообмотувальний трансформатор
multiplication	множення
multiplicity	складність, різноманітність
multiplier	множник
multiply	множити
multiplying	множення
multi-polar	багатополюсний

multi-position	багатопозиційний, східчастий, ступінчатий
multiturn	багатовитковий
multiwave	багатохвильовий
mutual	взаємний
mutual inductance	взаємна індуктивність
N	
naked	голий, неізольований
nature	характер, рід, сорт, тип
negative	від'ємний
negative sequence component	складова зворотної послідовності
negative sequence impedance	опір зворотної послідовності
nest	гніздо, блок, вузол
net	мережа
network	мережа, коло, схема
network with directly earthed neutral	мережа із глухозаземленою нейтраллю
network with isolated neutral	мережа з ізольованою нейтраллю
neutral	нейтраль, нейтральна точка, нульовий провід
neutral current	струм нульової послідовності
neutral point	нульова точка
neutral point connection	режим нейтралі
neutral point displacement voltage	напруга зсуву нейтралі
neutralising	нейтралізація
neutron	нейтрон
nipped	затиснутий, защемлений
node	вузол, точка перетину ліній
noise	шум, перешкоди, спотворювання
noiseless	безшумний, вільний від перешкод
noisefree	без перешкод
noise immunity	перешкодостійкість
nominal current	номінальний струм
nominal transformation ratio	номінальний коефіцієнт трансформації
nominal value	номінальна величина

nominal voltage	номінальна напруга
nominal voltage of a system	номінальна напруга електричної мережі
nominator	чисельник
non-dimensional	безрозмірний
non-linear	нелінійний
no-load	неробочій хід, нульове навантаження
no load current	струм неробочого ходу
no load operation	робота на неробочому ході
no-load power	потужність неробочого ходу
no-load test	випробування на неробочому ході
no-load voltage	напруга неробочого ходу
non-availability factor	коефіцієнт неготовності
non-linear distortion	нелінійне перекручування
non-linearity	нелінійність
non-linear resistor	нелінійний опір
non-reactive	нереактивний
non-saturated	ненасичений
non-sinusoidal current	несинусоїдний струм
non-steady	неусталений
non-synchronous	несинхронний, асинхронний
non-stationary	неусталений
non-type	нестандартний
normal	нормальний, нормаль, перпендикуляр
normalize	нормалізувати
normal operating condition	нормальний режим
nuclei, nucleus	ядро, центр
number	число, кількість, цифра, рахувати, нумерувати
numberless	незлічений
numerable	який піддається рахуванню
numerator	чисельник, нумератор
nut	гайка
O	
object	ціль, предмет, об'єкт
obstacle	перешкода, завада
obstruct	перепиняти, перешкоджати руху

observe	спостерігати, помічати
obtain	одержувати
obturate	затикати, закривати
occlude	закривати (*прохід*)
odd	непарний
offence	порушення
offset	зміщення, зсув, зміщати, зрушувати
open	відкритий, розімкнений, розмикати
open circuit	розімкнено коло, неробочий хід
open-delta connection	з'єднання в розімкнутий трикутник
opening	відключення
opening mechanism	механізм , що відключає
opening time	час відключення
open-loop control	керування в розімкнутому контурі
operated	керований
operating current	оперативний струм
operating impedance	робочий імпеданс (повний опір)
operating lag (of a relay)	затримка реле при спрацьовуванні
operating range	робочий діапазон
operating rules	правила експлуатації
operating time of protection	час спрацьовування захисту
operating times accuracy	точність часу спрацьовування
operating voltage (in a system)	робоча напруга електричної мережі
operating winding	робоча обмотка
operation	вплив, робота, дія (*мат.*)
operative current	оперативний струм
operative direction	напрямок дії
oppose	протидіяти
opposite	протилежний
opposition	протилежність, зсув за фазою
optimal	оптимальний
optimize	оптимізувати
order	черга, послідовність, степінь (*числа*)
ordered	упорядкований
ordinary	неособливий, звичайний, простий
origin	походження, початок
origin of coordinates	початок координат

oscillate	коливатися, вібрувати, генерувати
oscillation	коливання, качання
oscillograph	осцилограф
outage	вихід з ладу, аварійне відключення
outcome	результат, наслідок
outdoor apparatus	апаратура для зовнішньої установки
outdoor substation	відкрита підстанція
outflow	витікання
outgoing	вихідний
outlet	вихідний отвір, вихідний канал
outline	контур, обрис
out-of-operation	бездіяльний
out-of-order	несправний
out of service	вивести з роботи
output	потужність, вироблення (*електроенергії*)
output circuit	вихідне коло
output current	вихідний струм
output terminal	вихідний затискач
output value	вихідна величина
output winding	вихідна обмотка
outset	початок
outside	зовнішня частина, зовнішній
overcharge	перевантаження, перевантажувати
overcurrent protection	максимальний струмовий захист
overhead line	повітряна лінія
overhead line	капітальний ремонт
overhead system	повітряна електрична мережа
overload	перевантаження, перевантажувати
overload operation	режим перевантаження
overload protection	захист від перевантаження
overheating	захист від перегріву
overvoltage	перенапруга
overvoltage protection	захист максимальної напруги
oversize	збільшений розмір
P	
pair	пара, парний, спарювати

packet switch	пакетний вимикач
pairing, pair twisting	парна скрутка
parallel access -	паралельний доступ
parallel connection	паралельне з'єднання
parallel (shunt) circuit	паралельне коло
paralleling	включення на паралельну роботу
parallel operation	паралельна робота
parasite	пасивний відбивач (*ел.*), перешкоди
parity	рівність, парність
part	частина, частка, розділяти
partial	частковий
particle	частинка
particular	частковий
parting	розгалуження
partition	поділ, розчленування
partitive	роздрібнений, дробовий (*мат.*), частинний (*мат.*), частковий
pass	прохід, перехід; пропускати, приймати
passage	прохід, перехід
passing contact	контакт, що прослизає
passive	пасивний (*елемент*)
path	шлях, траєкторія, контур, вітка (*обмотки*)
pawl	заскочка, підпора, защіпка
peak	максимум, пік (*кривої*)
peak load	максимум навантаження
pendulate	качатися, коливатися
penetrability	проникність
penetrate	проникати
performance	характеристика, к.к.д.
period	період, цикл, коло
periodic component	періодична складова
periodicity	періодичність
permanent fault	стійке пошкодження
permeability	проникність
~ magnetic permeability	магнітна проникність

permissible error	припустима погрішність
persistence	сталість
perturbation	порушення, спотворення
pervade	проникати, насичувати
phase	фаза, фазовий
phase coincidence	збіг за фазою
phase current	струм фази, фазний струм
phase displacement	фазовий зсув
phase-earth coupling	однофазна індуктивність
phase lag	запізнювання за фазою
phase opposition	протифаза
phase segregated differential current position	пофазний диференціальний струмовий захист
phase sequence	послідовність фаз
phase sequence indicator	вказівник чергування фаз
phase sequence reversal	зміна порядку чергування фаз
phase-sequence test	перевірка порядку чергування фаз
phase voltage	фазна напруга
phase to earth voltage (line to ground voltage)	фазна напруга
phase-to-ground fault	замикання фази на землю, КЗ на землю
phase to phase voltage (line to line voltage)	лінійна напруга
phase to neural voltage (line to neural voltage)	фазна напруга
phase unbalance protection	захист від несиметрії фаз
phase voltage of a winding	фазна напруга обмотки
picture	графік, діаграма, рисунок, зображення
pivot	центр (вісь) обертання
place	місце, місцезнаходження, поміщати
~ decimal place	десятковий розряд
placed	розміщений
plane	площина, плоска поверхня
plate	пластина, електрод
plating	гальванічне покривання
plot	діаграма

plug	заглушка, штепсельна вилка (*до розетки*), вилочний контакт
plug-in unit	змінний блок
plus	плюс (*знак додавання*)
point	точка
~ break point	точка розриву безперервності
pointer	покажчик, стрілка
pointer stop	обмежувач ходу стрілки
point of connection	точка приєднання
polarity	полярність
pole	полюс
pole (of an equipment)	полюс (пристрою)
pole (of a d.c. system)	полюс (мережі постійного струму)
polyatomic	багатоатомний
polynomial	багаточлен, поліном
position	положення, розміщати
~ idle position	нейтральне положення, положення неробочого ходу
~ initial position	початкове положення
operating position	робоче положення
positive feedback	додатний зворотний зв'язок
positive sequence	пряма послідовність
positive sequence component	складова прямої послідовності
positive sequence impedance	повний опір прямої послідовності
positive voltage	додатна напруга
post emergency conditions	після аварійний режим
post-fault conditions	після аварійний режим
potential	потенціал
potential difference	різниця потенціалів
power	сила, потужність, енергія, здатність
~ active power	активна потужність
~ instantaneous power	миттьова потужність
~ rated power	номінальна потужність
~ reactive power	реактивна потужність
power factor	коефіцієнт потужності (cosφ)
power frequency	промислова частота
power input	вхідна потужність

power limitation	обмеження потужності
power output	вихідна потужність
power station	електростанція
power supply module	блок живлення
power unit	блок живлення
powerful	потужний
prearcing time	час плавлення
preparation	підготовка, приготування
press	прес, пресувати
press-button switch	кнопковий перемикач
pressure	тиск, стискання, напруга (*ел.*)
pressure monitoring device	реле контролю тиску
primary	первинний
primary circuit	первинне коло
primary winding	первинна обмотка
probability	імовірність
problem	задача (*мат.*), проблема
procedure	операція, метод, спосіб
product	виріб, результат, добуток (*мат.*)
production	виготування, продуктивність, вироблення
productivity	продуктивність
projection	проекція плавление
proof	доказ
property	властивість, характеристика, якість
proportion	пропорція, співвідношення
proportionality	пропорційність
protection	захист, запобігання
protection against short circuits	захист від короткого замикання
protection device	пристрій захисту
protection relay	реле захисту
protection system	система захисту
prove	доводити, перевіряти
pulsating current	струм
pulsating voltage	пульсуюча напруга
pulse-relay	імпульсне реле
pump	насос, помпа

purpose	ціль, призначення
Q	
quadrangle	чотирикутник
quadrate	квадрат, друга ступень
quadruple conductor	чотирьох провідний провідник (чотири проводи на фазу)
quality	якість, властивість, характеристика, дані
quantitative	кількісний
quantity	кількість, величина
quarter	чверть
quotient	частка (*мат.*)
R	
race	шлях, проточний канал
radial network	радіальна мережа
radiated electromagnetic field test	перевірка перешкодостійкості
radiation	випромінювання
radio link protection	захист з радіоканалом
radical	радикал, корінь (*мат.*)
raise	піднімання, піднімати
ramify	розгалужуватись
random synchronizing	включення без контролю синхронізму
range	ряд, лінія, дальність, далекість
range of adjustment	діапазон регулювань
range of regulation	зона регулювання
range of scale	діапазон шкали
ranging	вимірювання відстані
rapidity	швидкість
rate	стан, режим
rate of change of... relay	реле похідної
rated capacity	номінальна ємність
rated current of a contact	номінальний струм контакту
rated	номінальній, розрахунковий
ratio	співвідношення, ступень
ray	промінь, випромінювати
reactance	реактивний опір, реактанс
reaction	реакція, взаємодія

reactive energy	реактивна енергія
reactivity	реактивний
reactor	стабілізатор (*ел.*)
reading	показ (*приладу*)
real time	реальний час
receiver	приймач
receiving channel	канал прийому
recharge	перезаряджування
reclamation	виправлення, відновлювання
record	запис, записувати
rectangle	прямокутник
rectangular coordinates	прямокутні координати
rectification	випрямлення
rectifier	випрямляч (*ел.*)
rectify	випрямлювати (*струм*)
rectilinear	прямолінійний
reduced measuring error	відносна погрішність виміру
reduced voltage	знижена напруга
reduction	зниження, зменшення
reduplication	подвоєння
reference value	відносна величина
reference voltage	опорна напруга
reflected wave	відбита хвиля
reflection	відбивання
region	область, зона
regularity	регулярність
regularity condition	умова регулярності
regulate	регулювання
regulation	регулювання
reject	відхиляти
relate	бути зв'язаним
relation	співвідношення, зв’язок
relative error	відносна похибка
relay	реле, ставити реле
relay coil	котушка реле
relay protection	релейний захист
relay winding	обмотка реле

reliability	надійність
relocation	переміщення
repair	ремонт
replace	заміняти, заміщати
replacement	заміщення, заміна
reset	повернення
residual current	залишковий струм
residual magnetization	залишкове намагнічування
residual voltage	напруга нульової послідовності
resistance	опір
resistivity	питомий опір
resistor	опір, реостат
resonance	резонанс
resonant	резонансний
response	реагування, спрацьовування, характеристика
restart	заново запускати
restore	відновлювати
result	результат
reversal	реверсування, зворотний хід
reversal of magnetization	перемагнічування
reverse	зворотний (хід), реверсування
revolution	кругове обертання
rise in frequency	підвищення частоти
root	корінь, основа
root-mean-square deviation	середньоквадратичне відхилення
root-mean-square value	середньоквадратична величина
rotary	обертовий
rotate	обертатися
rotation	обертання
rotor winding	обмотка ротора
round	круг, коло, обхід
runaway	рознос (*ел. двигуна*)
S	
safe	безпечний, надійний
safeguard	запобіжник
safety	безпека

sample	вибірка
sampling cycle	цикл вимірів
saturate	насичувати
saturation	насичення
saturation region	область насичення
scale	масштаб, шкала
schematic diagram	структурна схема
scheme	схема, план
screen	екран, екранувати
screening	екранування
search	пошук
secondary	вторинний, вторинна обмотка
secondary circuit	вторинне коло
secondary voltage	вторинна напруга
secondary winding	вторинна обмотка
section	поперечний перетин
selection	вибір, відбір
self-capacitance	власна ємність
self-control	пряме управління
self-discharge	саморозряд
self-excitation	самозбудження
self-excited	з самозбудженням
self-extinguishing fault	ушкодження, що самоусувається
self holding contact	контакт, що сам втримується
self inductance	самоіндукція
self-regulation	саморегулювання
self-impedance	власний повний опір
self-inductance	індуктивність
self-induction	самоіндукція
semiconductor	напівпровідник
sensitive	чутливий
sensitive protection	чутливий захист (чого-небудь)
sensitivity	чутливість
sensitivity of a directional element	чутливість спрямованого елемента
separate	відділяти, розділяти
separate network	відособлена мережа
separated windings	роздільні обмотки

sequence	послідовність
series	ряд, набір, послідовне з'єднання
series compensation	послідовна компенсація
series connection	послідовне з'єднання
series-parallel connection	послідовне-паралельне з'єднання
set	набір, установлювання, розміщувати
shear	зрізування, зсув
sheet	лист, пластина
shift	зсув, зміщення
shifting	зміщення, переміщення
short	коротке замикання
short circuit	короткозамкнене коло
short circuited	короткозамкнений
short circuit and earth	коротке замикання на землю
short-circuit between phases	міжфазне коротке замикання
short-circuit characteristic	характеристика короткого замикання
short-circuit current	струм короткого замикання
short circuit current calculations	розрахунки струмів короткого замикання (КЗ)
short circuit earth current	струм КЗ на землю
short circuit protection	захист від КЗ
short-circuit power	потужність КЗ
short-circuit through an arc	дугове КЗ
short-circuit to earth	КЗ на землю
shock current	ударний струм
shunt	шунт, шунтувати
shunted	паралельне з'єднаний
shunting	шунтування
shunt-wound	з паралельним збудженням
shut-down	зупинка, неробочий період
shut-off	зупинка, вимикання
signal	імпульс, сигнал (*ел.*)
signal level	рівень сигналу
similar	подібний
simplification	спрощення
simplify	спрощувати
simulation	моделювання

single conductor	один провід
single-phase	однофазний
single phase automatic reclosing control equipment	однофазний пристрій автоматичного повторного включення
single-phase short circuit	однофазне КЗ
single-phase transformer	однофазний трансформатор
single-pole switch	однополюсний вимикач
slip	ковзання
slip frequency	частота сковзання
slope	нахил, падіння
slot	паз, канавка
slug	осердя (*ел.*)
slugged	інерційний
small-sized	малогабаритний
smelt	топлення, топити
socket	розетка, гніздо, цоколь
socle	цоколь (*ел.*)
software	програмне забезпечення
solid	твердий, тверде тіло
solid state switch	напівпровідниковий перемикач
solidly earthed neutral	глухо заземлена нейтраль
solution	рішення
source	джерело
space	відстань, проміжок
spare parts	запасні частини
sparking	іскріння, запалювання
speciality	особливість, *мн.* деталі
specified value	задана величина
speed	швидкість, число оборотів
speed governor	регулятор швидкості
speed regulator	регулятор частоти обертання
splash	сплеск
splicing	розгалужування
split-phase	розщеплена фаза
spread	розширювання, розширювати
spring	пружина
spurious opening	помилкове відключення

spurious tripping	помилкове відключення
square	*матем*. квадрат величини
squared	у квадраті
stability	стійкість
stabilize	стабілізувати
stable	сталий
stable conditions	сталий режим
stage	стадія, період, фаза
stand-by supply	резервне живлення
star	зірка (*ел. з'єднання*)
star connection	з'єднання зіркою
star-delta connection	з'єднання "зірка - трикутник"
star-delta switch	перемикання із зірки на трикутник
star-star connection	з'єднання зірка-зірка
starting	запуск, початок
starting relay	пускове реле
starting tests	пускові випробування
start of parallel operation	початок паралельної роботи
start operation (of a relay)	початок роботи (реле)
start-stop control	переривчасте керування
state	стан
statement	повідомлення, ствердження
static	статичний, нерухомий
static compensator	статичний компенсатор
static convertor	статичний перетворювач
static error	статична помилка
static excitation	статичне порушення
stationary	стаціонарний
station auxiliaries voltage	напруга власних потреб
stator winding	обмотка статора
steady	стійкий, постійний
steady-state	сталий, примушений
steady-state short-circuit current	сталий струм КЗ
steady-state stability	статична стійкість
step	ступень, шаг
step-down transformer	понижуючий трансформатор
stepless control	плавне регулювання

step response	перехідна характеристика
step-up transformer	підвищувальний трансформатор
storage	нагромаджування, нагромаджувач
storage element	нагромаджувач
store	запас, нагромаджувати
straight	прямий, лінійний
stream	потік, течія, теча
strength	сила, міцність
stress	напруга, зусилля
strike	запалювати дугу (*ел.*)
stroke	штрих, риса
structure	структура, пристрій
subsequent faults	послідовні ушкодження
substantial	міцний
substation	електрична підстанція
substitute	підставити
succession	послідовність
sudden change in frequency	раптова зміна частоти
sum	сума
superconductivity	надпровідність
superposition method	метод накладення
superposition	накладання
supply	постачання
supply voltage	напруга живлення
support	підтримувати
surface	поверхня
susceptance	реактивна провідність
~ capacitive susceptance	ємнісна провідність
~ inductive susceptance	індуктивна провідність
susceptibility	чутливість
sustained short-circuit current	сталий струм короткого замикання
sweep	розгорнення
swing blocking	блокування від хитань
switch	вимикач, ключ
~ switch in	вмикати
~ switch out (off)	вимикати
switching	переключення, комутація

switch-gear	розподільний пристрій
symbol	символ, знак
symmetrical component method	метод симетричних складових
symmetrical short circuit	симетричне коротке замикання
symmetrical voltage	симетрична напруга
synchronization	синхронізація
synchronism	синхронізм
synchronous compensator	синхронний компенсатор
synchronous generator	синхронний генератор
synchronous motor	синхронний двигун
synchronous operation	синхронна робота
synchronous reactance	синхронний реактанс
systematic error	систематична погрішність
system configuration	конфігурація електричної мережі
system control	керування енергосистемою
system diagram	схема системи (електричної мережі)
system operational diagram	оперативна схема електричної мережі
system	система
~ system of axis	система координат
~ system of equations	система рівнянь
~ system of units	система одиниць
T	
table	таблиця
tabulation	складання таблиць
tangent	тангенс, дотична
tangential	дотичний
tape	магнітна смужка
tapped line (teed line)	магістральна лінія
target	завдання, план
task	завдання
technique	техніка, обладнання, устаткування
telecommunication	зв'язок
telemetering	телевимірювання
temperature-dependant	залежний від температури
tension	*ел.* напруга
term	математичний член
terminal	затискач, клема

terminal voltage	напруга на затискачах
tertiary winding	третинна обмотка
test	перевіряння, перевірка, перевіряти
thermal power station	теплова електростанція
thermal relay	теплове реле
thermocouple	термопара
third harmonic	третя гармоніка
three-element relay	триелементне реле
three-phase	трифазний
three-phase fault	трифазне пошкодження
three windings transformer	триобмоточний трансформатор
threshold (of) sensitivity	поріг чутливості
threshold value	гранична величина
through fault current	наскрізний струм ушкодження (КЗ)
three-phase system diagram	схема електричної мережі в трифазному виконанні
three-phase transformer	трифазний трансформатор
three-pole	триполюсний
three windings transformer	триобмоточний трансформатор
three-wire	трипровідний
threshold (of) sensitivity	поріг чутливості
throttle	дросель
through fault current	наскрізний струм пошкодження (КЗ)
thyristor	тиристор
time characteristic	часова залежність
time constant	постійна часу
time delay	витримка часу
time-dependant	залежний від часу
time differential	ступень витримки часу
time	реле часу
time schedule	графік роботи
time to operate	час дії
timing element	елемент часу
toggle switch	перекидний перемикач
tolerance	допуск
total break time	повний час відключення
total current	сумарний струм

total loss of load	повне скидання навантаження
total solution	повне (загальне) рішення
trace	слід, залишати слід
track	слід, простежувати
transducer	датчик
transfer	переміщання, передавання
transfer function	передавальна функція
transform	трансформувати
transformation	трансформація, перетворювання
transformation of electricity	трансформація електричної енергії
transformer	трансформатор
transformer bank	трансформаторна група
transformer circuit-breaker	трансформаторний вимикач
transformer protection	релейний захист (РЗ) трансформатора
transformer ratio	коефіцієнт трансформації трансформатора
transformer substation	трансформаторна підстанція (ТП)
transformer winding	обмотка трансформатора
transformer with regulation in phase	трансформатор з регулюванням напруги по фазі
transforming station	трансформаторна підстанція
transient	перехідний процес
~ damping transient	затухаючий перехідний процес
transient analysis	аналіз перехідного процесу
transient fault	нестійке пошкодження
transient feedback	гнучкий зворотний зв'язок
transient performance	якість перехідного процесу
transient phenomenon	перехідний процес
transient function	перехідна функція
transient solution	перехідне (*нестійке*) рішення
transient reactance	перехідний реактанс
transient response	перехідна характеристика
transient short circuit	нестале КЗ
transient short circuit current	перехідний струм КЗ
transient stability	динамічна стійкість
transient state	несталий режим
transistor	транзистор

transition	*матем.* перетворювання
translator (isolating transformer)	ізолюючий трансформатор
transmission	передавання
transmission channel	передавальний канал
transmission data rate	швидкість передачі даних
transmission limit	межа переданої потужності
transmission line capability	пропускна спроможність ЛЕП
transmission of electricity	передача електроенергії
transmitter	передавач
transmissivity	проникність
transmit	передавати, надсилати
transport	перенесення, переміщення
transportation	транспортування
traveling wave	хвиля, що біжить
treating	обробка
treatment	обробка
triangle	трикутник
trigger	пусковий пристрій
trip	відключити
trip circuit	коло відключення
trip a circuit breaker	подати команду на відключення вимикача
trip coil	котушка відключення
triple conductor	три проводи у фазі
triple-frequency harmonic	третя гармоніка
triple unit (Motor, turbine)	строєний агрегат (двигун, турбіна)
tripping	вимикання, розмикання
tripping relay	реле, що відключає
tripping time	час спрацьовування
true value	істинне значення
trunk line	магістральна ЛЕП
trunk main	магістральна ЛЕП
turn-fault protection	захист від міжвіткових КЗ
tuned circuit	резонансний контур

tuning capacitor	підстроєчний конденсатор
tuning indicator	індикатор настройки
tuning range	діапазон настройки
turbine governor	регулятор турбіни
turbogenerator	турбогенератор
turbo-generator set -	паротурбінний агрегат
turn	оборот, повертати, виток (*дроту*)
turn-fault	міжвіткове пошкодження (КЗ)
turn-fault protection	захист від міжвіткових КЗ
turn-to-turn short circuit	міжвіткове коротке замикання
twin conductor	два проводи у фазі
twisted conductor	скручений багатожильний провідник (провід)
twist joint	з'єднання скруткою
two-layer winding	двошарова обмотка
two-way contact	двошаровий (перекидний) контакт
two-way feed	двошарове живлення
two-stage relay	двопозиційне реле
type	тип, вид
U	
ultra-high voltage	надвисока напруга
unattended substation	підстанція, що не обслуговується
unbalance current	струм небалансу
unbalanced conditions	несиметричний режим
unbalanced short circuit	несиметричне КЗ
unblocking	деблокіровка (розблокіровка)
uncharge	розряджати
unclose	відкривати, розмикати
underdamping	слабке загасання
underfrequency relay	реле захисту від зниження частоти
underground system	кабельна електрична мережа
underline	підкреслювати
undetermined	невизначений
uneven	непарний
unfaulted phase	неушкоджена фаза
unidirectional	одного напрямку

unidirectional current	струм одного напрямку
uniform	рівномірний
unify	поєднувати
uninterrupted operation	безперебійна робота
uninterruptible power supply	гарантоване енергопостачання
unit	одиниця, вузол
unity	одиниця (*мат.*)
unknown	невідома величина
unload	розвантаження
unloaded line	ненавантажена ЛЕП
unstable	несталий
upper harmonics	вищі гармоніки
uptake	вертикальний канал
urban network	міська розподільна мережа
use	застосування
used	використаний
user	споживач
V	
variable	змінна величина
variation	змінювання, переміна
variety	різноманітність
vector diagram	векторна діаграма
vector group (of a transformer)	група з'єднання трансформатора
vector product	векторний добуток
vectorial	векторний
velocity	швидкість
~ angular velocity	кутова швидкість
~ average velocity	середня швидкість
~ initial velocity	початкова швидкість
~ phase velocity	фазова швидкість
~ resultant velocity	підсумкова швидкість
~ wave velocity	швидкість поширювання хвилі
vibration	вібрація, коливання
view	вид, проекція
virtual	фактичний
visibility	видимість, обзір

visual	наочний
voltage	напруга
voltage balance	баланс напруг
voltage build	наростання напруги
voltage circuit	коло напруги
voltage deviation	відхилення напруги
voltage direction	полярність напруги
voltage divider	дільник напруги
voltage division	розподіл напруги
voltage drop	падіння напруги
~ external voltage	зовнішня напруга
~ impedance voltage	напруга короткого замикання
~ input voltage	вхідна напруга
~ internal voltage	внутрішня напруга
~ line voltage	лінійна напруга
~ open circuit voltage	напруга неробочого ходу
~ peak voltage	максимальна (пікова) напруга
~ phase voltage	фазова напруга
~ rated voltage	номінальна напруга
terminal voltage	напруга на затискачах
transient voltage	перехідна напруга
voltage level	ступень напруги
voltage limiting	обмеження напруги
voltage range	діапазон напруг
voltage regulator	регулятор напруги
voltage relay	реле напруги
voltage ripple	пульсація напруги
voltage rise	підйом напруги
voltage transformer	трансформатор напруги
voltage transformer error	погрішність трансформатору напруги
voltage trebling	потроєння напруги
volt-ampere characteristic	вольт-амперна характеристика (ВАХ)
volume	об’єм
vortical	віхровий
vorticity	інтенсивність вихору

W	
wattage	споживання активної потужності
watt consumption	споживання активної потужності
wave	хвиля
~ Hertz waves	електромагнітні хвилі
~ moving wave	біжуча хвиля
waveform	форма хвилі
wavelength	довжина хвилі
wear resistance	тривкість проти спрацьовування
weld	зварний шов
welding	зварювання, зварювальний
wheel	колесо, шестерня
wind	струм, потік
winding	обмотка (*ел.*)
winding path	напрямок намотування обмотки
wire	провід, дріт
wire-pilot	зв'язок проводом у диференціальному захисті
wire-wrap connection	накрутка
wire wrapping	накрутка
wiring	монтажна схема (монтаж)
wiring blemish	дефект монтажу
withdrawal from service	висновок з роботи
working zone	робоча зона
wye	зірка, з'єднання зіркою (*ел.*)
wye-delta	зірка-трикутник
wye-delta connection	з'єднання зірка - трикутник
wye-wye	зірка-зірка
Y	
yield	випуск, виробляти
Y-motion	рух у напрямку осі Y
yoke	обойма, ярмо
~ magnet yoke	ярмо магніту
Y-shaped	Y- подібний
Y-system	з'єднання зіркою

Z	
z-connection	з'єднання зиґзаґом
zero	нуль, нульова точка
zero crossing	перехід через нуль
zero drift	дрейф нуля
zero error	зрушення нуля
zero mark	нульова оцінка
zero offset	зсув нуля
zero phase-sequence voltage	напруга нульової послідовності
zero sequence component	складова нульової послідовності
zero sequence current	струм нульової послідовності
zero sequence impedance	імпеданс нульової послідовності
zero sequence voltage relay	реле напруги нульової послідовності
zone of protection	зона, що захищає
Z-motion	рух у напрямку осі Z
zone	зона, ділянка
~ zone of protection	захисна зона
~ active zone	активна зона
~ forbidden zone	заборонена зона
~ instability zone	зона нестійкості
~ neutral zone	нейтральна зона
~ safety zone	зона безпеки
~ stability zone	зона стійкості
~ transition zone	перехідна зона

LITERATURE

1. Англо-український словник з інформатики / уклад.: І. Білянкевич, Б. Шуневич; наук. ред. В. Перхач. - Львів, 1995. - 90 с.

2. Бессонов Л.А. Теоретические основы электротехники: Учебник для вузов. - М: Высшая школа, 1988. - 528 с.

3. Данько В.Г., Мілих В.І., Карпенко Ф.Т., Марков В.С. Збірник задач з електротехніки: Навч. посібник. - Х.: НТУ "ХПІ", 2004 - 119с.

4. Данько В.Г., Мілих В.І., Черкасов А.К. та ін. Електротехніка: Навчальний посібник для самостійної роботи студентів. - К.: УМК ВО, 1993 - 94 с.

5. Зевеке Г.В., Ионкин П.А., Нетушил А.В. и др. Основы теории цепей. - М.: Энергоатомиздат, 1987. - 528 с.

6. Качан Ю.Г. Лінійна електротехніка (теоретичні основи). - Запоріжжя: Видавництво ЗДІА, 1995. - 205 с.

7. Мілих В.І. Електротехніка та електромеханіка: Навч. посібник. - К: Каравела, 2005. - 376 с.

8. Мілих В.І., Шавьолкін О.О. Електротехніка, Електроніка та мікропроцесорна техніка: Навч. посібник. / За ред. В.І.Мілих. - К: Каравела, 2007. - 686 с.

9. Нейман Л.Р., Демирчян К.С. Теоретические основы электротехники. Ч. I, II. Учебник для студ. электротехнических и энергетических спец. вузов. Л.: Энергоиздат, 1983. - 533 с., 517 с.

10. Перхач В. С. Теоретична електротехніка. Лінійні кола : Підручник для студ. електроенерг. спец. вищ. навч. закл. - К.: Виша щк., 1992. - 439 с. - Бібліогр.: - 435 с.

11. Сборник задач по теоретическим основам электротехники: Учеб пособие для энерг. и приборостроит. спец. вузов. / Под ред.Л.А. Бессонова. - М.: Высш.шк., 1988. - 543 с.

12. Теоретические основы электротехники: Учебник для вузов. В трёх частях. Ч. 1. Линейные электрические цепи. / Г. И. Атабеков, С.Д. Купалян, А.Б. Тимофеев, С.С. Хухриков. Под редакцией Г. И. Атабекова. - М.: Энергия, 1984. - 425 с.

13. Чабан В. О. Анґло-український електротехнічний словник, - Львів., 1995. - 255 с.

14. Шебес М.Р., Каблукова М.В. Задачник по теории электрических цепей. - М.: Высшая школа, 1990. - 528 с.

15. Kozlov V.V., Nabokova O.V. Summary of lectures on Theoretical Electrical Engineering. Part I. - Zaporozhye: ZNTU, 2007. - 87 p.

16. Kozlov V.V., Nabokova O.V. Summary of lectures on Theoretical Electrical Engineering. Part II. - Zaporozhye: ZNTU, 2007. - 95 p.

17. Англійсько-російсько-український електротехнічний словник для студентів спеціальності "Технічний переклад". Укл. Набокова О.В. - Запоріжжя: ЗНТУ, 2007. - 50 с.

18. Methodical Instructions for independent students' work on Electrical Engineering course "Linear direct current and alternating current circuits" for specialities 7.092206, 7.092204. /Authors: Kozlov V.V., Nabokova O.V. - Zaporozhye: ZNTU, 2009. - 60 p.

19. Methodical Instructions for students' self-dependent calculation-graphic work on Electrical Engineering course "The three-phase circuits" for specialities 7.092206, 7.092204. /Authors: Kozlov V.V., Nabokova O.V. - Zaporozhye: ZNTU, 2010. - 38 p.

20. Methodical Instructions for students' self-dependent calculation-graphic work on theoretical engineering course "The transient processes in the linear electrical circuits" for specialities 7.092206, 7.092204. /Authors: Kozlov V.V., Nabokova O.V. – Zaporozhye: ZNTU, 2011. – 38 p.

21. S. Rosenberg. Basic Electricity. - Indianapolis, by Wiley Publishing, Inc., 2007. - 357 p.

22. Paul C.R., Nasar S.A., Unnewehr L E. Introduction to electrical engineering. - New York, McGraw-Hill Book Company, 1991. - 815 p.