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Затверджено
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SINGLE-PHASE SINUSOIDAL ALTERNATING CURRENT CIRCUITS

4.1. Sinusoidal Functions - Terminology

Appearing in Fig. 4.1 are two sinusoids - one denoting voltage and the other current. The voltage sinusoid is a sine function which is represented mathematically as

$$u = U_m \sin(\omega t + \varphi_u) \tag{4.1}$$

where $(\omega t + \varphi_u)$ is called the argument of the sine function. The *amplitude* of the sine function is U_m and it denotes the maximum value of the sine function. Equation (4.1) is often referred to as the expression for the *instantaneous voltage* because by insertion of any particular value for *t* the corresponding value of *u* is determined.

The equation which describes the instantaneous value of the current wave

$$i = I_m \sin(\omega t + \varphi_i)\,. \tag{4.2}$$

This current is termed a *sinusoidal alternating current*. In this equation and also in the graph of Fig. 4.1, *i* is the *instantaneous current* at the time ***t*****.** The value I_m is the *peak value* or *amplitude* of the alternating current (*ac* in abbreviated form), and *T* is the time occupied by one complete cycle of change, or the *period*. A word of caution is appropriate at this point concerning the units of the argument $(\omega t - \varphi_i)$. The form of Eq. (4.2) demands that the unit for this total quantity be radians.

There exist alternate forms with which to express sinusoids. These originate with Euler's identity, that is,

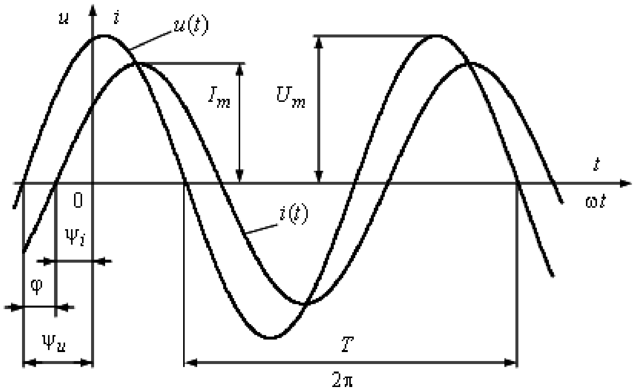
$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

It should be apparent from this expression that a cosine and a sine function can be expressed in terms of the exponential notation as follows:

$$\cos\omega t = \mathrm{Re}\left\lfloor e^{j\omega t} \right\rfloor$$

$$\sin\omega t = \mathrm{Im}\left\lfloor e^{j\omega t} \right\rfloor$$

where $\mathrm{Re}\left\lfloor e^{j\omega t} \right\rfloor$ denotes the real part of the expression in brackets and $\mathrm{Im}\left\lfloor e^{j\omega t} \right\rfloor$ the imaginary part.



The values of the argument of the sine function that lie between 0 and 2π radians, different values of the function result. Any complete set of positive and negative values of the function is called a *cycle*.

Another characteristic of the sinusoid is its *frequency*, which is denned as the number of cycles of the function which is traversed in 1 *s*. The time duration of one cycle is called the *period* of the function.

$$T = \frac{1}{f} \tag{4.3}$$

Moreover, a glance at Fig. 4.1 indicates that each cycle spans 2π radians. Hence, if this quantity is divided by the period, there results the *angular velocity* (or angular frequency) of the sine function. It is denoted by ω and has units of rad/second.

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4.4}$$

The second form of ω is obtained from Eq. (4.3).

The equation to be used to describe the second sinusoid appearing in Fig. 4.1, i.e., the current wave, obviously cannot be identical to that used for the voltage wave. The reason is that the two sinusoids are displaced from one another by some angles and so cannot have the same instantaneous value. Accordingly, the current wave is said to be *lagging* behind the voltage wave by the relative phase angle $\varphi = \psi_u - \psi_i$. *Relative phase* is the term used to denote the angular displacement between two sinusoids of the same frequency. Alternatively, we can say that the voltage wave has a *phase lead* of φ degrees relative to the current wave.

4.2. Average and Effective Values of Periodic Functions

The sinusoidal current is an *alternating* current, i.e., one which has positive and negative values. The term periodic is used rather than sinusoidal in order that the treatment is general. The sinusoid is only one example of a periodic function. Any function whose cycle is repeated continuously irrespective of waveform is called a periodic function.

The *average* (or *mean*) value of sinusoidal quantity over one cycle, or an integral number of cycles, is obviously zero. A significant mean will therefore be the average of the values prevailing during one positive (or negative) half-cycle. Thus, the average current during one half-cycle is

$$I_{av} = \frac{1}{T/2} \cdot \int_0^{T/2} I_m \sin \omega t \cdot dt = \frac{2}{\pi} \cdot I_m = 0.636\, I_m \tag{4.5}$$

that is, the average current during one positive (or negative) half-cycle is $2/\pi$ = 0.636 times the peak current. Similarly

$$E_{av} = \frac{2}{\pi} \cdot E_m; \; U_{av} = \frac{2}{\pi} \cdot U_m \tag{4.6}$$

Another point of interest is that the unit of the bracketed quantity is amperes squared. Accordingly, we can interpret this quantity as the square of an effective current, which, when multiplied by *R*, yields the average power. Expressing this mathematically, we can write

$$I_{ef}{}^2 = \frac{1}{T} \int_0^T i^2(t)\, dt = average \;\; i^2(t)$$

From this it follows that the effective current is the root mean square value. Thus

$$I_{ef} = \sqrt{average\ i^2(t)} = \sqrt{\frac{1}{T} \int_0^T i^2(t)} \tag{4.7}$$

For the practical purposes a convenient mean value is the *effective* value of current. It is the *square root of the mean of the currents squared* (which are all positive). It is the magnitude of that direct current which has the same heating effect in a given resistive circuit as the alternating current in question.

Hence the root mean square (r.m.s. or effective) value of current

$$I_{ef} = \sqrt{\frac{1}{T}\cdot\int_0^T i^2 dt} = \sqrt{\frac{1}{T}\cdot\int_0^T I_m{}^2 \sin^2 \omega t\cdot dt} = \frac{I_m}{\sqrt{2}} = 0.707\cdot I_m \quad (4.8)$$

Similarly,

$$E_{ef} = \frac{E_m}{\sqrt{2}} = 0.707\cdot E_m,\ U_{ef} = \frac{U_m}{\sqrt{2}} = 0.707\cdot U_m \quad (4.9)$$

4.3. Complex Representation of Sinusoidal Quantities

Often in determining the sinusoidal steady-state response of circuits it is necessary to perform algebraic operations such as addition, subtraction, multiplication, and division on two or more sinusoidal quantities of the same frequency. Usually, the sinusoids differ in amplitude and phase. Specifically consider the matter of adding two sinusoidal currents whose equations are

$$i_1 = I_{m_1} \sin\omega t,\quad i_2 = I_{m_2} \sin(\omega t + \psi_2) \quad (4.10)$$

The current i_2 leads i_1 by the relative phase angle ψ. The resultant current i_3 can obviously be written as

$$i_3 = I_{m_1} \sin\omega t + I_{m_2} \sin(\omega t + \psi_2) \quad (4.11)$$

The amplitude and phase of the resultant sinusoid can then be measured, thus allowing i_3 to be written in the more useful form

$$i_3 = I_{m_3} \sin(\omega t + \psi_3) \quad (4.12)$$

where ψ_3 is the phase angle measured with respect to the same reference point used for ψ_2. But this procedure is laborious and time-consuming.

We need a simpler and more direct method of treating sinusoidal quantities. In 1893 such a method was introduced, when Charles P. Steinmetz advanced the idea of using a constant-amplitude line rotating at a frequency ω to represent a sinusoid. Let us now see how such an idea is effective in simplifying the algebraic operations involving sinusoidal quantities.

Attention is first directed to the expression for i_1, as given by Eq. (4.10). By employing the notation of $\sin\omega t = \mathrm{Im}\left\lfloor e^{j\omega t}\right\rfloor$ we have

$$i_1 = I_{m_1} \sin\omega t = \mathrm{Im}\left\lfloor I_{m_1}\ e^{j\omega t}\right\rfloor \quad (4.13)$$

Keep in mind that the exponential function $e^{j\omega t}$ may be treated as a rotational operator. Its amplitude is always unity, but the cosine and sine components vary as time progresses. This is illustrated in Fig. 4.2 (a). As ωt moves through one full period of 2π radians (i.e., one complete cycle) the line *OA* makes one complete traversal of the circle in a counter clockwise direction. Line *OA* is fixed in value to the amplitude of the sine function it represents. Note that the vertical component of line *OA* is the sine function.

If we plot the vertical components of *OA* as it makes one complete revolution, the sine function shown in Fig. 4.2 (b). is generated. When $\omega t = 0$ the position of *OA* is on the horizontal axis directed towards the right. Its vertical component at this instant is zero, as it should be for the sine function. The current i_2 can be represented in a similar fashion. Thus in terms of the exponential notation, we have

$$i_2 = I_{m_2} \sin(\omega t + \psi_2) = \mathrm{Im}\left\lfloor I_{m_2}\ e^{j\psi_2 t}\, e^{j\omega t}\right\rfloor \qquad (4.14)$$

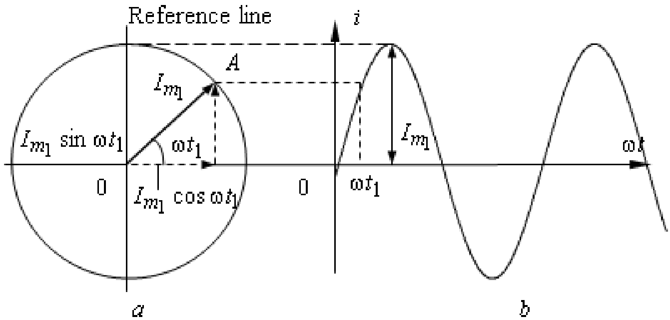


Fig. 4.2. Generating the sine function from the vertical component of the rotating line

To simplify the notation, we next define

$$\underline{I}_{m_2} = I_{m_2} e^{j\psi_2} = I_{m_2}\ (\cos\psi_2 + j\sin\psi_2) \qquad (4.15)$$

and denotes it as line *OB* in Fig. 4.3, *a*. It is this quantity - I_{m2} - which is called the *phasor* of the sinusoidal function of Eq. (4.14).

In general, the phasor can be represented by a *complex number* which is a result of locating a line in a plane. Thus, the phasor *OB* can be located by specifying its magnitude, I_{m2}, and its displacement from the horizontal axis ψ_2.

Note that *OB* can be located in terms of a horizontal (i.e., real-axis) component and a vertical (i.e., imaginary or *j*-axis) component as indicated by the second form of $\underline{I}_{m_2}$ in Eq. (4.15). For a given ψ_2 the real part of the complex number is $I_{m2}\cos\psi_2$ and the imaginary or *j* part is $I_{m2}\sin\psi_2$. It is important that the position of *OB* in Fig. 4.3, *a* corresponds to time $t = 0$ in Eq. (4.14). The corresponding value of i_2 at this instant must be the projection of *OB* on the vertical reference line, which clearly is $I_{m2}\sin\psi_2$.

The phasor of current i_1 is $\underline{I}_{m1} = I_{m1}e^{j0^o} = I_{m_1}$. Hence it has no vertical {or *j*) component and so initially (i.e., at $t = 0$) must lie along the horizontal axis as shown by line *OA* in Fig. 4.3, *a*. Note too in this figure that the phasors $\underline{I}_{m1}$ and $\underline{I}_{m2}$ are displaced from one another by ψ_2. This is entirely consistent with Eqs. (4.10).

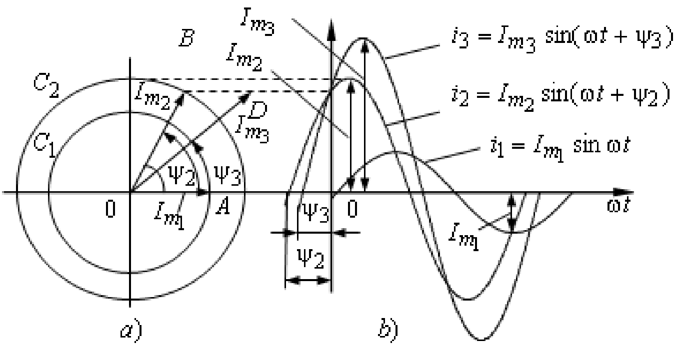


Fig. 4.3. Illustration of the addition of two sinusoids, *a*) position of the phasor for $t = 0$; *b*) sinusoidal function for increasing time

Now, as time elapses, both these phasors revolve at a constant frequency of ω radians per second, and a continuous plot of the vertical components of both phasors results in the sinusoids shown in Fig. 4.3, *b*. The magnitude of the line must be equal to the amplitude of the sinusoid.

How does the phasor representation of sinusoids simplify the procedure for adding two sinusoidal quantities?

To understand this, let us find $i_3 = i_1 + i_2$ by rewriting Eq. (4.11) in exponential form. Thus

$$i_3 = i_1 + i_2 = \mathrm{Im}\left[\underline{I}_{m_1}e^{j\omega t}\right] + \mathrm{Im}\left[\underline{I}_{m_2}e^{j\omega t}\right] = \mathrm{Im}\left[\left(\underline{I}_{m_1} + \underline{I}_{m_2}\right)e^{j\omega t}\right] \qquad (4.16)$$

Expressed in words, Eq. (4.16) states:

To add two sinusoidal quantities, it is necessary merely to add the corresponding phasors.

Accordingly, if we call the phasor quantity of the resultant $\underline{I}_{m3} = I_{m3} e^{j\psi_3}$, we may write

$$\underline{I}_{m_3} = \underline{I}_{m_1} + \underline{I}_{m_2} \tag{4.17}$$

$$I_{m_3} e^{j\psi_3} = I_{m_1} + I_{m_2} e^{j\psi_2} = (I_{m_1} + I_{m_2} \cos\psi_2) + jI_{m_2} \sin\psi_2 \tag{4.18}$$

The right side of the last equation involves all known quantities. Moreover, since it is complex number having horizontal and vertical components, it follows that the magnitude of the component is

$$I_{m_3} = \sqrt{(I_{m_1} + I_{m_2} \cos\psi_2)^2 + (I_{m_2} \sin\psi_2)^2} \tag{4.19}$$

And the angle is

$$\psi_3 = \tan^{-1} \frac{I_{m_2} \sin\psi_2}{I_{m_1} + I_{m_2} \cos\psi_2} \tag{4.20}$$

The resultant phasor I_{m3} is depicted in Fig. 4.3 as line OD. When the value of I_{m3} has been established, the corresponding time expression for i_3 readily follows from Eq. (4.16). Hence

$$i_3 = \operatorname{Im}\left[\underline{I}_{m_3} e^{j\omega t}\right] = \operatorname{Im}\left[I_{m_3} e^{j\psi_3} e^{j\omega t}\right] = \operatorname{Im}\left[I_{m_3} e^{j(\omega t + \psi_3)}\right] =$$

$$= \operatorname{Im}\left[I_{m_3} \cos(\omega t + \psi_3) + jI_{m_3} \sin(\omega t + \psi_3)\right] =$$

$$= I_{m_3} \sin(\omega t + \psi_3) \tag{4.21}$$

The set of vectors of electrical magnitudes of the same frequency represented in uniform axes is called *a vector diagram*. If there are some vectors of various electrical magnitudes (usually a current and a voltage) in a diagram, then such a figure is called a combined diagram Fig. (4.4).

At calculation of sinusoidal current circuit, one must constantly carry out the operations of addition (or subtraction) of instantaneous values of currents, voltages and e.m.f.s.

Vector diagrams allows replacing addition and subtraction of instantaneous values by addition and subtraction of corresponding vectors.

The vector which represents a total sinusoid is equal to the sum of the vectors which represent composed items (Fig. 4.5).

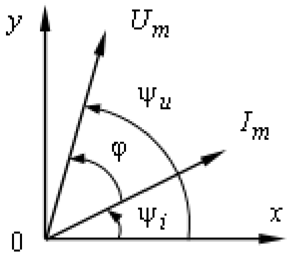


Fig. 4.4. A combined diagram

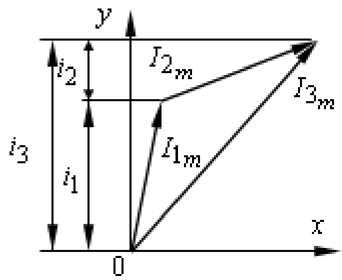


Fig. 4.5. The additional of vectors

Example 4.1. Two sinusoidal currents are described as follows: $i_1 = 10\sqrt{2}\,\sin\omega t$ and $i_2 = 20\sqrt{2}\,\sin(\omega t + 60^o)$. Find their sum.

Solution: The solution is found by performing phasor addition in the manner indicated by Eq. (4.17).

In passing, note that i_1 and i_2 have effective values of 1o and 20 *A*, respectively. The phasor quantities are

$$\underline{I}_{m_1} = I_{m_1} = 10\sqrt{2}$$

$$\underline{I}_{m_2} = 20\sqrt{2}\,(\cos 60^o + j\sin 60^o) = 20\sqrt{2}\left(\frac{1}{2} + j\,\frac{\sqrt{3}}{2}\right) = 10\sqrt{2} + j10\sqrt{6}\,.$$

Hence

$$\underline{I}_{m_3} = \underline{I}_{m_1} + \underline{I}_{m_2} = 10\sqrt{2} + 10\sqrt{2} + j10\sqrt{6} = 20\sqrt{2} + j10\sqrt{6}\,.$$

So that

$$I_{m_3} = \sqrt{800 + 600} = 37.4\,,\ \psi_3 = \tan^{-1}\frac{10\sqrt{6}}{20\sqrt{2}} = 41^o\,.$$

Therefore,

$$\underline{I}_{m_3} = 37.4e^{j41^o} \text{ and } i_3 = 37.4\sin(\omega t + 41^o)\,.$$

As we have already known the most important and useful, quantity of a sinusoidal function is its effective value. Although knowledge of the peak value of a sinusoid can be useful, it is not nearly so useful as the effective value.

To express the addition of two sinusoids in terms of the effective values of the phasors, it follows readily from Eq. (4.18) upon dividing each term in the expression by $\sqrt{2}$. Thus

$$\frac{\underline{I}_{m_3}}{\sqrt{2}} = \frac{\underline{I}_{m_1}}{\sqrt{2}} + \frac{\underline{I}_{m_2}}{\sqrt{2}}$$

or, more simply,

$$\underline{I}_3 = \underline{I}_1 + \underline{I}_2 \,. \tag{4.22}$$

Let us consider *multiplication and division of complex quantities*.

In the interest of illustrating how the product of two complex numbers is obtained, consider the following two complex numbers. One is denoted by the phasor $\underline{I} = Ie^{j\psi}$ the other is represented by the operator $\underline{Z} = Ze^{j\varphi}$. The product is desired. As the first step in the procedure, formulate the product in terms of the exponential form.

Accordingly, we can write

$$\underline{I} \cdot \underline{Z} = Ie^{j\psi} Ze^{j\varphi} = IZ\, e^{j(\psi+\varphi)} \tag{4.23}$$

Therefore, the product of two complex numbers is found by taking the product of their magnitudes and the sum of their angles.

The division of one complex quantity by another is treated in a similar fashion. To illustrate, let it be required to divide the complex quantity $\underline{U} = U\, e^{j\psi}$ into the value $\underline{Z} = Ze^{j\varphi}$. We shall call the quotient $\underline{I}$. Thus

$$\underline{I} = \frac{\underline{U}}{\underline{Z}} = \frac{Ue^{j\psi}}{Ze^{j\varphi}} = \frac{U}{Z} e^{j(\psi-\varphi)} \,. \tag{4.24}$$

Therefore, the division of one complex number by another involves the division of their magnitudes to yield the magnitude of the quotient and the difference of their phase angles to yield the phase of the quotient. For the sake of completeness, we give the corresponding time expression for the phasor of equation (4.24), which is

$$i = \sqrt{2} \frac{U}{Z} \sin\left(\omega t + \psi - \varphi\right) .$$

The angular frequency ω is the same as that for u .

4.4. The Receipt of the Sinusoidal Electromotive Force

Sinusoidal e.m.f.s in technical devices are obtained by various methods in electric machines, electronic oscillators and other devices.

Let's observe a cross-section of the simplified model of a monophase sinusoidal current generator (Fig. 4.6).

The right-angled conductor frame (or convolution) with cross -section *S*, is uniformly revolving with a constant angular frequency *ω* in a magnetic field between poles *N* and *S* of a constant magnet. We will assume that the excited magnetic field is constant and uniform in the area of rotation. The value of a magnetic induction is equal to *B*.

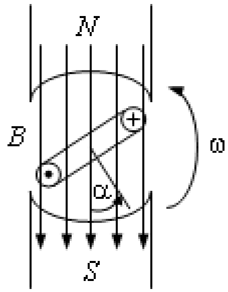


Fig. 4.6. A cross-section of a generator

The basic energy sources are electromechanical generators which transform the energy of a rotatory movement into electrical one. The principle of action of such generators is based on the effect of excitation of vortical electric field when a magnetic field changes. This effect leads to origin of the induced electromotive force in a closed loop at a magnetic flux modification through the surface restricted with this contour.

When the conductor frame revolves the value of the magnetic flux, passing through its plane, changes. This magnetic flux is maximum when the frame is perpendicular to the magnetic lines of the field.

$$\Phi_m = B \cdot S$$

In the process of rotation of the frame from this position, it decreases and becomes zero when the frame plane settles down along field lines. Then the flux direction changes its sign, and the flux starts to increase. Thus, the magnetic flux piercing the frame, changes depending on the angle of its rotation by the law:

$$\Phi = \Phi_m \cos\alpha$$

where α is an angle between the direction of magnetic field lines and a normal to the frame plane. Hence the magnetic flux varies in a time according to the equation

$$\Phi = \Phi_m \cos(\omega t + \psi_e) = B \cdot S \cos(\omega t + \psi_e)$$

where Φ is the magnetic flux through an area *S*, and *B* is a magnetic induction.

Experiments show that the induced e.m.f. is directly proportional to the rate of change of flux linkages:

$$e(t) = -\frac{d\Phi}{dt} = B \cdot S \cdot \omega \sin(\omega t + \psi_e) = E_m \sin(\omega t + \psi_e)\ .$$

The e.m.f. reaches an amplitude value when the frame plane is parallel to the lines of magnetic induction. At such position of the frame the flux piercing it, is equal to zero, and the velocity of intersection of magnetic lines is maximum. This e.m.f. through the contact rings and motionless brushes is transmitted to output terminals of a generator where it creates the sinusoidal voltage which is equal to e.m.f. If some load is connected to the brush terminals there will be an electric current in this formed circuit.

4.5. A Sinusoidal Current Through a Resistance

Assume a sinusoidal forcing function which can be described by

$$u = U_m \sin \omega t = \mathrm{Im}\left[U_m e^{j\omega t}\right] \tag{4.25}$$

Because the circuit is linear, we know that the steady-state response must also be a sinusoid having the same frequency as the source function. Therefore, we can say that the form of the response must be

$$i = I_m \sin(\omega t + \psi) = \mathrm{Im}\left[I_m e^{j\psi} e^{j\omega t}\right] \tag{4.26}$$

where I_m and ψ are the amplitude and phase of current. The equation in this instance is KVL applied to the circuit of Fig. 4.7. Thus

$$u = Ri = RI_m \sin \omega t = R \times \mathrm{Im}\left[I_m e^{j\psi} e^{j\omega t}\right] \tag{4.27}$$

This expression is Ohm's law but that sinusoidal quantities are involved for the variables *u* and *i*. Rewrite the equation in a simpler form by dropping the notation "imaginary part of…". After algebraic manipulations it must be remembered that the actual solution is found by taking only the *j* part of the exponential solution. Eq. (4.27) can be written as

$$R I_m e^{j\psi} e^{j\omega t} = U_m e^{j\omega t}$$

Since the time factor $e^{j\omega t}$ is common to both sides of the equation, it may be suppressed, thus leading to

$$I_m e^{j\psi} = \frac{U_m}{R} U_m\, e^{j0^o} \tag{4.28}$$

On the left side of this equation appear the two unknown quantities I_m and $\psi = 0$, while on the right side appear the known quantities.

Although both sides of this equation represent complex numbers, it is clear from the right side that in this case we have just a real number. This is characteristic of purely resistive circuit. A comparison of amplitudes and angles of the left and right sides of Eq. (4.26) leads to

$$I_m = \frac{U_m}{R} \text{ and } \psi = 0 \tag{4.29}$$

Substituting this information into Eq. (4.26) yields the final expression for the solution. Hence

$$i = \frac{U_m}{R} \sin \omega t \tag{4.30}$$

Comparing Eq. (4.30) with Eq. (4.25) reveals that *for a resistive circuit the current and voltage are in time phase*, i.e., the peak values and the zero values occur at the same time instants. The instantaneous values of current, voltage and power are illustrated by plots in Fig. 4.7, *a*. The instantaneous power is

$$p = U_m I_m \sin \omega t \sin \omega t = \frac{U_m I_m}{2}(1 - \cos 2\omega t) \tag{4.31}$$

One can very simply describe the current response caused by a sinusoidal voltage of effective value U applied to a circuit of resistance R as

$$\underline{I} = \frac{\underline{U}}{R} \tag{4.32}$$

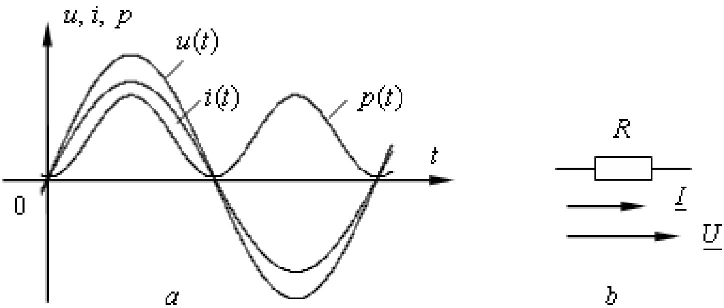


Fig. 4.7. Resistive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex current and voltage

Referring to Fig. 4.7, *b*, the complex current $\underline{I}$ and the complex voltage $\underline{U}$ are in phase. Appearing in Fig. 4.7, *b*, is the phasor diagram for the resistive circuit.

4.6. A Sinusoidal Current Through an Inductance

The same method of analysis is used to find the response of a purely inductive circuit to a sinusoidal source function in the steady state. For the inductive circuit the defining equation is

$$u = L\frac{di}{dt} \tag{4.33}$$

Substituting Eqs. (4.25) and (4.26) gives

$$\mathrm{Im}\left[U_m e^{j\omega t}\right] = L\frac{di}{dt}\,\mathrm{Im}\left[I_m e^{j\psi} e^{j\omega t}\right] \tag{4.34}$$

where I_m and ψ are different from the values found for the resistive circuit. Since the defining equation involves a derivative of the current. Performing the differentiation and suppressing $e^{j\omega t}$ leads to

$$I_m e^{j\psi} = \frac{U_m}{j\omega L} \tag{4.35}$$

Remember that factor *j* is a rotational operator defined as

$$j = e^{j90^o} \tag{4.36}$$

Any real number which is multiplied by *j* changes its position from the horizontal axis to the vertical axis and Eq. (4.35) can be rewritten

$$I_m e^{j\psi} = \frac{U_m}{\omega L} e^{-j90^o} \tag{4.37}$$

Therefore,

$$I_m = \frac{U_m}{\omega L} \text{ and } \psi = -90^o . \tag{4.38}$$

Expressed in terms of effective values, the solution for the response can be found conveniently by writing Eq. (4.37) as

$$I = \frac{U}{\omega L} = \frac{U}{X_L} \quad \text{and } \psi = \text{-}90^{\text{o}} \tag{4.39}$$

where $X_L = \omega L$ and is called the *inductive reactance* of the coil. The term reactance is used to distinguish it from resistance.

It is always true that for sinusoidal source functions *the current flowing through an inductor always lags behind the potential difference across the inductor terminals by* 90°.

The phasors of Fig. 4.8, *a* are rotating counterclockwise (the assumed positive direction) at the angular frequency ω radians/second. The phasor diagram for the inductive circuit is illustrated in Fig. 4.8, *b*.

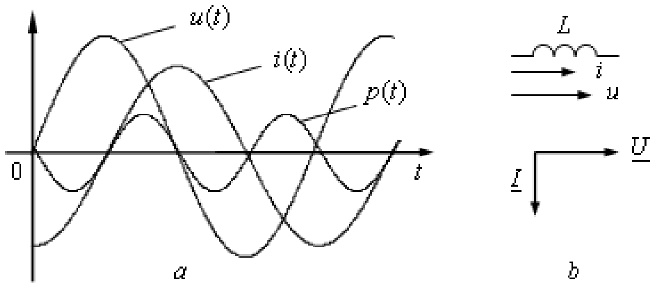


Fig. 4.8. Inductive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex voltage and current

The instantaneous power

$$p = ui = -\frac{U_m I_m \sin 2\omega t}{2}.$$

4.7. A Sinusoidal Voltage Across a Capacitance

The current which flows through the capacitor is related *u* by

$$i = C\frac{du}{dt}$$

As $u = U_m \sin \omega t$ the current can be represented by $i = I_m \sin(\omega t + \psi)$. In exponential forms of voltage and current

$$I_m e^{j\psi} e^{j\omega t} = C\frac{d}{dt}\left[U_m e^{j\omega t}\right] \tag{4.41}$$

Performing the differentiation leads to

$$I_m e^{j\psi} = j\omega C U_m \quad \text{or} \tag{4.42}$$

$$I_m e^{j\psi} = \frac{U_m}{1/\omega C} e^{j90^o} \tag{4.43}$$

Equating the right and left sides then yields

$$I_m = \frac{U_m}{1/\omega C} \quad \text{and} \quad \psi = 90^o \tag{4.44}$$

This equation can be rewritten in terms of effective values:

$$\underline{I} = \frac{\underline{U}}{1/j\omega C} \tag{4.45}$$

It states that the phasor current $\underline{I}$ is equal to the phasor potential difference $\underline{U}$ across the capacitor divided by the quantity $1/j\omega C$.

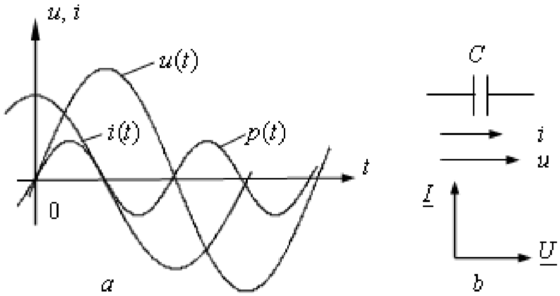


Fig. 4.9. Capacitive circuit: *a*) The instantaneous values of current, voltage and power; *b*) the complex current and voltage

The unit of this value is volt/ampere, which is ohm. The value $1/j\omega C$ is called *a capacitive reactance* because a capacitor is involved. Equation (4.45) often appears in the form

$$\underline{I} = \frac{\underline{U}}{X_C e^{-j90^o}} = \frac{\underline{U}}{X_C} e^{j90^o} \tag{4.46}$$

Where X_C is the capacitive reactance in ohms.
Therefore, for the capacitive circuit

$$I = \frac{U}{X_C} = U\omega C \text{ and } \psi = 90^o. \tag{4.47}$$

Figure 4.9, b, depicts the phasor diagram of a purely capacitive circuit The corresponding current response is illustrated in Fig. 4.9, a. The time-domain equation is obtained in the usual way and found to be

$$i = \sqrt{2}\,\frac{U}{X_C}\sin(\omega t + \psi) = \frac{U_m}{X_C}\sin(\omega t + 90^o) \tag{4.48}$$

The important characteristic of the capacitive circuit: *the current flowing through a capacitor in the sinusoidal steady state always leads the potential difference across the capacitor by 90°.*

4.8 The Series *RL* Circuit

Appearing in Fig. 4.10 is a diagram of the series *RL* circuit. One should find the steady-state current response assuming that *R*, *L*, and the forcing function are known. The differential equation for the circuit is

$$u = iR + L\frac{di}{dt}$$

This expression can be written in terms of effective values as

$$\underline{U} = \underline{I}\,(R + j\omega L) \tag{4.49}$$

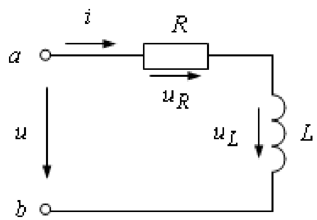


Fig. 4.10. The series *RL* circuit

Then current becomes

$$\underline{I} = \frac{\underline{U}}{R + j\omega L} \tag{4.50}$$

where *R* is *a resistance* and ωL is the *inductive reactance*.

Information about the magnitude and relative phase angle of the response then readily follows when Eq. (4.50) is rewritten as

$$I = \frac{U}{\sqrt{R^2 + \omega^2 L^2}} \text{ and } \psi = \tan^{-1}(\omega L/R) \tag{4.51}$$

where ψ is the angle between a voltage and a current.

The potential difference across the resistive element is clearly

$$\underline{U}_R = \underline{I}R \tag{4.52}$$

Because there is no phase shift associated with a resistor, it follows that this voltage drop must be in phase with the current. Hence the voltage drop across a resistance is drawn on the same line as the current. Equation (4.52) is represented by in the phasor diagram in Fig. 4.11.

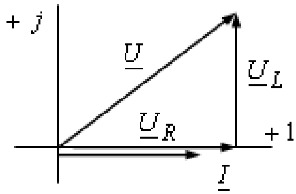


Fig. 4.11 *A* vector diagram

Similarly, the potential difference across the inductive element is shown by the second term of the right side of Eq. (4.49) to be expressed as

$$\underline{U}_L = j\omega L\underline{I} \qquad (4.53)$$

This quantity is to be rotated in the positive (or counter clockwise) direction through 90°. So, this value is perpendicular to the current in Fig. 4.11. The sum of two voltages must be equal to the input voltage.

4.9. The Series *RC* Circuit

In solving this problem, we proceed directly with Kirchhoff's voltage law for the circuit of Fig. 4.12.

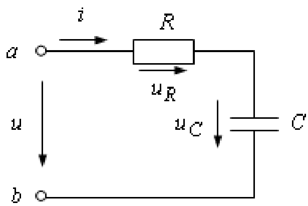


Fig. 4.12. The series *RC* circuit

Thus $\underline{U} = \underline{I}R + \underline{I}\dfrac{1}{j\omega C}$ (4.54)

Rewriting Eq. (4.54) gives

$$\underline{U} = \underline{I}\left(R + \frac{1}{j\omega C}\right) = \underline{I}\left(R - j\frac{1}{\omega C}\right) \qquad (4.55)$$

where R is a resistance and $1/\omega C$ is the capacitive reactance.

Equation (4.55) is KVL written in terms of phasor quantities. Upon transposing the equation for the current becomes

$$I = \frac{U}{\sqrt{R^2 + 1/\omega^2 C^2}} \text{ and } \psi = \tan^{-1}\frac{1}{\omega RC} \qquad (4.56)$$

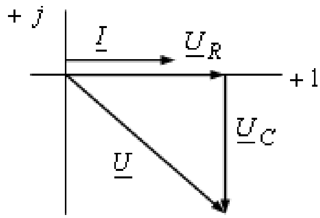


Fig. 4.13 *A* vector diagram

The phasor diagram for the series *RC* circuit is depicted in Fig. 4.13. The current now leads the voltage in time - a result which is consistent with Eq. (4.56) showing a positive value for the phase angle.

4.10. The Series *RLC* Circuit. Ohm's Law for A.C. Circuits

The circuit for series combination of the three parameters is shown in Fig. 4.14. For this network the instantaneous voltage

$$u = u_R + u_L + u_C \text{ or} \tag{4.57}$$

$$iR + L\frac{di}{dt} + \frac{1}{C}\int idt = u \tag{4.58}$$

But *ac* circuit problem is solved by the method of complex numbers.

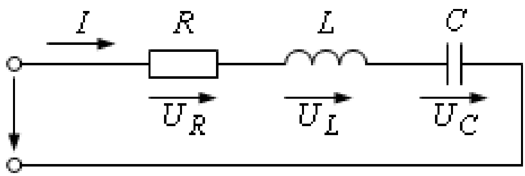


Fig. 4.14. The series *RLC* circuit

More specifically, in any Kirchhoffian equation written for a steady state, the instantaneous current *i* is replaced by the complex peak current $\underline{I}_m$; the instantaneous voltage across a resistance (*RI*) by the complex voltage drop $R\underline{I}_m$ which is in phase with the current $\underline{I}_m$; the instantaneous voltage across an inductance (u_L) with the complex quantity $\underline{I}_m j\omega L$ which leads the current by 90°; the instantaneous voltage across a capacitance (u_C) with the complex quantity $\underline{I}_m(-j/\omega C)$ lagging behind the current by 90°; and the instantaneous input voltage *u* with the complex value $\underline{U}_m$.

In complex notation,

$$\underline{I}_m R + \underline{I}_m j\omega L + \underline{I}_m \frac{-j}{\omega C} = \underline{U}_m$$

Putting before the brackets gives

$$\underline{I}_m \left(R + j\omega L - \frac{j}{\omega C} \right) = \underline{U}_m \tag{4.59}$$

Thus, for the current of Fig. 4.14

$$\underline{I}_m = \frac{\underline{U}_m}{R + j\omega L - \dfrac{j}{\omega C}} \tag{4.60}$$

The complex-number method is also called the *symbolic method* for the reason that the currents and voltages are represented by their complex transforms or symbols.

The operator in the denominator of Eq. (4.60) marked with letter $\underline{Z}$ is called the *complex impedance*, the unit being the *ohms*. Thus

$$\underline{Z} = R + j\omega L - \frac{j}{\omega C}\,. \tag{4.61}$$

The value

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \tag{4.62}$$

is called the *impedance* of an electric circuit.

Then Eq. (4.60) may be written thus

$$\underline{I}_m \cdot \underline{Z} = \underline{U}_m$$

Dividing the last equation by $\sqrt{2}$ we get

$$\underline{I} = \frac{\underline{U}}{\underline{Z}} \tag{4.63}$$

Eq. (4.63) is *Ohm's law* for alternating current circuits.

Writing KVL expressed in terms of effective values we have

$$\underline{U} = \underline{I}\left(R + j\omega L + \frac{1}{j\omega C} \right)$$

From here

$$\underline{I} = \frac{\underline{U}}{R + j\omega L - \dfrac{j}{\omega C}} . \qquad (4.64)$$

Therefore, the magnitude and phase of the effective current are

$$I = \frac{U}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \text{ and } \psi = \tan^{-1}\frac{\omega L - 1/\omega C}{R} \qquad (4.65)$$

Appearing in Fig. 4.15, *a*, *b*, is the phasor (or vector) diagram for the circuit of Fig. 4.14.

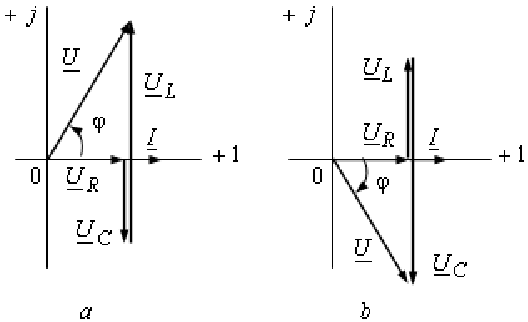


Fig. 4.15 *a*) The phasor diagram for a case $X_L > X_C$;

b) the phasor diagram for a case $X_L < X_C$

Now we will pass from vector diagrams (Fig. 4.15, *a*, *b*) to impedance triangles. As it follows from the formula (4.62) the impedance, the resistance and the reactance may be depicted by a rectangular triangle, similar to the triangle of voltage vectors Fig. (4.15, *a*, *b*).

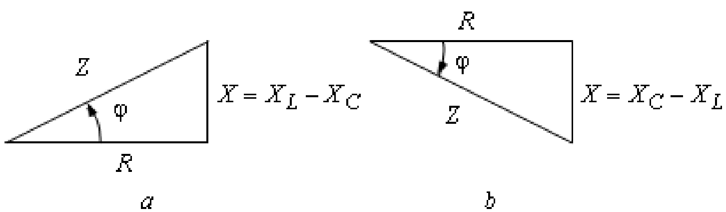


Fig. 4.16. The impedance triangles

As follows from Fig. 4.16 resistance and reactance are connected with impedance by relationships $R = Z\cos\varphi; \quad X = Z\sin\varphi$

4.11. The Voltage Resonance

When a source of a.c. is connected across a two-terminal network containing one or several inductances and one or several capacitances in additional to a resistance, phenomena arise under certain conditions, referred to as *resonance*. When an inductive reactance and a capacitive reactance are joined in series (as in Fig. 4.14) conditions may be obtained under which *voltage (series) resonance* will take place.

The expression for the effective current flow

$$\underline{I} = \frac{\underline{U}}{R + j\omega L - \frac{j}{\omega C}} = \frac{\underline{U}}{\underline{Z}} \tag{4.66}$$

The condition for voltage resonance is thus

$$\omega_o L = \frac{1}{\omega_o C} \tag{4.67}$$

from where

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{4.68}$$

The frequency ω_o is called the *resonant frequency* of the circuit.

At resonance the impedance of the circuit is a minimum, and specifically it is equal to R:

$$Z_o = \sqrt{R^2 + (X_{Lo} - X_{Co})^2} = R \tag{4.69}$$

Consequently, when a series *RLC* circuit is at resonance, the current is a maximum and is also in time phase with the voltage

$$\underline{I}_o = \frac{\underline{U}}{\underline{Z}} = \frac{\underline{U}}{R} \tag{4.70}$$

where $\underline{I}_o$ denotes the current at resonance.

The resistances of reactive elements at the resonant frequency are equal to each other

$$\begin{aligned} X_{Lo} &= \omega_o L = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}}, \\ X_{Co} &= \frac{1}{\omega_o C} = \frac{\sqrt{LC}}{C} = \sqrt{\frac{L}{C}}. \end{aligned} \tag{4.71}$$

The value $\rho = \sqrt{\frac{L}{C}}$ is called a *wave resistance* (in Ω).

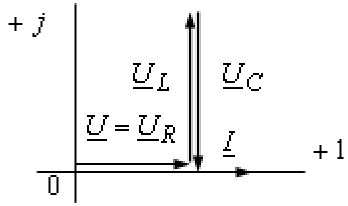


Fig. 4.17. The vector diagram for a series resonance

The vector diagram is shown in Fig. 4.17. $\underline{U}_L$ and $\underline{U}_C$ are equal to each other, and their vectors are in phase opposition (phase shift = π). The voltage across the resistance R is equal to the input voltage U.

When the ratio of inductive reactance to resistance in a series *RLC* circuit is high, a very rapid rise of current to the maximum value occurs in the vicinity of the resonant frequency. The ratio

$$\frac{\omega_o L}{R} = \frac{\sqrt{\frac{L}{C}}}{R} = \frac{\rho}{R} = Q \tag{4.72}$$

is termed the *Q*-factor or the *quality factor* of a resonant circuit. It shows how many times the voltage across the reactive component exceeds the applied voltage at resonance. In practical circuits, *Q* is often relatively high.

Fig. 4.18, *a*, shows entering current and voltage across a resistor. Fig. 4.18, *b* - voltage across inductive and capacity elements. Such curve lines are called *resonance characteristics.*

They are expressed by the following equations:

$$I(\omega) = \frac{U}{Z} = \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}};$$

$$U_R(\omega) = \frac{UR}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}; \tag{4.73}$$

$$U_L(\omega) = \frac{U\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}};$$

$$U_C(\omega) = \frac{U}{\omega C\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

Referring to Fig. 4.18, *a*, both the voltage across the inductance and the voltage across the capacitance are maximum at frequencies other than the resonant frequency. The former is maximum at a frequency higher than ω_o, and the latter at a frequency which is lower than the resonant frequency. To describe the width of the resonance curve the term *bandwidth* is used. For the series *RLC* circuit bandwidth is defined as the range of frequency for which the power delivered to *R* is greater than or equal to $P_o/2$ where P_0 is the power delivered to *R* at resonance.

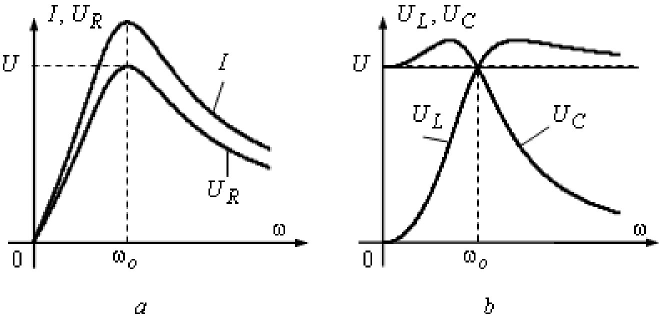


Fig. 4.18 *a*) The dependence of I and inductive U on frequency; *b*) the voltage across the inductance and the voltage across the capacitance

From the shape of the resonance curve, it should be clear that there are two frequencies for which the power delivered to *R* is half the power at resonance. For this reason, these frequencies are referred to as those corresponding to the half-power points. The magnitude of the current at each half-power point is the same. It follows then that

$$I_1 = I_2 = \frac{I_o}{\sqrt{2}} = 0.707\ I_o \tag{4.74}$$

In Fig. 4.19 for Q_1, the frequency at the lower half-power point is denoted ω_1, and that of the upper half-power point is denoted ω_2. Hence the bandwidth is $\omega_2 - \omega_1$. Calling the reactance at the lower half-power frequency X_1, we have

$$X_1 = \omega_1 L - \frac{1}{\omega_1 C} = -R \tag{4.75}$$

The minus sign appears on the right side of the equation because below resonance the capacitive reactance exceeds the inductive reactance.

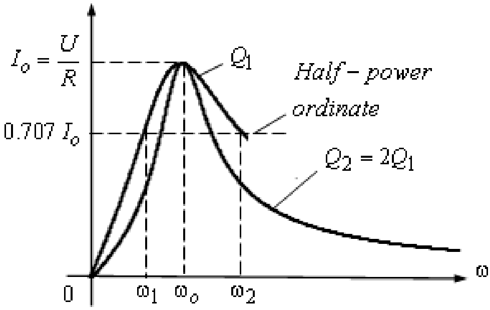


Fig. 4.19. Variation of current in a series *RLC* circuit with frequency

Rearranging Eq. (4.75) leads to

$$\omega_1{}^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

The roots are therefore

$$(\omega_1)_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 + \omega_o{}^2}$$

where $\alpha = R/2L$. Although the previous equation provides two solutions, only one of these is physically realizable. The negative frequency is meaningless and so may be discarded.

Hence the expression for the lower half-power frequency is

$$\omega_1 = -\alpha \pm \sqrt{\alpha^2 + \omega_o{}^2} \tag{4.76}$$

The upper half-power frequency is found in a similar fashion. In this instance we have

$$X_2 = \omega_2 L - \frac{1}{\omega_2 C} = +R$$

which leads to $\qquad \omega_2{}^2 + \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$

The expression for the useful ω_2 is then

$$\omega_2 = \alpha \pm \sqrt{\alpha^2 + \omega_o{}^2} \tag{4.77}$$

Therefore, the expression for the bandwidth becomes

$$\omega_{bw} = \omega_2 - \omega_1 = 2\alpha = \frac{R}{L} \tag{4.78}$$

This expression is significant because it reveals that the bandwidth of the series *RLC* circuit depends solely upon the *R/L* ratio.

By forming the ratio of the resonant frequency to the bandwidth we obtain a factor which is a measure of the selectivity or sharpness of tuning of the series *RLC* circuit. As this quantity is called the *quality factor* of the circuit and is expressed as

$$Q_o = \frac{\omega_o}{\omega_{bw}} = \frac{\omega_o}{R/L} = \frac{\omega_o L}{R} = \frac{X_{LO}}{R} \tag{4.79}$$

A glance at the curves appearing in Fig. 4.19 should make it clear as to why the quality factor Q_o is used as a measure of the selectivity or sharpness of tuning. Note how much narrower the resonance curve is for Q_2 than for Q_1. Equation (4.78) shows that the value of *C* in no way influences the bandwidth but it does alter the value of ω_o.

4.12. The Parallel *RLC* Circuit

The circuit of parallel *RLC* connection appears in Fig. 4.20.

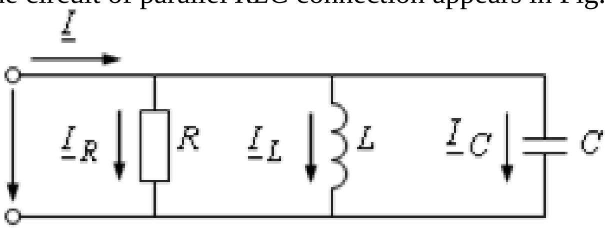


Fig. 4.20. The parallel *RLC* connection

If we apply sinusoidal voltage $u = U_m \sin \omega t$ to input terminals, in a non-ramified part it appears the current whose instantaneous value according to KCL law is equal to an algebraic sum of branch currents

$$i = i_R + i_L + i_C \tag{4.80}$$

Thus, the current through a resistance i_R coincides in phase with the voltage, the current through an inductance i_L lags behind the voltage by 90°, and the current through a capacitance i_C leads the voltage by 90°. We can write this mathematically as

$$i_R = I_{Rm} \sin \omega t = \frac{U_m}{R} \sin \omega t,$$

$$i_L = I_{Lm} \sin\left(\omega t - \frac{\pi}{2}\right) = -\frac{U_m}{X_L} \cos \omega t,$$

$$i_C = I_{Cm} \sin\left(\omega t + \frac{\pi}{2}\right) = \frac{U_m}{X_C} \cos \omega t.$$

Then we can get the expression for the current in a non-ramified part of the circuit as follows:

$$\begin{aligned} i &= \frac{U_m}{R} \sin \omega t - \frac{U_m}{X_L} \cos \omega t + \frac{U_m}{X_C} \cos \omega t = \\ &= \frac{U_m}{R} \sin \omega t - \left(\frac{1}{X_L} - \frac{1}{X_C}\right) U_m \cos \omega t = \\ &= U_m G \sin \omega t - (B_L - B_C) U_m \cos \omega t = \\ &= U_m \left(G \sin \omega t - B \cos \omega t\right) = I_m \sin(\omega t - \varphi) \end{aligned} \tag{4.81}$$

where G is called a *conductance*, B_L is called the *inductive susceptance*, B_C is called the *capacitive susceptance*.

The value B is called the *susceptance*.

$$B = B_L - B_C \qquad (4.82$$

The *complex admittance* of the circuit

$$Y = G + j(B_L - B_C) \tag{4.83}$$

The equation (4.81) represents a trigonometrical notation of KVL. Rewrite Eq. (4.81) in complex notation,

$$\underline{I}_m = \underline{U}_m \left[G + j(B_L - B_C)\right] = \underline{U}_m Y \tag{4.84}$$

Dividing Eq. (4.84) through by $\sqrt{2}$ we have got the equation which presents Ohm's law for parallel connection of *RLC*:

$$\underline{I} = \underline{U} \left[G + j(B_L - B_C)\right] = \underline{U} \cdot \underline{Y} \tag{4.85}$$

Appearing in Fig. 4.21, is the phasor (or vector) diagram for the circuit of Fig. 4.20. Fig. 4.21, *a* is drawn for the case where $B_L > B_C$. Hence the current phasor must lag behind the voltage phasor.

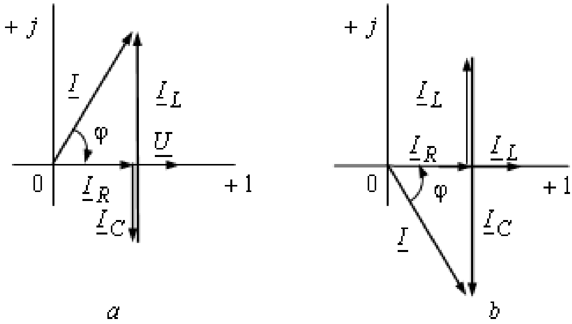


Fig. 4.21. Vector diagram for parallel *RLC* connection

From Fig. 4.21 we will pass to triangles of conductivities.

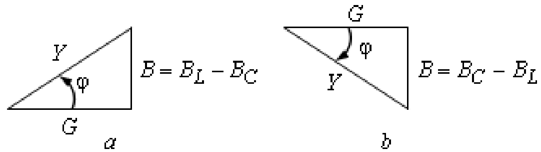


Figure 4.22, *a* shows a right-angled triangle, called admittance triangle, for a case of is active-inductive load of an electric circuit, and Fig. 4.22, *b* -for a case of active-capacity load.

The magnitude of the complex admittance is

$$Y = \sqrt{G^2 + B^2} = \sqrt{G^2 + (B_L - B_C)^2} \tag{4.86}$$

In general, *Y* is a complex number which often is denoted as

$$\underline{Y} = G - jB \tag{4.87}$$

where *G* is the real part of the admittance and is called the *conductance*, and *B* is the quadrature component and is called *susceptance*.
Then we can write that for the parallel *RLC* case

$$G = \frac{1}{R} \text{ and } B = \frac{1}{X}, \tag{4.88}$$

where $B = B_L - B_C = 1/(\omega L) - \omega C$ and $X = X_L - X_C = \omega L - 1/\omega C$.

The active and reactive conductivities of a circuit are connected with admittance by the following relationships

$$G = Y\cos\varphi,$$

$$B = Y\sin\varphi.$$

Since the impedance of a series *RLC* circuit is $\underline{Z} = R + jX$, the equivalent admittance is expressed as the reciprocal of this quantity^

$$\underline{Y} = \frac{1}{\underline{Z}} = \frac{1}{R + jX}$$

To express this in terms of an equivalent conductance and susceptance this equation is rationalized as follows:

$$\underline{Y} = \frac{1}{R + jX} \cdot \frac{R - jX}{R - jX} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

Recalling that $Y = G - jB$, we may rewrite the last equation as

$$G - jB = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

Equating the real and imaginary parts then yields

$$G = \frac{R}{R^2 + X^2} \text{ and } B = \frac{X}{R^2 + X^2} \qquad (4.89)$$

where *G* is the conductance and *B* is the susceptance of the equivalent parallel combination of the series *RLC* configuration.

The corresponding equivalent impedance may be expressed as

$$\underline{Z} = \frac{1}{\underline{Y}} = \frac{1}{G - jB}$$

Express this in terms of equivalent resistance and reactance:

$$\underline{Z} = \frac{1}{G - jB} \cdot \frac{G + jB}{G + jB} = \frac{G}{G^2 + B^2} + j\frac{G}{G^2 + B^2}$$

Equating the real and imaginary parts then yields

$$R = \frac{G}{G^2 + B^2} \text{ and } X = \frac{G}{G^2 + B^2}. \qquad (4.90)$$

4.13. A Current Resonance

Let's consider the circuit (see Fig. 4.23, *a*) where the conditions may be obtained under which current resonance (or parallel resonance) takes place. Let one branch contain a resistance R_1 and an inductive reactance ωL, and the other branch, a resistance R_2 and a capacitive reactance $1/\omega C$. Current $\underline{I}_1$ in the 1st branch lags behind voltage $\underline{U}$

$$\underline{I}_1 = \underline{U} \cdot Y_1 = \underline{U} \cdot (G_1 - jB_1)$$

The complex current $\underline{I}_2$ in the second branch leads the voltage

$$\underline{I}_2 = \underline{U} \cdot Y_2 = \underline{U} \cdot (G_2 + jB_2)$$

The total current in the circuit

$$\underline{I} = \underline{I}_1 + \underline{I}_2 = \underline{U} \cdot [(G_1 + G_2) - j(B_1 - B_2)] = \underline{U}\,\underline{Y}$$

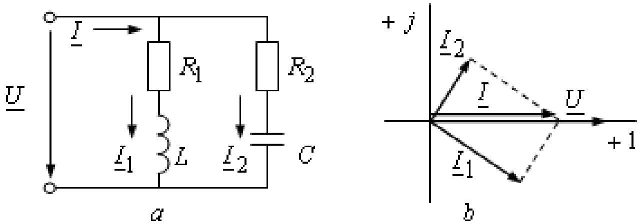


Fig.4.23. The circuit with parallel connection of elements

Current resonance occurs if inductive susceptance is equal to capacitive susceptance:

$$B_L = B_C \text{ or } B = B_L - B_C = 0 \tag{4.91}$$

As we have active resistances in branches, there are the following equations for active conductances

$$G_1 = \frac{R_1}{Z_1^2} = \frac{R_1}{R_1^2 + \omega^2 L^2}; \quad G_2 = \frac{R_2}{Z_2^2} = \frac{R_2}{R_2^2 + \dfrac{1}{\omega^2 C^2}}; \tag{4.92}$$

and for reactive susceptances

$$B_L = \frac{X_L}{Z_1^2} = \frac{\omega L}{R_1^2 + \omega^2 L^2}; \quad B_C = \frac{X_{C2}}{Z_2^2} = \frac{\dfrac{1}{\omega C}}{R_2^2 + \dfrac{1}{\omega^2 C^2}} \tag{4.93}$$

From the resonance condition

$$\frac{\omega L}{R_1^2 + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{R_2^2 + \dfrac{1}{\omega^2 C^2}} \tag{4.94}$$

From Eq. (4.94), we get *resonance angular frequency*:

$$\omega_o = \frac{1}{\sqrt{LC}} \sqrt{\frac{\dfrac{L}{C} - R_1^2}{\dfrac{L}{C} - R_2^2}} = \omega_o \sqrt{\frac{\rho^2 - R_1^2}{\rho^2 - R_2^2}} \tag{4.95}$$

The vector diagram for a parallel resonance is shown in Fig. 4.24. Active and reactive current components:

$$I_{a1} = U\,G_1; \quad I_{p1} = U\,B_L;$$

$$I_{a2} = U\,G_2; \quad I_{p2} = U\,B_C.$$

The reactive currents $\underline{I}_1$ and $\underline{I}_2$ are equal, and their vectors are in phase opposition (a phase shift between them is equal π) and compensate each other, the resonance in a parallel circuit is named a current resonance.

As a result, the entering current is the sum of active components of branch currents. From a vector diagram one can see, that at a resonance the input current can be much less than branch currents.

This property allows using a parallel contour as a current amplifier.

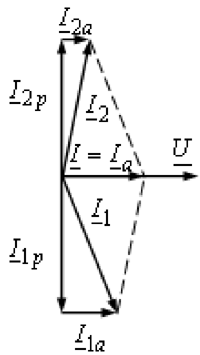


Fig. 4.24

In an ideal parallel contour (see Fig. 4.25, *a*) any resistances are absent, i.e. $R_1 = R_2 = 0$. In this case $I_{a1} = I_{a2} = 0$. At a resonance $I_1 = I_2 = I_{p1} = I_{p2}$, and a total (input) current $I = I_1 + I_2 = 0$, what we can see in the vector diagram in Fig. 4.25, *b*.

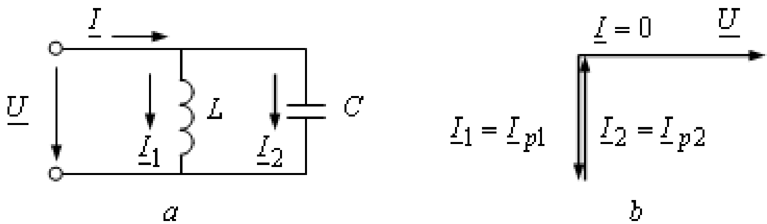


Fig. 4.25. Ideal parallel contour (*a*), and the vector diagram for it (*b*)

Let's observe the frequency characteristics of branch susceptances and the input susceptance for an ideal parallel contour (Fig. 4.26, *a*), and also a resonance characteristic of an input current (Fig. 4.26, *b*). They are expressed by following formulas

$$B_L(\omega) = \frac{1}{\omega L}; \qquad B_C(\omega) = \omega C;$$

$$B(\omega) = \frac{1}{\omega L} - \omega C; \qquad I(\omega) = \left|\frac{1}{\omega L} - \omega C\right| \cdot U.$$

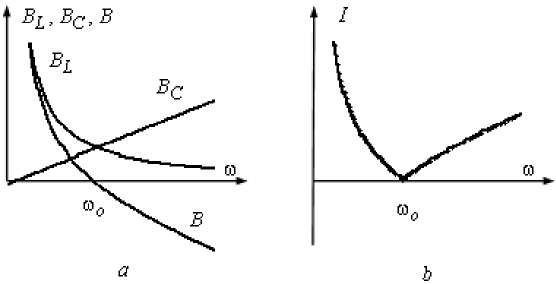


Fig. 4.26. Frequency characteristics for ideal parallel contour (*a*), and a resonance characteristic of an input current (*b*)

The current resonance in contrast to a series resonance is the safety condition for electropower installations. As currents in branches are mutually independent, their values are defined by input voltage and impedances of branches (according to Ohm's law).

4.14. Kirchhoff's Laws in Complex Notation

By Kirchhoff's first law, the algebraic sum of the instantaneous currents at any load of a network is zero, or $\sum i_k = 0$.

Substituting $\underline{I}_k e^{j\omega t}$ for i_k in the previous equation and putting $e^{j\omega t}$ before the brackets, we have

$$e^{j\omega t}\sum \underline{I}_k = 0$$

Since $e^{j\omega t} \neq 0$ for any *t*, it follows that

$$\sum \underline{I}_k = 0 \qquad (4.96)$$

Equation (4.96) is KCL in complex form.

Consider Kirchhoff's voltage law in complex form.

KVL gives the instantaneous values of currents, voltages and e.m.f.s for any closed loop in any network carrying an alternating sinusoidal current. Let a mesh has *n* branches and each (k -th) branch in the general case contains an electromotive force (e_k), a resistance (R_k), an inductance (L_k), and a capacitance (C_k), with a current i_k flowing through all of them. Then by Kirchhoff's voltage law

$$\sum_{k=1}^{n}\left(i_k R_k + L_k \frac{di_k}{dt} + \frac{1}{C_k}\int i_k dt \right) = \sum_{k=1}^{n} e_k$$

In accordance with Sec. 4.10, each term on the left-hand side may be replaced by $\underline{I}_k \underline{Z}_k$, and each term on the right-hand side - by $\underline{E}_k$. Then KVL for the network with *n* branches is written in complex form thus

$$\sum_{k=1}^{n} \underline{I}_k \underline{Z}_k = \sum_{k=1}^{n} \underline{E}_k \qquad (4.97)$$

4.15. A Vector Diagram and its Application to Sinusoidal Circuit Problems

A vector diagram is a graphic representation of the vectors of sinusoidal quantities in a complex plane. Let us consider the example.

Example 4.1. In the network of Fig. 4.27 e.m.f. is $\dot{E} = 100V$, and the parameters are $L = 0.0955H$, $R = 30\Omega$, $C = 53.08 \cdot 10^{-6} F$. Find current and voltages across the circuit elements.

Solution: Calculate inductive and capacitive reactances. Thus

$$X_L = 2\pi f L = 314 \times 0.0955 = 30 \ \Omega,$$

$$X_c = \frac{1}{2\pi f C} = \frac{1}{314 \times 53.08 \cdot 10^{-6}} = 60 \ \Omega..$$

Now one can determine the complex impedance of the circuit as

$$\underline{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + j(X_L - X_C) =$$

$$= 30 + j(30 - 60) = 30 - j30 = 42.43 \ e^{-j45^o} \ \Omega$$

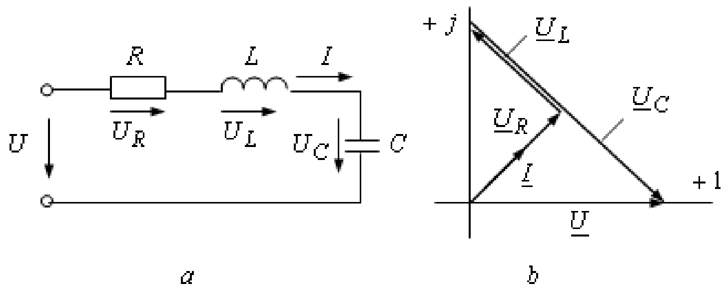


Fig. 4.27

We write down the equation by Kirchhoff's voltage law in complex form as $\underline{I}\,\underline{Z} = \underline{U}$. Thus, complex current in the loop

$$\dot I = \frac{\dot E}{Z} = \frac{100}{42.43e^{-j45^o}} = 2.36\ e^{j45^o} A.$$

The voltage across resistance

$$\dot U_R = R\dot I = 30 \cdot 2.36e^{j45^o} = 70.8e^{j45^o} V.$$

The voltage across inductance

$$\dot U_L = jX_L \dot I = 30e^{j90^o} \cdot 2.36e^{j45^o} = 70.8e^{j135^o} V.$$

The voltage across capacitance

$$\dot U_c = -jX_c \dot I = 60e^{-j90^o} \cdot 2.36e^{j45^o} = 141.6e^{-j45^o} V.$$

The vector diagram for this case is shown in Fig. 4.28, *b*.

4.16. Instantaneous and Average Power. Power Factor

The voltage and current sinusoid are shown in Fig. 4.28. It follows then that the expression for the instantaneous power is

$$\begin{aligned} p = ui = U_m \sin \omega t I_m \sin(\omega t - \varphi) = U_m I_m \sin \omega t \sin(\omega t - \varphi) = \\ = \frac{U_m I_m}{2}(\cos\varphi - \cos(2\omega t - \varphi)) = UI \cos\varphi - UI \cos(2\omega t - \varphi) \end{aligned} \quad (4.98)$$

A plot of Eq. (4.98) appears in Fig. 4.28.

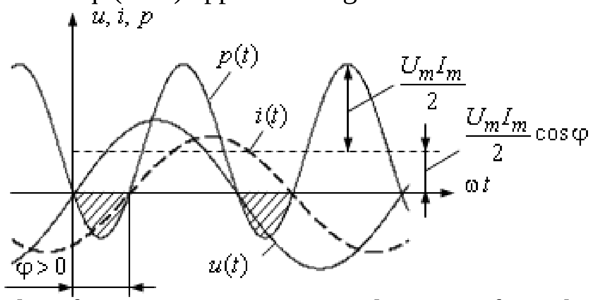


Fig. 4.28. Plot of instantaneous power as determine from the given voltage and current sinusoid

Note that for a fixed *φ* the instantaneous power consists of two components - a constant part and a time-varying part.

A glance at Fig. 4.28 shows that the instantaneous power is negative whenever the voltage and current are of opposite sign. However, for the case plotted in Fig. 4.28 notice that the positive area under the *p*(*t*) curve exceeds the negative area. So, the average power is positive and finite, and specifically it is equal to the constant term of Eq. (4.98).

Since each cycle is repeated, whatever is done for one cycle applies equally as well for each succeeding cycle. As the average power is given by the constant term of Eq. (4.98), from the general definition of the average value over one cycle, we have

$$P = \frac{1}{T}\int_0^T uidt \tag{4.99}$$

It follows that the average power called *active* or *true* is

$$P = UI\cos\varphi \tag{4.100}$$

In the interest of introducing another term of electric-circuit theory rewrite Eq. (4.96) as follows

$$\cos\varphi = \frac{P}{UI} \tag{4.101}$$

The product UI is called *apparent power*. $\cos\varphi$ is called the *power factor* of the circuit.

Physically, active power is the energy dissipated in unit time as heat in a portion of a circuit of resistance *R*. As $U\cos\varphi = IR$, it follows

$$P = I^2R \tag{4.102}$$

The active-power can be determined by other formulas:

$$P = I^2Z\cos\varphi = U^2Y\cos\varphi = I^2R = U^2G\,. \tag{4.103}$$

The active power is measured in *watts*.

The expression for the instantaneous inductive power is

$$p_L = ui = \frac{U_mI_m}{2}\left[\cos\frac{\pi}{2} - \cos\left(2\omega t + \frac{\pi}{2}\right)\right] = \\ = UI\sin 2\omega t = I^2\omega L\sin 2\omega t \tag{4.104}$$

This power p_L can also be expressed in terms of the effective values of current and voltage as well as the relative phase angle. The peak of circulating power is given by

$$Q = UI\sin\varphi \tag{4.105}$$

Since $U\sin\varphi = IX$, it follows that

$$Q = I^2X \tag{4.106}$$

This equation may be rewritten for inductive reactance as

$$Q = I^2X_L \tag{4.107}$$

and for capacitive reactance

$$Q = I^2(-X_C) \qquad (4.108)$$

In a general case, when $\varphi \neq 0$, and the circuit contains capacitance or inductance, or both, the product *UI* gives the *apparent* (*total*) power

$$S = UI \qquad (4.109)$$

This power makes no physical sense, but it can be defined as the maximum possible active power at given values of input current and voltage, i.e. an active power at cos φ = 1. It is measured in *volt-amperes*.

The true, reactive and apparent powers are related thus

$$P^2 + Q^2 = S^2. \qquad (4.110)$$

It may be graphically presented as a power triangle (Fig. 4.29).

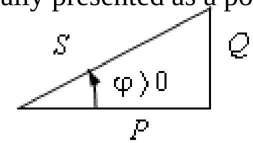


Fig. 4.29. A power triangle

The nameplate of energy source bears the value of *S*, that is the energy that source can deliver to the load if the circuit is a purely resistive one.

4.17. Power in Complex Form

The correct result is obtained if the complex voltage is multiplied by the conjugate of the complex current:

$$\begin{aligned} \underline{S} &= \tilde{S} = \underline{U}\,\underline{I}^* = U e^{j\psi_u}\left(I e^{j\psi_i}\right)^* = U e^{j\psi_u} I e^{-j\psi_i} = \\ &= UI e^{j(\psi_u - \psi_i)} = UI e^{j\varphi} = UI \cos\varphi + jUI \sin\varphi = \\ &= S\cos\varphi + jS\sin\varphi = P + jQ, \end{aligned} \qquad (4.111)$$

The "~" sign over *S* denotes that the product $\underline{U}\,\underline{I}^*$ gives the vector power and not its conjugate. To determine a complex power of some passive element it is more convenient to take advantage of other equation

$$\underline{S} = \underline{Z}I^2 = (R + jX)I^2 = P + jQ \qquad (4.112)$$

From Eq. (4.111) it follows that the active power is the real part, and the reactive power is the imaginary part of the product $\underline{U}\,\underline{I}^*$:

$$P = \mathrm{Re}\,\underline{U}\,\underline{I}^*; \quad Q = \mathrm{Im}\underline{U}\,\underline{I}^* \qquad (4.113)$$

Example 4.2. Consider circuit (Fig. 4.30). The instantaneous input voltage is u = 141 sin ωt where ω = 314 s^{-1}.The circuit resistances $R_1 = 10\ \Omega$, $R_2 = 40\ \Omega$, $R_3 = 20\ \Omega$; inductances and capacitances of reactive elements $L_1 = 15{,}9\ mH$, $L_2 = 95{,}5\ mH$, $C_1 = 212\ \mu F$, $C_3 = 79{,}6\ \mu F$. Find branch currents, voltages across the sections, and write them in instantaneous forms. Determine the powers of the supply and the load. Draw a vector diagram of currents and voltages.

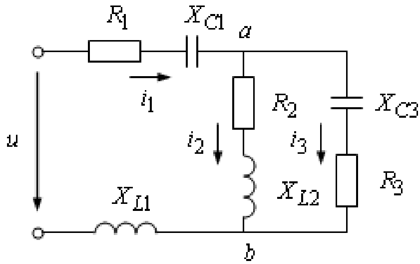


Fig. 4.30

Determine reactive susceptances of branches:

$$X_{L1} = \omega L_1 = 314 \cdot 15.9 \cdot 10^{-3} = 5\ \Omega;$$

$$X_{L2} = \omega L_2 = 314 \cdot 95{,}5 \cdot 10^{-3} = 30\ \Omega;$$

$$X_{C1} = \frac{1}{\omega C_1} = \frac{1}{314 \cdot 212 \cdot 10^{-6}} = 15\ \Omega;$$

$$X_{C3} = \frac{1}{\omega C_3} = \frac{1}{314 \cdot 79{,}6 \cdot 10^{-6}} = 40\ \Omega.$$

All complex values must be written both in algebraic and in exponential form. The complex effective input voltage

$$\underline{U} = \frac{U_m}{\sqrt{2}} e^{j\psi_e} = \frac{141}{\sqrt{2}} e^{j0^o} = 100 e^{j0^o} = 100 + j0\ V.$$

The complex impedances of branches:

$$\underline{Z}_1 = R_1 + j(X_{L1} - X_{C1}) = 10 + j(5 - 15) = 10 - j10 = 14{,}1 e^{-j45^o}\ \Omega;$$

$$\underline{Z}_2 = R_2 + jX_{L2} = 40 + j30 = 50{,}0 e^{j37^o}\ \Omega;$$

$$\underline{Z}_3 = R_3 - jX_{C3} = 20 - j40 = 44{,}7 e^{-j63^o}\ \Omega.$$

Now we pass from the initial circuit to the circuit with complex impedances (Fig. 4.31).

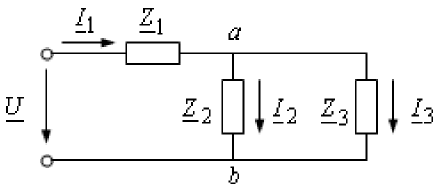


Fig. 4.31

For calculation of the circuit, we use a method of equivalent transformations. The section *ab* with a parallel connection of branch impedances $\underline{Z}_2$ and $\underline{Z}_3$ is changed into equivalent complex impedance

$$\underline{Z}_{ab} = \frac{\underline{Z}_2 \underline{Z}_3}{\underline{Z}_2 + \underline{Z}_3} = \frac{(40 + j30)(20 - j40)}{40 + j30 + 20 - j40} =$$

$$= \frac{(40 + j30)(20 - j40)}{60 - j10} = 35{,}1 - j10{,}8 = 36{,}8e^{-j17^o}\ \Omega.$$

Then the equivalent impedance of the circuit

$$\underline{Z}_o = \underline{Z}_1 + \underline{Z}_{ab} = 10 - j10 + 35{,}1 - j10{,}8 =$$

$$= 45{,}1 - j20{,}8 = 49{,}7e^{-j25^o}\ \Omega.$$

The current in a non-ramified part of the electric circuit is found by Ohm's law:

$$\underline{I}_1 = \frac{\underline{E}_1}{\underline{Z}_o} = \frac{100 + j0}{45{,}1 - j20{,}8} = 1{,}83 + j0{,}84 = 2{,}01e^{j25^o}\ A.$$

The voltages across circuit sections:

$$\underline{U}_1 = \underline{I}_1 \underline{Z}_1 = (1{,}83 + j0{,}84)(10 - j10) = 26{,}7 - j9{,}9 = 28{,}5e^{-j20^o}\ V;$$

$$\underline{U}_{ab} = \underline{I}_1 \underline{Z}_{ab} = (1{,}83 + j0{,}84)(35{,}1 - j10{,}8) = 73{,}3 + j9{,}7 = 73{,}9e^{j8^o}\ V.$$

The currents through the parallel branches

$$\underline{I}_2 = \frac{\underline{U}_{ab}}{\underline{Z}_2} = \frac{73{,}3 + j9{,}7}{40 + j30} = 1{,}29 - j0{,}72 = 1{,}48e^{-j29^o}\ A;$$

$$\underline{I}_3 = \frac{\underline{U}_{ab}}{\underline{Z}_3} = \frac{73{,}3 + j9{,}7}{20 - j40} = 0{,}54 + j1{,}56 = 1{,}65e^{j71^o}\ A.$$

The solution can be check up using Kirchhoff's voltage and current-laws

$$\underline{I}_1 = \underline{I}_2 + \underline{I}_3 = 1{,}29 - j0{,}72 + 0{,}54 + j1{,}56 = 1{,}83 + j0{,}84 = 2{,}01e^{j25^o}\ A;$$

$$\underline{E} = \underline{U}_1 + \underline{U}_{ab} = 26{,}7 - j9{,}9 + 73{,}3 + j9{,}7 = 100{,}0 - j0{,}2 = 100{,}2e^{j0^o}\ V.$$

The relative error does not exceed one per cent.

Let's pass from complex currents and voltages to their instantaneous values:

$$u_1 = \sqrt{2}U_1 \sin(\omega t + \psi_{U_1}) = \sqrt{2} \cdot 28{,}5 \sin(314t - 20^0) =$$
$$= 40{,}3 \sin(314t - 20^o)\ V;$$
$$u_{ab} = \sqrt{2}U_{ab} \sin(\omega t + \psi_{U_{ab}}) = \sqrt{2} \cdot 73{,}9 \sin(314t + 8^0) =$$
$$= 104{,}5 \sin(314t + 8^o)\ V;$$
$$i_1 = \sqrt{2}I_1 \sin(\omega t + \psi_{I_1}) = \sqrt{2} \cdot 2{,}01 \sin(314t + 25^0) =$$
$$= 2{,}84 \sin(314t + 25^o)\ A;$$
$$i_2 = \sqrt{2}I_2 \sin(\omega t + \psi_{I_2}) = \sqrt{2} \cdot 1{,}48 \sin(314t - 29^0) =$$
$$= 2{,}09 \sin(314t - 29^o)\ A;$$
$$i_3 = \sqrt{2}I_3 \sin(\omega t + \psi_{I3}) = \sqrt{2} \cdot 1{,}65 \sin(314t + 71^0) =$$
$$= 2{,}33 \sin(314t + 71^o)\ A.$$

Draw a combined vector diagram of branch currents and voltages (in corresponding scale)

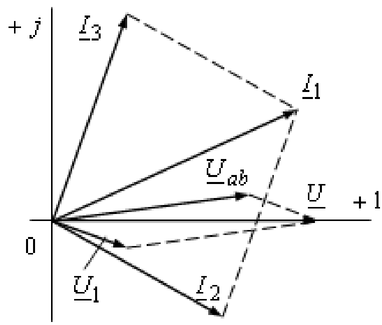


Fig. 4.32

The complexes of apparent powers of the energy source and the branches of this electric circuit:

$$\underline{S}_E = \underline{E} \cdot \overset{*}{\underline{I}}_1 = 100 \cdot (1{,}83 - j0{,}84) = 183 - j84 = 201e^{-j25^o}\ VA;$$

$$\underline{S}_1 = I_1^2 \underline{Z}_1 = 2{,}01^2 \cdot (10 - j10) = 40{,}4 - j40{,}4 = 57{,}1e^{-j45^o}\ VA;$$

$$\underline{S}_2 = I_2^2 \underline{Z}_2 = 1{,}48^2 \cdot (40 + j30) = 87{,}6 + j65{,}7 = 109{,}5e^{j37^o}\ VA;$$

$$\underline{S}_3 = I_3^2 \underline{Z}_3 = 1{,}65^2 \cdot (20 - j40) = 54{,}5 - j108{,}9 = 121{,}8e^{-j63^o}\ VA.$$

Then we can write active powers for the branches of the electric circuit and for the supply:

$$P = 183\ Wt;\ P_1 = 40{,}4\ Wt;\ P_2 = 87{,}6\ Wt;\ P_3 = 54{,}5\ Wt.$$

The reactive powers for the branches of the electric circuit and for the supply:

$$Q = -84\ \text{var}; \qquad Q_1 = -40{,}4\ \text{var}$$

$$Q_2 = 65{,}7\ \text{var}; \qquad Q_3 = -108{,}9\ \text{var}.$$

Check up the balance of powers:

$$P = P_1 + P_2 + P_3 = 40.4 + 87.6 + 54.5 = 182.5 \approx 183\ Wt$$

$$Q = Q_1 + Q_2 + Q_3 = -40.4 + 65.7 - 108.9 = -83.6 \approx -84\ \text{var}$$

4.18. Mutual Inductance in a Circuit

The coils of an electric circuit can place in space in such a way that the magnetic flux created by one coil partially links the other. The change in current in one coil is accompanied by a change in its magnetic field which induces an e.m.f. in the other. Such an electromotive force is called an *e.m.f. of mutual inductance*. When two coils are mutually connected, they are also named *magnetically coupled coils*.

Let two coils be in proximity to each other. The first carries a current i_1, and the second, a current i_2. Then we can write

$$M_{12} = \frac{\Psi_{12}}{i_2};$$
$$M_{21} = \frac{\Psi_{21}}{i_1}. \qquad (4.114)$$

The coefficients M_{12} and M_{21} are numerically equal:

$$M_{12} = M_{21} = M$$

The coefficient M is termed *the mutual inductance* and is measured as inductance L in *Henry* (*H*). It is connected with the inductances of the coils by the following relationship

$$M = k\sqrt{L_1 L_2} \tag{4.115}$$

where k is a coefficient of inductive coupling.

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

It cannot be greater than unity $0 \le k < 1$. The coefficient of coupling will be equal to unity only if all of the flux produced by the first circuit links the second.

The coils in Fig. 4.33, *a* are so connected that their fluxes are aiding, and the coils in Fig. 4.33, *b* are connected so that their fluxes are bucking each other. It is convention is to mark the like terminals (stars of the coil) with an asterisk or a dot.

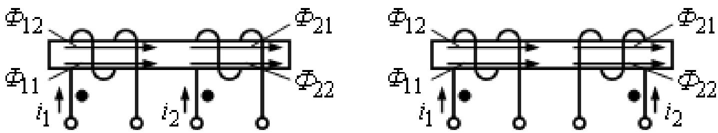


Fig. 4.33. Coupled coils

The dot convention is shown in the diagram of Fig. 4.34, *a* which is fully analogous with that of Fig. 4.33, *a*, and in the diagram of Fig. 4.34, *b* which is fully analogous with that of Fig. 4.33, *b*.

Fig. 4.34. Marking of coupled coils

If the currents in a mutual inductance are flowing towards (or away from) the like terminals, the corresponding coils are said to be connected *aiding*. If the current in one coil flows towards to the terminal marked with a dot, and that in the other coil flows away from the like terminal, the coils are said to be connected *bucking* or *in opposition*.

The circuits containing mutual inductances (or simply, *coupled circuits*) are analyzed by the method of complex numbers.

To define a position of similar terminals of two coils is possible on the basis of a simple experiment for which a direct current voltage source and an amperemeter of electro-magnetic system are necessary (see Fig. 4.35). One of coils is joined to the amperemeter, another - to the voltage source. At closure of key *Q* it arrears a short-term current which relaxes the magnetic field created by the current.

Hence, at the moment of turning on of the power supply currents i_1 and i_2 direct to opposite sides concerning similar terminals.

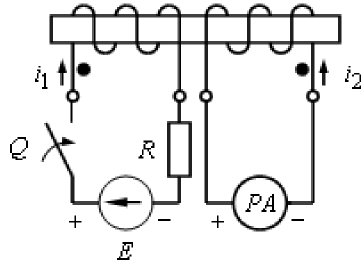


Fig. 4.35. The experiment for defining a position of similar terminals

The direction of current i_1 is determined by the polarity of the voltage source. A short-term diversion of an arrow of the amperemeter defines the direction of the current. If the arrow is declined towards a dial, the current i_2 directs to a positive terminal of the amperemeter. For this the terminals of coils which are joined to the terminals of the amperemeter and the voltage source are similar.

4.19. Mutual Inductance in Series

At aiding connection at any moment of time the currents in both coils of the circuit direct equally concerning the like terminals (Fig. 4.35), therefore, the fluxes of self-induction and mutual induction are summed up. So, it is a series aiding connection.

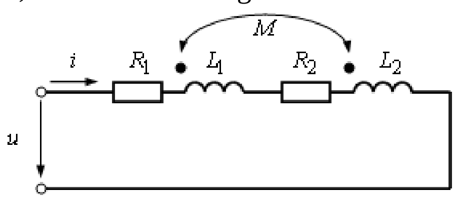


Fig. 4.35. The coils connected in series

When writing equations for a *coupled* circuit we assign positive directions to the currents in the branches of the circuit and take the direction for summation round them. Let's work out the equation for this electric circuit by KCL for instantaneous values remembering that the same current flows through both coils

$$R_1 i + L_1 \frac{di}{dt} + M \frac{di}{dt} + R_2 i + L_2 \frac{di}{dt} + M \frac{di}{dt} = u\,, \qquad (4.116)$$

or in complex notation

$$\underline{I}(R_1 + R_2 + j\omega(L_1 + L_2 + 2M)) = \underline{U}\,. \qquad (4.117)$$

The equivalent impedance of the circuit at aiding connection

$$\underline{Z}_{aid} = R_1 + R_2 + j\omega(L_1 + L_2 + 2M)\,. \qquad (4.118)$$

Fig. 4.37, *a*, shows a vector diagram for series aiding connection, where $\underline{U}_1 = \underline{I}(R_1 + j\omega L_1 + j\omega M)$ is the complex voltage across the first coil, and $\underline{U}_2 = \underline{I}(R_2 + j\omega L_2 + j\omega M)$ is the complex voltage across the second coil.

At bucking connection at any moment of time the currents in both coils of the circuit direct equally concerning the different terminals (Fig. 4.36), therefore, the fluxes of mutual induction are subtracted from the fluxes of self-induction. So, for a series opposition

$$R_1 i + L_1 \frac{di}{dt} - M \frac{di}{dt} + R_2 i + L_2 \frac{di}{dt} - M \frac{di}{dt} = u \qquad (4.119)$$

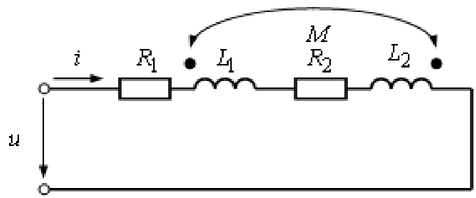


Fig. 4.36. A series opposition

In complex notation

$$\underline{I}(R_1 + R_2 + j\omega(L_1 + L_2 - 2M)) = \underline{U} \qquad (4.120)$$

where the equivalent impedance of the circuit at series opposition

$$\underline{Z}_{op} = R_1 + R_2 + j\omega(L_1 + L_2 - 2M) \qquad (4.121)$$

Fig. 4.37, *b*, shows a vector diagram for series opposition.

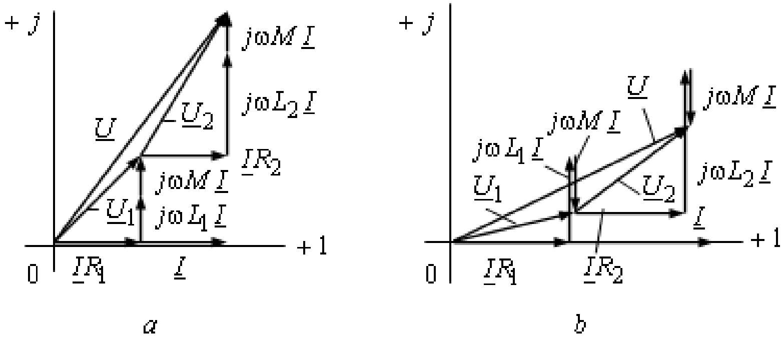


Fig. 4.37. A plot of vector diagram

4.20. Mutual Inductance in Parallel

Let's consider the electric circuit with parallel coupled coils (see Fig. 4.38). Let's work out a system of equations by Kirchhoff's laws for instantaneous values of currents and voltages

$$\begin{cases} i = i_1 + i_2 \\ e = R_1 i_1 + L_1 \dfrac{di_1}{dt} \pm M \dfrac{di_2}{dt} \\ e = R_2 i_2 + L_2 \dfrac{di_2}{dt} \pm M \dfrac{di_1}{dt} \end{cases} \tag{4.122}$$

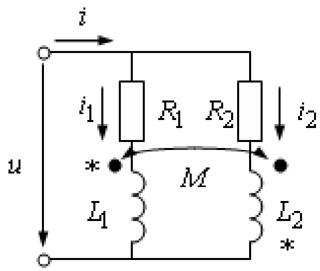


Fig.4.37

In Eq. (4.122) and other equations in this subsection the upper mark corresponds to aiding connection (the similar terminals are marked by dots • in Fig. 4.37), and the lower mark corresponds to the bucking connection (the similar terminals are marked by asterisk ∗ in Fig. 4.37).

Rewrite the system of equations in complex notation

$$\begin{cases} \underline{I} = \underline{I}_1 + \underline{I}_2; \\ \underline{E} = (R_1 + j\omega L_1)\underline{I}_1 \pm j\omega M \underline{I}_2; \\ \underline{E} = (R_2 + j\omega L_2)\underline{I}_2 \pm j\omega M \underline{I}_1. \end{cases} \tag{4.123}$$

On solving the system of equations, we will find currents

$$\begin{gathered} \underline{I}_1 = \frac{\underline{Z}_2 \mp \underline{Z}_M}{\underline{Z}_1\underline{Z}_2 - \underline{Z}_M^2}\underline{E}; \\ \underline{I}_2 = \frac{\underline{Z}_1 \mp \underline{Z}_M}{\underline{Z}_1\underline{Z}_2 - \underline{Z}_M^2}\underline{E}; \\ \underline{I} = \frac{\underline{Z}_1 + \underline{Z}_2 \mp 2\underline{Z}_M}{\underline{Z}_1\underline{Z}_2 - \underline{Z}_M^2}\underline{E}, \end{gathered} \tag{4.124}$$

where $\underline{Z}_1 = R_1 + j\omega L_1$; $\underline{Z}_2 = R_2 + j\omega L_2$ are complex impedances of branches, and $\underline{Z}_M = jX_M = j\omega M$ is the complex impedances of mutual inductance. The input complex impedance of the circuit

$$\underline{Z} = \frac{\underline{E}}{\underline{I}} = \frac{\underline{Z}_1\underline{Z}_2 - \underline{Z}_M^2}{\underline{Z}_1 + \underline{Z}_2 \mp 2\underline{Z}_M}. \tag{4.125}$$

4.21. Experimental Determination of Mutual Inductance

The difference of inductive reactance at aiding and bucking connections allows measuring their mutual inductance.

The first method. We make two experiments. In the 1st we connect two coils in series aiding and measure the current and voltage at the input and the active power in the circuit. In the 2nd, we connect two coils in series bucking and also measure the current, voltage and the active power in circuit. If both experiments are spent at the same input voltage the smaller current will correspond to aiding connection. It allows marking the similar terminals of the coils.

We can find reactances of coils by the results of measurements

$$X_{aid} = \omega(L_1 + L_2 + 2M) = \frac{U}{I_{aid}} \sin\arccos\left(\frac{P_{aid}}{UI_{aid}}\right);$$

$$X_{op} = \omega(L_1 + L_2 - 2M) = \frac{U}{I_{op}} \sin\arccos\left(\frac{P_{op}}{UI_{op}}\right).$$

The difference of these two reactances $X_{aid} - X_{op} = 4\omega M$ allows finding the mutual inductance

$$M = \frac{X_{aid} - X_{op}}{4\omega}. \tag{4.126}$$

The second method. We connect the first coil to a source of sinusoidal *emf* through an amperemeter (see Fig. 4.38), and the other coil is connected across a high-impedance voltmeter. Then we measure current I_1 and voltage U_2. The effective value of voltage across the second coil $U_2 = \omega M I_1$. Hence, the mutual inductance is found as

$$M = \frac{U_2}{\omega I_1} \tag{4.127}$$

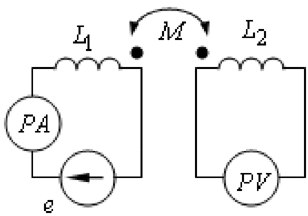


Fig. 4.38

4.22. The calculation of circuits with mutual inductance

One can calculate complex electric circuits with mutual inductance composed equations according to Kirchhoff's current and voltage laws. In the equations by KVL one must add the voltage of mutual induction. Or, we can apply mesh-current method as it is based by Kirchhoff's voltage law which takes into account the e.m.f. of a mutual induction.

Example 4.3. Consider ramified circuit (Fig. 4.39). Work out the system of equations by Kirchhoff's laws. Choose current directions and clockwise direction of summation round two meshes as shown in Fig. 4.39.

$$-i_1 + i_2 + i_3 = 0;$$

$$R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_2 \frac{di_2}{dt} + M\frac{di_3}{dt} + R_2 i_2 = e_1 + e_2;$$

$$-\left(R_2 i_2 + L_2 \frac{di_2}{dt} + M\frac{di_3}{dt}\right) + L_3\frac{di_3}{dt} + M\frac{di_2}{dt} + R_3 i_3 = -e_2 + e_3$$

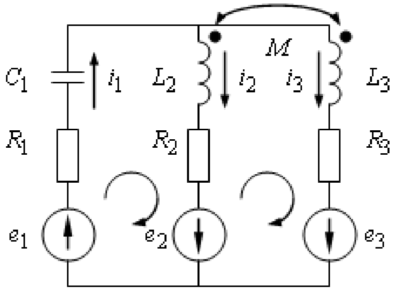


Fig. 4.39

In complex notation

$$\begin{cases} -\underline{I}_1 + \underline{I}_2 + \underline{I}_3 = 0; \\ R_1 \underline{I}_1 - j\dfrac{1}{\omega C_1}\underline{I}_1 + j\omega L_2 \underline{I}_2 + j\omega M\,\underline{I}_3 + R_2 \underline{I}_2 = \underline{E}_1 + \underline{E}_2; \\ -\left(R_2 \underline{I}_2 + j\omega L_2 \underline{I}_2 + j\omega M\,\underline{I}_3\right) + j\omega L_3 \underline{I}_3 + j\omega M\,\underline{I}_2 + R_3 \underline{I}_3 = \\ = -\underline{E}_2 + \underline{E}_3. \end{cases}$$

Further the problem can be solved by any known method.

Example 4.4. Consider the same circuit (Fig. 4.39) and work out the set of equations by mesh-current method. We choose the directions of mesh currents and the direction of summation clockwise.

When calculating self-impedances ($\underline{Z}_{11}$; $\underline{Z}_{22}$) and mutual impedances $\underline{Z}_{12}$; $\underline{Z}_{21}$ of independent contours it is necessary to take into account the presence of inductive couplings between the coils.

Then self-impedances of meshes would be as follows

$$\underline{Z}_{11} = R_1 + R_2 + j\left(\omega L_2 - \frac{1}{\omega C_1}\right);$$

$$\underline{Z}_{22} = R_2 + R_3 + j\left(\omega L_2 + \omega L_3 - 2\omega M\right).$$

The mutual impedances of contours look like

$$\underline{Z}_{12} = \underline{Z}_{21} = -\left(R_2 + j\omega L_2\right) + j\omega M$$

In the last equation the complex impedance $(R_2 + j\omega L_2)$ is taken with a minus sign by a general rule for mutual impedances of loops as two mesh currents direct in the opposite directions through the given impedance. The value $j\omega M$ is taken with a plus sign because the mesh current $\underline{I}_{11}$ in the second coil, and the mesh current $\underline{I}_{22}$ in the third coil are equally oriented concerning the similar terminals of two coils.

As the network consists of two meshes, the set of equations would be as follows

$$\begin{cases} \left(R_1 + R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_1}\right)\right)\underline{I}_{11} - \left(R_2 + j\left(\omega L_2 - \omega M\right)\right)\underline{I}_{22} = \underline{E}_1 + \underline{E}_2; \\ -\left(R_2 + j\left(\omega L_2 - \omega M\right)\right)\underline{I}_{11} + \left(R_2 + R_3 + j\left(\omega L_2 + \omega L_3 - 2\omega M\right)\right)\underline{I}_{22} = \\ = -\underline{E}_2 + \underline{E}_3. \end{cases}$$

On solving the given set by any method, we can define currents in all branches of the electric circuit as

$$\underline{I}_1 = \underline{I}_{11};\ \ \underline{I}_2 = \underline{I}_{11} - \underline{I}_{22};\ \ \underline{I}_3 = \underline{I}_{22}$$

4.23 Decoupling of Coupled Circuits

Sometimes it is necessary to decouple coupled circuits to simplify the calculation of a circuit with a mutual inductance.

This is done by placing additional inductances or modifying those already present in the original coupled circuits so that there is no coupling between any inductances in the transformed circuit.

Generally decoupling of any two coupled coils which are connected in one junction (Fig. 4.40, *a)*, one can realise by means of passage to the equivalent circuit which presented in Fig. 4.40, *b*.

Thus, we will simultaneously consider two cases: when two coupled coils are connected by similar terminals and when they are connected by different terminals.

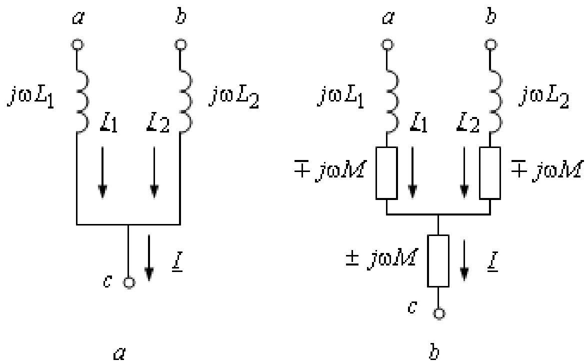


Figure 4.40

For the circuit with inductive coupled coils (Fig. 4.40, *a)*, we have the following equations

$$\begin{cases} \underline{U}_{ac} = j\omega L_1 \underline{I}_1 \pm j\omega M \underline{I}_2 \\ \underline{U}_{bc} = j\omega L_2 \underline{I}_2 \pm j\omega M \underline{I}_1 \end{cases} \quad (4.128)$$

For the circuit without coupled coils (Fig. 4.40, *b)*, we get

$$\begin{cases} \underline{U}_{ac} = j\omega (L_1 \mp M) \underline{I}_1 \pm j\omega M \underline{I} \\ \underline{U}_{bc} = j\omega (L_2 \mp M) \underline{I}_2 \pm j\omega M \underline{I} \end{cases} \quad (4.129)$$

The signs of additional inductances are determined only by the way of joint of inductive coupled elements and do not depend on the chosen directions of currents.

Comparing the circuits of Fig. 4.40, *a* and Fig. 4.40, *b,* we note that L_1 has been replaced by $L_1 \mp M$ and L_2 by $L_2 \mp M$, while the other arm has been extended to include an inductance $\pm M$.

In Eq. (4.129) upper marks correspond to a case when two coupled coils are connected by similar terminals in a common junction, and the lower marks correspond to the case for different terminals.

4.24. Power Transfer to an A. C. Load

Let us consider how the energy is transmitted between two coupled circuits. In any linear a. c. circuit, the sum of the active powers due to the voltage sources is equal to the sum of the active powers dissipated by the loads, and the sum of the reactive powers due to the voltage sources is equal to the sum of the reactive powers stored by the loads.

The sum of the reactive powers in this theorem is taken to be the sum of the squared branch currents times the branch reactances computed neglecting the mutual inductances, plus the algebraic sum of the power transferred by the magnetic fluxes from one set of branches to the other through mutual induction.

Let two currents $\underline{I}_1 = I_1 e^{j\psi_1}$ and $\underline{I}_2 = I_2 e^{j\psi_2}$ are known for two coupled coils. Work out the expressions for complex powers of the first and the second coils which are caused by a mutual induction in a case of their aiding connection

$$\underline{S}_{M1} = \underline{U}_{M1}\overset{*}{\underline{I}}_1 = j\omega M\,\underline{I}_1\overset{*}{\underline{I}}_1 = e^{j\frac{\pi}{2}}\omega M I_1 e^{j\psi_1} I_2 e^{-j\psi_2} =$$

$$= \omega M I_1 I_2 e^{j\left(\frac{\pi}{2}+\psi_1-\psi_2\right)} = \omega M I_1 I_2 \cos\left(\frac{\pi}{2}+\psi_1-\psi_2\right) +$$

$$+ j\omega M I_1 I_2 \sin\left(\frac{\pi}{2}+\psi_1-\psi_2\right) = \omega M I_1 I_2 \sin(\psi_1-\psi_2) +$$

$$+ j\omega M I_1 I_2 \cos(\psi_1-\psi_2) = P_{1M} + jQ_{1M};$$

$$\underline{S}_{M2} = \underline{U}_{M2}\overset{*}{\underline{I}}_2 = j\omega M\,\underline{I}_2\overset{*}{\underline{I}}_2 = e^{j\frac{\pi}{2}}\omega M I_2 e^{j\psi_2} I_2 e^{-j\psi_1} =$$

$$= \omega M I_1 I_2 e^{j\left(\frac{\pi}{2}+\psi_2-\psi_1\right)} = \omega M I_1 I_2 \cos\left(\frac{\pi}{2}+\psi_2-\psi_1\right) +$$

$$+ j\omega M I_1 I_2 \sin\left(\frac{\pi}{2}+\psi_2-\psi_1\right) = -\omega M I_1 I_2 \sin(\psi_1-\psi_2) +$$

$$+ j\omega M I_1 I_2 \cos(\psi_1-\psi_2) = P_{2M} + jQ_{2M}$$

From the last equations it follows

$$\begin{aligned} P_{1M} &= -P_{2M} = \omega M I_1 I_2 \sin(\psi_1-\psi_2); \\ Q_{1M} &= Q_{2M} = \omega M I_1 I_2 \cos(\psi_1-\psi_2). \end{aligned} \tag{4.130}$$

Thus, the total active power which is caused by a mutual induction, is equal to zero, that is $P_M = P_{1M} + P_{2M} = 0$. It means, that coupled coils do not influence on the general balance of an active power.

If $0 < \psi_1 - \psi_2 < \pi$, then $P_{1M} > 0$, $P_{2M} < 0$. On such a condition the energy is transmitted from a circuit into a magnetic field through the first coil and returned back into the circuit through the second coil. If $-\pi < \psi_1 - \psi_2 < 0$, then $P_{1M} < 0$, $P_{2M} > 0$, and the energy is transmitted from a circuit into a magnetic field through the second coil and returned back into the circuit through the first one.

The total reactive power which is caused by a mutual induction is

$$Q_M = 2\omega M I_1 I_2 \cos(\psi_1 - \psi_2)$$

The energy reserved by two inductive coils with mutual induction

$$W_M = M i_1 i_2$$

The results are true and in such a case if the coupled circuits are not connected electrically.