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STRENGHT OF MATERIALS

LECTURE NOTES

Запоріжжя • 2024

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National University “Zaporizhzhia Polytechnic”

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LECTURE NOTES ON THE DISCIPLINE

STRENGTH OF MATERIALS

for students of engineering specialities of all forms of studying

Zaporizhzhia
STATUS
2024

UDC 531 (075.8)

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Lecture notes on the discipline "Strength of materials" for students of engineering specialities of all forms of studying / Compiled by O. S. Omelchenko, A. A. Skrebtsov, P. K. Shtanko. — Zaporizhzhia : National University "Zaporizhzhia Polytechnic", 2024. — 308 p.

ISBN 978-617-8040-84-0

The lecture notes are intended for effective self-study of students with minimal time, for organizing and consolidating the acquired knowledge, as well as for distance learning. The effective application of the presented materials is based on the general preliminary study of courses in advanced mathematics and theoretical mechanics. The outline meets the standards of educational programs on the resistance of materials for students of mechanical specialties and is adapted to the educational level of applicants.

Approvedat the meeting of the Department of Theoretical and Applied Mechanics
Protocol № __2__ from "_16__" __09_____ 2024.

CONTENT

FOREWORD

Material strength is one of the most important disciplines that forms the engineering thinking of students of mechanical and civil engineering specialties of higher technical educational institutions and is the theoretical foundation of their technical education. These lecture notes are relevant and contain the main provisions, definitions, rules, theorems, calculation formulas and methods of strength of materials, presented in a summary form according to the course sections. The methodology of deriving the calculation formulas is given without detailing the intermediate results.

The material is divided into sections, each of which consists of a theoretical part and control questions.

The lecture notes are intended for effective self-study of students with minimal time, for organizing and consolidating the acquired knowledge, as well as for distance learning. The effective application of the presented materials is based on the general preliminary study of courses in advanced mathematics and theoretical mechanics. The outline meets the standards of educational programs on the strength of materials for students of mechanical specialties and is adapted to the educational level of applicants.

The lecture notes are based on modern textbooks on the strength of materials.

1. INTRODUCTION. COURSE OBJECTIVES. BASIC CONCEPTS

1.1 The science of material strength. Objects of study

Strength of materials is the science of engineering methods for calculating the strength, stiffness and stability of elements of machines and structures.

Strength is defined as the ability of a structure, its parts and components to withstand a certain load without collapsing.

Stiffness is the ability of a structure and its elements to resist deformation (changes in shape and size) under the influence of external loads.

Stability is the ability of a structure or its elements to maintain the original form of elastic equilibrium.

The strength of materials in the theoretical part is based on theoretical mechanics and mathematics, and in the experimental part on physics and materials science.

Thus, strength of materials is the most general science about the strength of structural elements, machines and structures. It is closely related to other disciplines, such as structural mechanics of rod systems, elasticity theory and plasticity theory. The main role in solving strength problems belongs to the strength of materials.

All the various types of structural elements used in structures, buildings and machines can be reduced to the basic forms: rods, shells, plates and massive bodies.

A rod or bar is a body in which one dimension (length) is significantly greater than the other two (cross) dimensions. Rods can be straight or curved, prismatic or of variable cross-section (Figure 1.1). These are shafts, axles, beams, lifting hooks, chain rings, etc.

a) b) c) d)

Figure 1.1 - Types of rods

Rods whose wall thickness is significantly less than the overall cross-sectional dimensions are called thin-walled (Fig. 1.1, d).

A shell is a body limited by curved surfaces that are close together. The surface equidistant from the solid surfaces of the shell is called **the median surface.** According to the shape of the median surface, the shells can be cylindrical (a), conical (b), and spherical (c) (Fig. 1.2). If the median surface is a plane, the calculated object is called **a plate**. Plates can be round, rectangular and other shapes.

a) b) c) d)

Figure 1.2 - Types of shells

Examples are boiler tanks, building domes, fuselage skins, and aircraft wings.

Bodies in which all three dimensions are of the same order are called massive bodies. They include building foundations, retaining walls, etc.

The emergence of the science of strength of materials is associated with the name of the Italian scientist **Galileo Galilei**, who conducted research on the study of strength, although the sources of this science can be found in the works of **Leonardo da Vinci.**

The rapid development of the science of material strength began in the late 18th century due to the rapid development of industry and transport.

The problems of strength were studied by Leonard Euler, Academician of the St. Petersburg Academy of Sciences, prominent scientists M.O. Belelyubsky, M.G. Bubnov, A.M. Voropaev, A.V. Gadolin, X.S. Golovin, D.I. Zhuravsky, F.S. Yasinsky, and others.

In the XX century, textbooks by prominent scientists V.L. Kirpichov, S.P. Timoshenko, M.M. Belyaev, O.O. Umansky, V.I. Feodosiev, O.A. Ilyushin, I.A. Birger, etc. played a significant role in the development of mechanics and the dissemination of scientific knowledge in the field of strength of materials,

In 1678, the English scientist **Robert Hooke** established **the law of deformation of elastic bodies**, according to which the deformation of an elastic body is proportional to the force acting on it. This **law is fundamental in the theory of material strength**

$$\boldsymbol{\sigma} = E \cdot \boldsymbol{\varepsilon}. \tag{1.1}$$

1.2 Coordinates and coordinate systems

The coordinates of a point are the values that determine the position of this point (in space, on a flat or curved surface, on a straight or curved line).

A rectangular coordinate system on a plane.

The position of a point on the plane is determined by two coordinates.

Two mutually perpendicular lines $x'x$ and $y'y$ (Figure 1.3). They are called **coordinate axes.** One of them, $x'x$ (often horizontal), is called the **abscissa axis**, and the other $y'y$ – **the ordinate axis.** The point where they intersect is called the origin and is denoted by the letter O.

To measure segments on the coordinate axes, the scale units are selected (freely, but the same for both axes).

The positive direction on the axis is indicated by an arrow. It is common to choose a positive direction so that $0x$ after turning 90° counterclockwise, combined with the positive "+" direction of the $0y$.

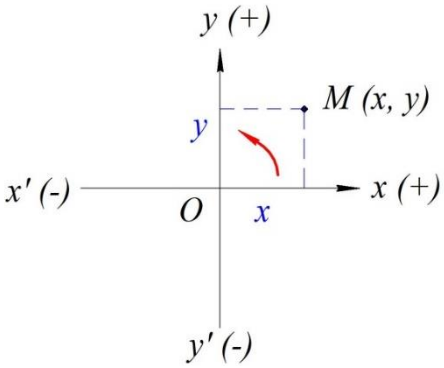
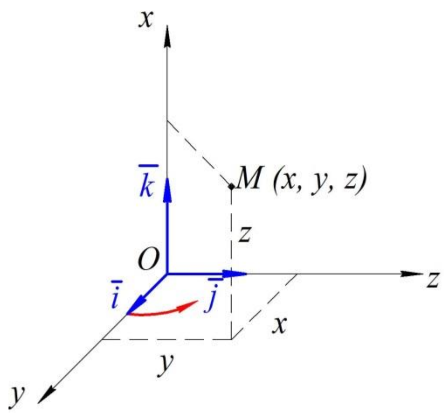


Figure 1.3 - Flat rectangular coordinate system

Rectangular spatial coordinate system.

The three mutually perpendicular axes Ox, Oy, Oz Figure 1.7), which pass through the point O, form a rectangular spatial coordinate system. The point O – **the beginning of the coordinate system,** the lines Ox, Oy, Oz – the coordinate axes (Ox – **the abscissa axis**, Oy – **the ordinate axis,** Oz – **the applicate axis**). The scale for all axes is the same.



By putting the segments OA, OB, OC on the axes in the positive direction, which are equal to the scale unit, we obtain three vectors $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$.

Figure 1.4 - Rectangular spatial coordinate system

They are called principal vectors or single orths (see Figure 1.4).

Positive directions on the axes are usually chosen so that the turn on 90°, which combines the positive direction Ox with the direction Oy (see Fig. 1.4), is counterclockwise when viewed from the side of the segment Oz. Oz. **This**

coordinate system is called the right-hand coordinate system. In **the left** coordinate system**,** such **rotation is clockwise.**

1.3 Classification of external forces

The forces of interaction between a structural element and the environment or other bodies are called external forces. They are classified in the following way.

1. **By the nature of the forces applied:**

– static (no inertial forces are involved);

– dynamic (forces of inertia are involved):

– instantly applied;

– percussion;

– repeatedly - variable.

2. **By change in time:**

– constant

– variable;

– for a certain period of time.

3. **By the nature of distribution:**

- volumetric or mass (own weight, force of inertia);

- surface

- concentrated;

- distributed over a surface or line.

The equivalent of the distributed load is numerically equal to the area of the figure of its epure and is applied at the center of gravity of this figure (Fig. 1.5).

$$Q = \frac{1}{2} qH \, . \tag{1.2}$$

In addition, there are loads that can be represented as a concentrated

moment (pair of forces) with the dimension (Nm, kNm, MNm).

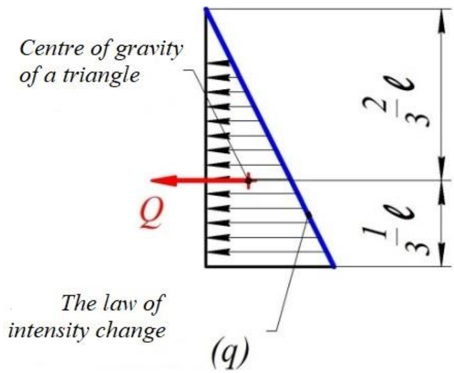


Figure 1.5 - Scheme for determining the equivalent distributed load

1.4 Types of rods deformation. The concept of deformed state

Real bodies change their shape and size as a result of external forces or temperature changes, i.e. they are deformed. When a body is deformed, its points, as well as imaginary lines or sections, move in the plane or in space relatively to their original position.

Deformations can be elastic, i.e. those that disappear after the forces that caused them cease to act, and plastic, or residual, deformations that do not disappear.

The main object considered in the study of materials is **a rod with a straight axis**. The following basic types of deformation of a rod are studied in the field of material strength: **stretching, compression, shear, torsion, and bending.**

More complex deformations, which are the result of a combination of several basic types of deformation, are also considered.

Stretching (compression) occurs when oppositely directed forces are applied to the rod along the axis (Fig. 1.6).

The change $\Delta\ell$ in the initial length ℓ of the rod is called absolute lengthening in stretching or absolute shortening in compression.

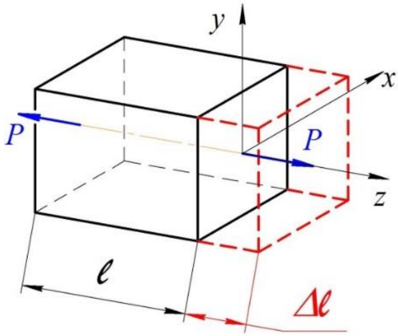


Figure 1.6 - Stretching the rod

The ratio of absolute lengthening (shortening) $\Delta\ell$ to the initial length ℓ of the rod is called the average relative lengthening (shortening) along the length ℓ

$$\varepsilon_{\text{cp}} = \frac{\Delta\ell}{\ell}. \tag{1.3}$$

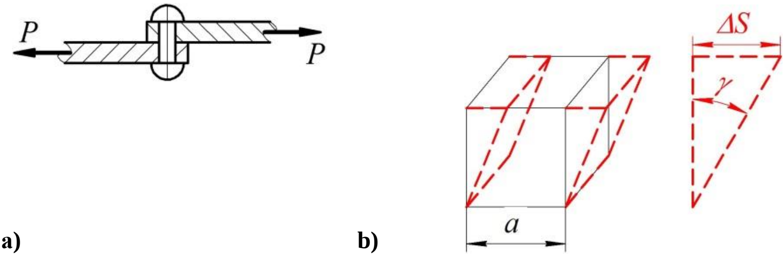
Many structural elements operate in stretching or compressive conditions: truss rods, columns, piston rod, tie bolt, etc.

A shear or a cut occurs when external forces shift two parallel plane sections of a rod relative to each other with a constant distance between them (Fig. 1.7, a). **The displacement ΔS is called the absolute shear. To determine the relative shear, we select an elementary volume with dimensions** $a \times a \times a$ (Fig. 1.7, b).

For example, rivets and bolts that fasten elements that external forces tend to move relative to each other work in shear or a cut.

The ratio of the absolute shear ΔS to the distance *a* between the displaced planes (the tangent of the angle γ), is called the relative shear. Due to the smallness of the angle γ its tangent is equal to the value of the angle $(tg\gamma \approx \gamma)$. So the relative shear

$$\gamma = \frac{\Delta S}{a}. \tag{1.4}$$



a) b)

Figure 1.7 - Shear deformation

Torsion occurs when external forces (applied tangentially) act on a rod, generating a torque relative to the rod axis (Figure 1.8).

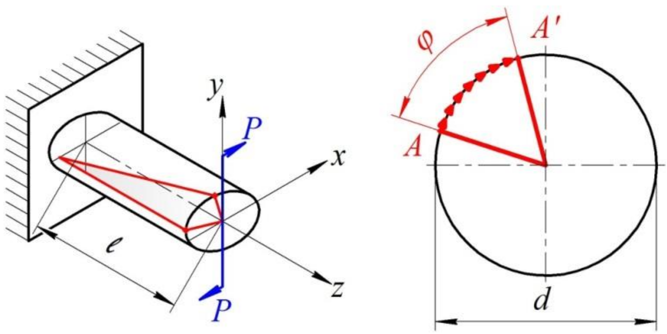


Figure 1.8 - Torsional deformation

The torsional deformation is accompanied by a rotation of the cross-sections of the rod relative to each other around its axis. The angle φ of this rotation is called **the torsion angle at length ℓ. The ratio of the torsion angle φ to ℓ is called the relative torsion angle θ**

$$\theta = \frac{\varphi}{\ell}. \tag{1.5}$$

Shafts, spindles of lathes and drilling machines and other details are worked in torsion.

Bending deformation consists of a curvature of the axis of a straight rod or a change in the curvature of a curved rod (Figure 1.9).

In straight rods, **the displacements of points that are perpendicular to the axis are called deflections and are denoted by the letter w.**

w **– the distance between the axes.** In bending, the sections are rotated from their original positions. **The angles of rotation of the sections relative to the initial position are denoted by the letter φ.**

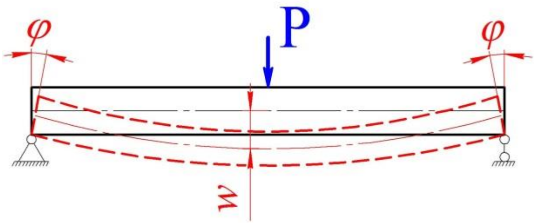


Figure 1.9 - Bending deformation

Beams of interfloor ceilings, bridges, railway car axles, leaf springs, shafts, gear teeth, wheel spokes, levers and many other parts are bent.

As a result of the simultaneous action of forces on the body that cause different types of deformation, a more complex deformation occurs, i.e. bending with torsion, or bending with stretching, etc.

1.5 Basic hypotheses of the science of material strength

In order to solve practical problems of material strength, some hypotheses are proposed regarding the structure and properties of materials, as well as the nature of deformation.

1. The material integrity hypothesis. It is believed that the material completely fills the body shape. There are no internal defects (pores, flocks, cavities, cracks).

2. Homogeneity and isotropy hypothesis. The material is considered to be homogeneous and isotropic, i.e. in any volume and in any direction the material properties are considered to be the same (for parts made by deformation). Sometimes the isotropy assumption is not valid for some materials. For example: wood, reinforced composite materials, etc.

3. The hypothesis of the smallness of deformations. It is assumed that deformations are small compared to the size of the body. This makes it possible to largely neglect changes in the location of external forces relative to individual parts of the body and to do the equations of statics as for an undeformed body.

4. The hypothesis of perfect elasticity of a material. It is assumed that all bodies are perfectly elastic. A body that has been deformed, after the elimination of the causes that caused it, completely restores its original shape.

Accepting the hypotheses of the smallness of deformations and the linear relationship between deformations and forces, it is possible to apply the principle of superposition (the principle of independence and addition of forces) to solve most problems of material strength. The results of the calculations are in a very good match with the data from practice.

These hypotheses, as well as some others (the Saint-Venant principle), discussed below, allow us to solve a wide range of strength, stiffness and stability problems.

1.6 Control questions

1. What is the strength of materials?
2. The concepts of strength, stiffness and stability.
3. The main types of structural elements.
4. What is the difference between the right and left spatial coordinate systems?
5. Determination of the equivalent distributed load.
6. What are the main types of deformation caused by external forces?
7. Define the deformations of stretching-compression, torsion, shear, bending and torsion angle.
8. What is called absolute and relative lengthening (shortening)?
9. What is called absolute and relative displacement?
10. Write the formulas for relative lengthening, shearing and torsion.
11. Basic hypotheses and assumptions of material strength.

2. GEOMETRICAL CHARACTERISTICS OF FLAT SECTIONS

The strength and stiffness of structural elements are determined by the size of their cross-sections and depend significantly on their shape. The cross-sectional area of a rod determines material consumption, but is not a measure of reliability.

These features are taken into account in the formulas of material strength by special cross-sectional characteristics (geometrical characteristics), which will be studied in this section. The cross-sections of rods can be represented as plane shapes, and the geometrical characteristics of plane shapes are considered.

2.1 Static moments of the area. Centre of gravity of the cross-section

Let's consider an arbitrary figure (cross-section of a rod) with coordinate axes x, y (Fig. 2.1). Let us select from this figure a platform with an elementary area dF with coordinatesx, y. Analogously to the expression for the moment of force relative to an axis (from the course of theoretical mechanics), an expression can be written for the moment of area, which is called **the static moment.**

Thus, the product of the elementary area by the distance $\boldsymbol{y}$ from the $\boldsymbol{Ox}$ $dS_x = ydF$ is called **the static moment of the elementary area relative to the $\boldsymbol{Ox}$ axis.**

Similarly $dS_y = xdF$.

Adding such products on the area F of an arbitrary shape, we obtain the static moments with respect to the x and y axes, respectively:

$$S_x = \int_F ydF, \quad S_y = \int_F xdF. \tag{2.1}$$

Mark x_C, y_C **the coordinates of the centre of gravity of the figure.**

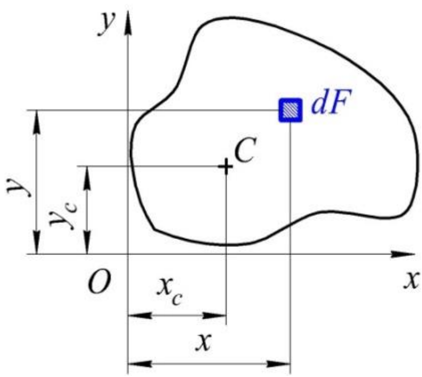


Figure 2.1 - Arbitrary figure (cross-section)

Continuing the analogy with the moments of forces, on the basis of the theorem of the moment of equivalent, we can write: $S_x = F y_C$, $S_y = F x_C$, and then determine **the coordinates of the centre of gravity.**

$$x_C = \frac{S_y}{F}, \quad y_C = \frac{S_x}{F}, \tag{2.2}$$

where $F = \sum F_i$ – the area of the entire figure.
With parallel axis transfer $x_1 = x - a$, $y_1 = y - b$ (Figure 2.2).

Then

$$S_{x_1} = \int_F (y - b)\, dF, S_{y_1} = \int_F (x - a)\, dF \tag{2.3}$$

or

$$S_{x_1} = S_x - bF, S_{y_1} = S_y - aF.$$

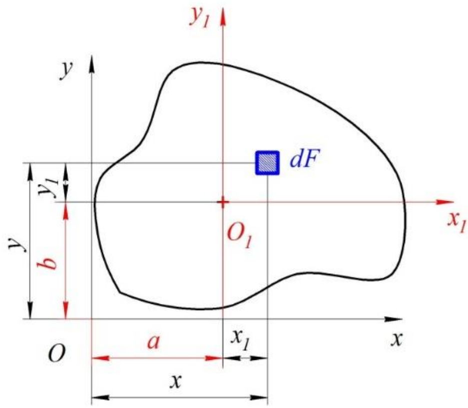


Figure 2.2 - An arbitrary figure with parallel axis transfer

To determine the static moments of a complex figure, it is divided into simple parts, for each of which the area F_i, is known, and the position of the centre of gravity x_{C_i}, y_{C_i} is known. The static moment of the area of the whole figure relative to a given axis is determined as the sum of the static moments of each part:

$$S_x = F_1 \cdot y_{C1} + F_2 \cdot y_{C2} + \cdots F_n \cdot y_{Cn} = \sum_1^n (F_i \cdot y_{Ci});$$

$$S_y = F_1 \cdot x_{C1} + F_2 \cdot x_{C2} + \cdots F_n \cdot x_{Cn} = \sum_1^n (F_i \cdot x_{Ci}).$$

Using the formulas (2.2), it is easy to find the coordinates of the centre of gravity of a figure:

$$x_C = \frac{S_y}{F} = \frac{1}{\sum F_i} \cdot \sum_1^n (F_i \cdot y_{Ci});$$

$$y_C = \frac{S_x}{F} = \frac{1}{\sum F_i} \cdot \sum_1^n (F_i \cdot x_{Ci}). \tag{2.4}$$

In accordance with the tables of the assortment, the static moment is expressed in cm^3. Determining the static moment is important for asymmetric shapes because important conclusions can be drawn from the given definitions:

– if the centre of mass of a figure coincides with an axis, then the static moment about this axis is zero; such axes are called **central axes;**

– if a figure has an axis of symmetry, and this axis passes through the centre of mass, then the static moment of such a figure is zero.

The results of the static moments of some simple shapes are shown in table 2.1.

To determine the position of the centre of gravity of shapes and bodies of complex geometric shape, they are decomposed into such parts of simple shape (if possible) for which the positions of the centres of gravity are known. Then determine the position of the centre of gravity of the entire figure or body using formulas (2.4), understanding in these formulas that F_i s the area of the parts into which the body is decomposed, and x_{Ci}, y_{Ci} – are the coordinates of the centres of gravity of these parts. It is necessary to take into account the signs when adding or subtracting static moments and planes.

2.2 Moments of inertia of flat figures

The axial moments of inertia are included in the formula for the stress and deflection of beams in bending. The moment of inertia of a section determines the stability of a compressed rod.

The axial moment of inertia of a figure is the integral of the products of the areas of the elementary areas by the squares of their distances from the axis lying in the plane of the figure (Figure 2.3).

$$\mathcal{J}_x = \int_F y^2 dF;$$
$$\mathcal{J}_y = \int_F x^2 dF; \qquad (2.5)$$
$$\mathcal{J}_\rho = \int_F \rho^2 dF.$$

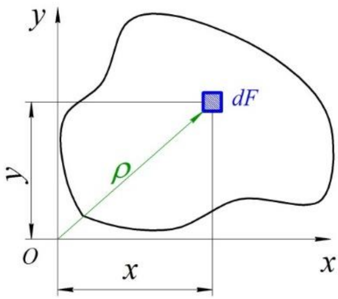


Figure 2.3 – **An arbitrary figure**

The polar moment of inertia $\mathcal{J}_\rho$ is used in the formulas for torsional stress and strain.

The polar moment of inertia of the area of a figure relative to the pole O is the integral of the products of the areas of the elementary areas by the squares of their distances from the pole.

If a system of rectangular axes is drawn through the pole, then $\rho^2 = x^2 + y^2$, then

$$\mathcal{J}_\rho = \int_F (x^2 + y^2)\mathrm{d}F = \int_F x^2 dF + \int_F y^2 dF = \mathcal{J}_y + \mathcal{J}_x. \quad (2.6)$$

Note that axial and polar moments of inertia can only take on positive values.

The centrifugal moment of inertia is the integral of the products of the areas of the elementary platforms at their distances from the x and y:

$$\mathcal{J}_{xy} = \int_F xydF. \quad (2.7)$$

Depending on the position of the axes, the centrifugal moment of inertia can be **positive** or **negative** or **equal to zero**. Obviously, when the axes are gradually rotated, it is possible to find a position at which the centrifugal moment of inertia is zero. Such axes are called the **principal axes of inertia.**

Two mutually perpendicular axes, of which at least one is the axis of symmetry of the figure, will always be the main axes of inertia, since in this case every positive value of $xydF$ corresponds to an equally negative value on the other side of the symmetry axis and their sum over the entire area of the figure is zero. The main axes passing through the centre of gravity of the cross-section are called **the main central axes.**

Determining the moments of inertia of complex cross-sections, the latter can be decomposed into simple parts whose moments of inertia are known. From the basic property of the sum integral, it follows that the moment of inertia of a complex figure is equal to the sum of the moments of inertia of its component parts.

The unit of measurement for moments of inertia is – см^4.

2.3 Moments of inertia relative to parallel axes

If we know the moments of inertia of the figure relative to the central axes $x,\ y$ and need to determine the moments of inertia relative to the axes parallel to the central axes (Fig. 2.4), then considering that

$x_1 = x + b$ and $y_1 = y + a$ we can substitute these values and integrate them by parts:

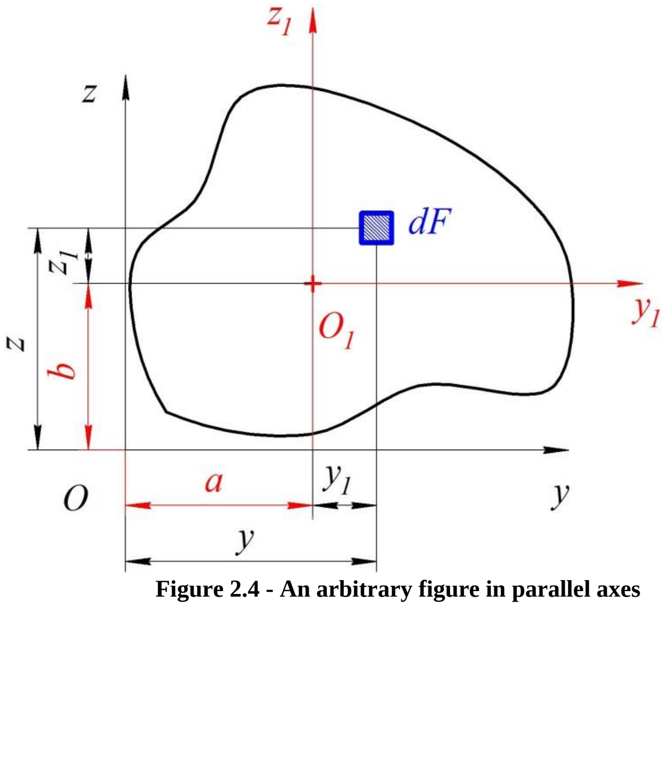


Figure 2.4 - An arbitrary figure in parallel axes

$$\mathcal{J}_{x_1} = \int_F y_1^2 dF = \int_F (y-b)^2\, dF = \int_F y^2\, dF + b^2 \int_F dF - 2b \int_F ydF\,;$$

$$\mathcal{J}_{y_1} = \int_F x_1^2 dF = \int_F (x-a)^2\, dF = \int_F x^2\, dF + a^2 \int_F dF - 2a \int_F xdF\,;$$

$$\mathcal{J}_{x_1 y_1} = \int_F x_1 y_1\; dF = \int_F (x-a)(y-b) dF =$$

$$= \int_F xydF + ab \int_F dF + a \int_F xdF - b \int_F ydF.$$

The integrals $\int_F\ xdF = S_y$ and $\int_F\ ydF = S_x$ are zero as static moments of the area relative to the central axes. Therefore, we obtain:

$$\mathcal{J}_{x_1} = \mathcal{J}_x\ + b^2 F;\quad \mathcal{J}_{y_1} = \mathcal{J}_y\ + a^2 F;\ \mathcal{J}_{x_1 y_1} = \mathcal{J}_{xy}\ + abF. \quad (2.8)$$

2.4 Dependence between the moments of inertia when the coordinate axes are rotated

Suppose that the moments of inertia of an arbitrary figure are known relative to the coordinate axes $x,\ y$ (2.5).

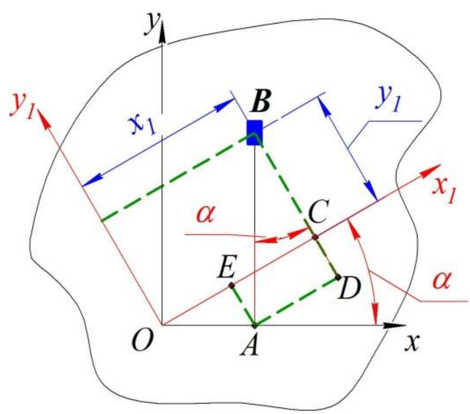


Figure 2.5 - An arbitrary figure when the axes are rotated

If you rotate the $x,\ y$ axes by an angle α counterclockwise, assuming the angle of rotation of the axes in this direction is positive, you can find the moments of inertia of the cross-section relative to the rotated x_1, y_1, axes by expressing the coordinates of an arbitrary elementary platform using trigonometric functions (Figure 2.5):

$$x_1 = OC = OE + AD = x \cdot cos\alpha + y \cdot sin\alpha;$$
$$y_1 = BC = BD - EA = y \cdot cos\alpha - x \cdot sin\alpha. \qquad (2.9)$$

Moment of inertia of the cross-section relative to the rotated axes x_1y_1:

$$\mathcal{J}_{x_1} = \int_F y_1{}^2\, dF;\ \mathcal{J}_{y_1} = \int_F x_1{}^2\, dF;\ \mathcal{J}_{x_1y_1} = \int_F x_1y_1\, dF. \qquad (2.10)$$

Substituting (2.9) into formula (2.10), we obtain:

$$\mathcal{J}_{x_1} = \int_F (ycos\alpha - xsin\,\alpha)^2 dF = cos^2\alpha \int_F y^2\, dF +$$

$$+sin^2\alpha \int_F x^2 dF - sin2\alpha \int_F xydF;$$

$$\mathcal{J}_{y_1} = \int_F (xcos\alpha - ysin\,\alpha)^2 dF = sin^2\alpha \int_F y^2\, dF +$$

$$+cos^2\alpha \int_F x^2 dF + sin2\alpha \int_F xydF;$$

$$\mathcal{J}_{x_1y_1} = \int_F (xcos\alpha + ysin\alpha)(ycos\alpha - xsin\alpha)dF =$$

$$= (cos^2\alpha - sin^2\alpha) \int_F xydF + \frac{1}{2} sin2\alpha (\int_F y^2 dF - \int_F x^2 dF).$$

Taking into account the formulas (2.5), we finally get the following:

$$\left.\begin{aligned} \mathcal{J}_{x_1} &= \mathcal{J}_x cos^2\alpha + \mathcal{J}_y sin^2\alpha - \mathcal{J}_{xy} sin2\alpha; \\ \mathcal{J}_{y_1} &= \mathcal{J}_x sin^2\alpha + \mathcal{J}_y cos^2\alpha - \mathcal{J}_{xy} sin2\alpha; \end{aligned}\right\} \qquad (2.11)$$

$$\mathcal{J}_{x_1y_1} = \mathcal{J}_{xy} cos2\alpha - \frac{1}{2}(\mathcal{J}_y - \mathcal{J}_x) sin2\alpha. \qquad (2.12)$$

Note that these formulas, obtained by rotating an arbitrary system of rectangular axes, are also valid for central axes.

Adding the equations (2.11) by steps, we find:

$$\mathcal{J}_{x_1} + \mathcal{J}_{y_1} = \mathcal{J}_x + \mathcal{J}_y = \mathcal{J}_\rho. \qquad (2.13)$$

Thus, when the rectangular axes are rotated, the sum of the axial moments of inertia does not change and is equal to the polar moment of inertia relative to the beginning of the coordinates.

When the axes are rotated by an angle $\alpha = 90°$

$$\mathcal{J}_{x_1} = \mathcal{J}_y; \quad \mathcal{J}_{y_1} = \mathcal{J}_x; \quad \mathcal{J}_{x_1 y_1} = -\mathcal{J}_{xy}.$$

2.5 Determining the direction of the main axes of inertia. Main points of inertia

Recall that **the principal axes** are those about which the centrifugal moment is zero and the axial axes take on extreme values. The principal central axes are of the greatest practical importance. Let's denote them by the letters $\boldsymbol{u}$ and $\boldsymbol{v}$. Thus, $\mathcal{J}_{uv} = 0$.

To determine the position of the main central axes of a plane figure, we rotate the initial system of central axes xy (Figure 2.6) by some angle α_0, at which the centrifugal moment of inertia is zero ($\mathcal{J}_{uv} = 0$).

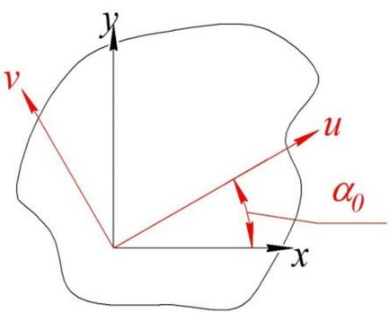


Figure 2.6 - An arbitrary figure with principal axes

Then we get from formula (2.12):

$$\mathcal{J}_{uv} = \mathcal{J}_{x_1y_1} = \mathcal{J}_{xy}cos2\alpha_0 - \frac{\mathcal{J}_y - \mathcal{J}_x}{2}sin2\alpha_0, \tag{2.14}$$

from which

$$tg2\alpha_0 = \frac{2\mathcal{J}_{xy}}{\mathcal{J}_y - \mathcal{J}_x}. \tag{2.15}$$

The values of the angle α_0 obtained from formula (2.15) differ by 90° and give the positions of the main axes. The smaller of these angles does not exceed $^{\pi}/_{4}$. Further, we use only the smaller angle. The principal axis drawn at this angle (positive or negative) is denoted by the letter $\boldsymbol{u}$.

The principal moments of inertia can be found from the general formulas for the transition to rotated axes (2.11), taking $\alpha = \alpha_0$. By substitutions and transformations, we find expressions for the principal moments of inertia that do not contain trigonometric functions:

$$\mathcal{J}_u = \frac{1}{2}\left(\mathcal{J}_x + \mathcal{J}_y\right) \pm \sqrt{\left(\mathcal{J}_x - \mathcal{J}_y\right)^2 + 4\mathcal{J}_{xy}^2}\,; \tag{2.16}$$

$$\mathcal{J}_v = \frac{1}{2}\left(\mathcal{J}_x + \mathcal{J}_y\right) \mp \sqrt{\left(\mathcal{J}_x - \mathcal{J}_y\right)^2 + 4\mathcal{J}_{xy}^2}\,. \tag{2.17}$$

Moreover, the upper signs are taken when $\mathcal{J}_x > \mathcal{J}_y$, and the lower signs when $\mathcal{J}_x < \mathcal{J}_y$.

Thus, formulas (2.15), (2.16) and (2.17) allow us to determine the position of the main axes and the main central moments of inertia.

It is important to note that the main moments of inertia calculated by formulas (2.16) and (2.17) have the property of extremality.

2.6 Radius of inertia. Ellipse of inertia

The moment of inertia of a figure relative to any axis can be expressed as the product of the figure's area multiplied by the square of a certain value, called **the radius of inertia**:

$$\mathcal{J}_z = \int_F y^2 dF = F i_z^2, \qquad (2.18)$$

where i_z — the radius of inertia relative to the $z-$axis. It appears from expression (2.18) that

$$i_z = \sqrt{\frac{\mathcal{J}_z}{F}}, \quad i_z^2 = \frac{\mathcal{J}_z}{F}. \qquad (2.19)$$

Similarly, **the radius of inertia of the cross-sectional area relative to the y-axis:**

$$i_y = \sqrt{\frac{\mathcal{J}_y}{F}} \quad ; \quad i_y^2 = \frac{\mathcal{J}_y}{F}. \qquad (2.20)$$

The main central axes of inertia correspond to the main radii of inertia:

$$i_u = \sqrt{\frac{\mathcal{J}_u}{F}}; \quad i_u^2 = \frac{\mathcal{J}_u}{F}. \qquad (2.20')$$

The concept of an ellipse of inertia is of great importance in mechanics.

The ellipse of inertia can be used to determine the radius of inertia of an area relative to any axes passing through the centre of the ellipse. Using the ellipse of inertia, you can graphically find the radius of inertia for any axis.

Let's draw an ellipse with semi-axes equal to the main radii of inertia on the main central axes of inertia, and along the **u-axis** we put segments i_v, and along the **v-axis** segments i_u (Figure 2.7).

This ellipse is called the **ellipse of inertia**. The radius of inertia relative to an arbitrary axis **u** is determined by the perpendicular OA, placed from the centre of the ellipse to the tangent to the ellipse parallel to this axis.

To determine the point of contact, it is enough to draw any chord parallel to the z-axis. The point of intersection of the ellipse with the line connecting the centre O with the middle of the chord is the point of contact. By measuring the segment OA, we find the moment of inertia.

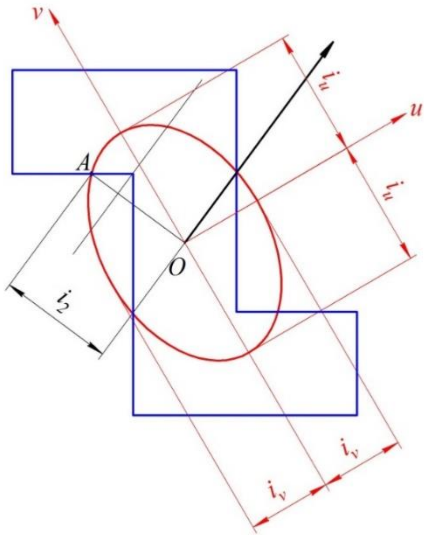


Figure 2.7 - Ellipse of inertia

The ellipse of inertia for the centre of gravity is called the central ellipse.

2.7 Moments of strength of flat cross-sections

Moment of strength – a geometric quantity that characterises the strength of a body (rod, beam, shaft) to stress depending on the shape and dimensions of its cross-section. The unit of measurement is – $см^3$.

Moments of strength characterize the strength of a cross-section. The moment of strength of a complex cross-section is determined for the entire figure as a whole as the ratio of the main moment of inertia to the distance from the main axis to the most distant point of the cross-section.

$$W_x = \frac{\mathcal{J}_x}{y_{max}}; \quad W_y = \frac{\mathcal{J}_y}{x_{max}}; \quad W_\rho = \frac{\mathcal{J}_\rho}{\rho_{max}}. \qquad (2.21)$$

Moments of strength of some simple figures.

Rectangle

$$W_x = \frac{bh^2}{6}; \ W_y = \frac{hb^2}{6}, \text{см}^3.$$

Triangle

$$W_{x_\text{C}} = \frac{bh^2}{24}, \quad \left(\text{у}_{max} = \frac{2}{3}h\right), \text{см}^3.$$

Circle

$$W_x = \frac{\pi d^2}{32}, \qquad W_\rho = \frac{\pi d^3}{16}, \ \left(\rho_{max} = \frac{d}{2}\right), \text{см}^3.$$

Ring

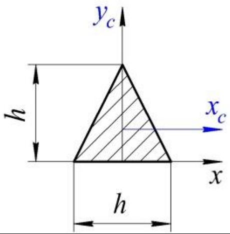
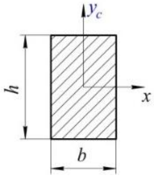
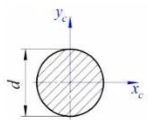
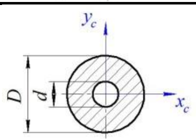
$$W_x = \frac{\pi d^3}{32}(1 - a^4), \ W_\rho = \frac{\pi d^3}{16}(1 - a^4), \ \text{см}^3 \left(a = \frac{d}{D}\right).$$

The geometric characteristics of simple shapes are shown in Table 2.1

2.8 Control questions

1. How is the static moment of a figure determined? What is the dimension of static momentum?
2. Formulas for determining the coordinate of the centre of gravity of a figure.
3. How to determine the coordinates of the centre of gravity of a complex figure?
4. What are the axial, polar and centrifugal moments of inertia? What is their dimension?
5. Which of the moments of inertia are always positive?
6. How does the centrifugal moment of inertia change when the coordinate axes are rotated by 90°?
7. What axes are called the main central axes of inertia?
8. Why is the axis of symmetry of a figure always one of the principal axes of inertia?
9. What is the radius and ellipse of inertia?
10. What is the moment of strength of a cross-section?
11. Moments of inertia relative to parallel axes.

Table 2.1 - Geometric characteristics of simple figures

Figure	S_X	S_y	$\mathcal{J}_X$	$\mathcal{J}_y$	$\mathcal{J}_\rho$	W_X	W_y	W_ρ
	$\frac{bh^2}{6}$		$\frac{bh^3}{36}$	$\frac{hb^3}{48}$		$\frac{bh^2}{24}$	$\frac{hb^2}{6}$	
			$\frac{bh^3}{12}$	$\frac{hb^3}{12}$		$\frac{bh^2}{6}$	$\frac{hb^2}{6}$	
			$\frac{\pi d^4}{64}$	$\frac{\pi d^4}{64}$	$\frac{\pi d^4}{32}$	$\frac{\pi d^3}{32}$	$\frac{\pi d^3}{32}$	$\frac{\pi d^3}{16}$
			$\frac{\pi d^4}{64}(1-a)$	$\frac{\pi d^4}{64}(1-a)$	$\frac{\pi d^4}{32}(1-a$	$\frac{\pi d^3}{32}(1-a^4)$	$\frac{\pi d^3}{32}(1-a^4)$	$\frac{\pi d^3}{16}(1-a^4$

INTERNAL FORCES. CROSS-SECTIONAL METHOD. STRESSES AND INTERNAL FORCE FACTORES

3.1 Internal forces. The method of cross-sections

There are always certain forces of interaction between neighbouring parts of a body (crystals, molecules, atoms), i.e. **internal forces.** In the strength of materials, internal forces acting in a body in its natural (unloaded) state are not considered or taken into account, and only those internal forces that appear as a result of loading the body are studied and determined. Internal forces are often called **efforts**. To detect and then determine internal forces in the strength of materials, **the cross-sectional method** is widely used (Fig. 3.1).

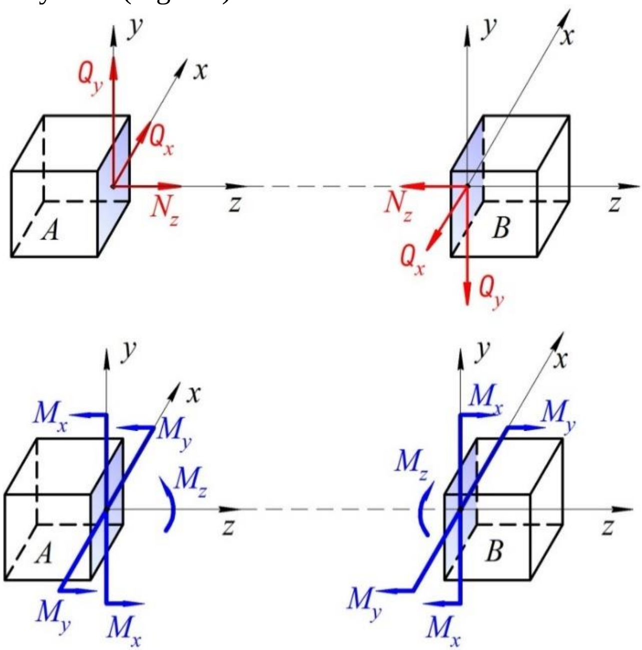


Figure 3.1 - Cross-sectional method

Let's look at an arbitrary body loaded with a self-balanced system of forces. Let's divide it into two parts A and B (see Fig. 3.1 a, b).

According to Newton's third law, the internal forces acting in the cross-section of part A are equal in modulus and opposite in direction to the internal forces of part B. They can be summarised as **the principal vector and principal momentum**.

If the principal vector and principal moment are projected onto the rod **z-axis** and the main central axes of the cross-section $\boldsymbol{y}$ and $\boldsymbol{x}$, then on each side of the cross-section we have six internal force factors: three forces (N_z, Q_y, Q_z) and three moments $\left(M_x, M_y, M_z\right)$ $(Fig. 3.1\ b)$.

These quantities are called **internal force factors** in the cross-section of the rod. Moment forces in the cross-section have the following definitions:

N_z – is the longitudinal force; it is equal to the sum of the projections of all internal forces acting in the cross-section on the normal to the cross-section;

Q_x та Q_y – transverse forces, which are equal to the sum of projections of all internal forces in the cross-section on the main central axes of the cross-section x and y;

M_x та M_y – bending moments, defined as the sum of the moments of all internal forces in the cross-section relative to the main central axes of the cross-section x and y;

$M_z = M_{\text{кр}}$ – torsional moment, equal to the sum of the moments of all internal forces in the cross-section relative to the rod z-axis:

where $N_z = \sum P_{iz}\,; Q_x = \sum Q_{ix}\ ; Q_y = \sum Q_{iy}\,;$

$$M_x = \sum_1^n M_{ix};\quad M_y = \sum_1^n M_{iy};\quad M_{\text{кр}} = \sum_1^n M_{iz}.$$

For the practical calculation of forces and moments in a cross-section, the equilibrium equations of an arbitrary spatial system of forces are used:

$$\begin{gathered}
\sum_1^n M_x(\bar{P}_i) + M_x = 0\,;\quad \sum_1^n P_{ix} + Q_x = 0; \\
\sum_1^n M_y(\bar{P}_i) + M_y = 0\,;\quad \sum_1^n P_{iy} + Q_y = 0; \\
\sum_1^n m_z(\bar{P}_i) + M_z = 0\,;\quad \sum_1^n P_{iz} + N_z = 0.
\end{gathered} \tag{3.1}$$

The cross-sectional method makes it possible to find the internal forces and moments in any cross-section of a rod along the entire length of the rod under any load.

The forces and moments in different cross-sections of the same rod are different. **Graphical representations showing how internal forces change from one section to another are called epures.** The number of sections is determined by the number of force sections. **A force section** is the part of the rod between the points of application of concentrated forces, or where the distributed load changes according to the same law. The number of force sections is most easily determined by the number of dimensions shown in the diagram. In practical problems, the construction of epure is required to determine the position of the most loaded (dangerous) section of the rod.

Rules for making epures:

1. The axis (base) on which the epure is built is always chosen so that it is parallel or simply coincides with the axis of the rod.
2. The ordinates of the epure are drawn with the sign perpendicular to the axis of the epure.
3. Hatch the epure with lines perpendicular to the base.
4. For forces and moments, select a certain scale. Put a force sign in the field of the epure.
5. At the point where the force or moment is applied, a jump (step) occurs on the epure by the value of this force or moment.
6. A longitudinal force is considered positive if it causes stretching and negative if it causes compression.

Rules for drawing stretching (compression) epures.

1. External loads that stretch the rod cause positive internal stress; those that compress cause negative stress. Positive values are set upwards from the baseline, negative values are set downwards.
2. At the point of application of the longitudinal force, a jump ("step") in the value of this force appears on the epure.

3. In a section with a distributed load, the epure is bounded by a straight, inclined line. If the section does not have a distributed load, the epure is bounded by a straight line parallel to the baseline.

3.2 Stress in the cross-section

As already mentioned in the cross-section, the internal forces can be summarised as the principal vector $\bar{R}$ and the principal moment $\bar{M}$, whose projections on the main central axes give the values $N_z;\ Q_y;\ Q_x;\ M_x;\ M_y;\ M_z$ – which are called **forces and moments in the cross-section, or internal force factors.**

Let's look at an infinitely small element of area dF (Figure 3.2).

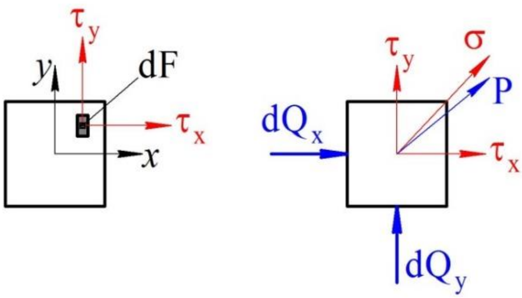


Figure 3.2 - Stresses in a cross-section

Due to the smallness of the element dF the forces on this area will be the same in module and direction. Their equivalent dR will pass through the centre of gravity of the area dF The projections of dR along the x, y, z axes will be **the elementary forces** $dN_z,\ dQ_x,\ dQ_y$. Then the relation

$$\sigma = \frac{dN_z}{dF}, \qquad \tau_y = \frac{dQ_y}{dF}, \qquad \tau_x = \frac{dQ_x}{dF} \tag{3.2}$$

is called the stress at the point with coordinates x and y, where σ – **normal stress**; τ – **tangential stress**. They are usually expressed in

$\text{Pa} = \frac{H}{\mathcal{M}^2}$ or kPa; MPa (kPa= $10^3 Pa$; MPa= $10^6 Pa$).

Thus, **stress is the internal force per unit area at the point of section under consideration.** Sometimes, in addition to the normal σ and tangential τ_x, τ_y **the total stress** $= \frac{dR}{dF}$, per unit area is also considered

$$p = \sqrt{\sigma^2 + \tau_y^2 + \tau_x^2}. \qquad (3.3)$$

So, based on the definition of forces and moments (see Section 3.1) and taking into account formulas (3.2), we have:

$$N_z = \int_F dN_z = \int_F \sigma dF; \quad M_y = \int_F xdN = \int_F \sigma xdF;$$

$$Q_y = \int_F dQ_y = \int_F \tau_y dF; \quad M_x = \int_F ydN = \int_F \sigma ydF; \quad (3.4)$$

$$Q_x = \int_F dQ_x = \int_F \tau_x dF;$$

$$M_{\text{кр}} = \int_F \left(ydQ_y - xdQ_x\right) = \int_F \left(y\tau_x - x\tau_y\right) = \int_F \rho\tau dF, \qquad (3.5)$$

where τ – **full tangential stress:**

ρ – distance from the centre of gravity of the cross-section to the line of action dQ.

$$\tau = \frac{dQ}{dF} = \frac{\sqrt{dQ_{y+}^2 dQ_x^2}}{dF} = \sqrt{\tau_y^2 + \tau_x^2}, \qquad (3.6)$$

The formulas (3.5, 3.6) defining the relationship between stresses and internal forces are called **static equations** or **integral equilibrium equations.**

The following scheme is always used to derive the formulas for studying the stress state (of rods):

1. The static aspect of the problem is considered, i.e., the static equations that are needed to solve the problem are written down.

2. **The geometric aspect of the problem** is considered: based on the experimental study of the deformation of the rod and certain hypotheses (in particular, the hypothesis of flat cross-sections), the dependencies between the displacement of the points of the rod and their position in the

cross-section relative to the selected coordinate system are set. These dependencies are called **geometric equations.**

3. **The physical aspect** of the problem is considered: based on experimental studies of the physical properties of the material, the relationships between stresses and deformations (or displacements) are determined. These relationships are called **physical equations.**

4. **Synthesise**, i.e. solve all the equations obtained in the previous steps together. By eliminating deformations (or displacements), we determine the formulas that express stresses in terms of forces or moments in the cross-section.

3.3 The Saint-Venant principle

When determining stresses under various types of deformation, the Saint-Venant principle is widely used in the strength of materials, which is as follows: if a body is loaded by statically equivalent systems of forces, i.e., those in which the main vector and principal moment are the same, and the size of the zone of applied loads is small compared to **the size of the body, in cross-sections that are sufficiently distant from the places where the forces are applied, the stresses have little dependence on the method of loading.** Let us explain the principle using the following example.

If you take a rod and apply a load to it according to different schemes (Fig. 3.3), the stress in the cross-section at a distance *b* will be almost the same. There is no general theoretical justification for the Saint-Venant principle, but its validity is confirmed by numerous theoretical and experimental studies.

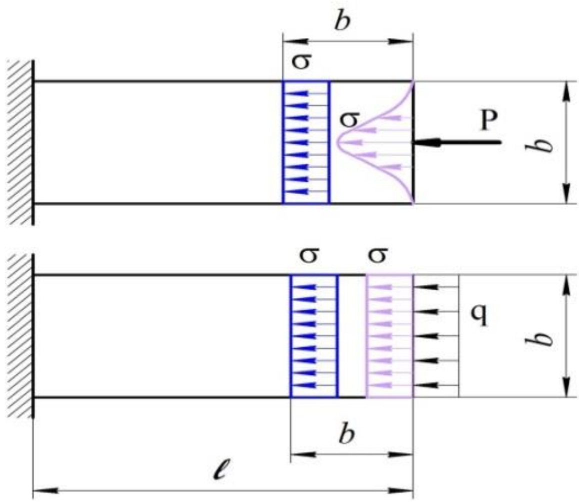


Figure 3.3 - Schemes for applying loads to the rod

3.4 Control questions

1. The concept of internal force factors.
2. What are epures and what are they for?
3. Rules for building epures.
4. Normal and tangential stresses (formulas).
5. The principle of Saint Venant.
6. What causes internal forces?
7. What is the method of cross-sections used to identify?
8. The third law of Newton.
9. How many and what internal force factors will we have on each side of the cross-section?
10. Write static or integral formulas for forces and moments.
11. What are the static, geometric, and physical aspects of deriving formulas for studying the stress state (of rods)?

3. STRETCHING AND COMPRESSION

4.1 Stress and deformations during the stretching and compression

Simple stretching or compression of a rod is caused by forces acting along its axis. In this case, only one of the six internal force factors, the longitudinal (axial) force N_z, appears in the cross-sections of the rod (Fig. 4.1). An example of drawing a longitudinal force epure is shown in Section 3.1.

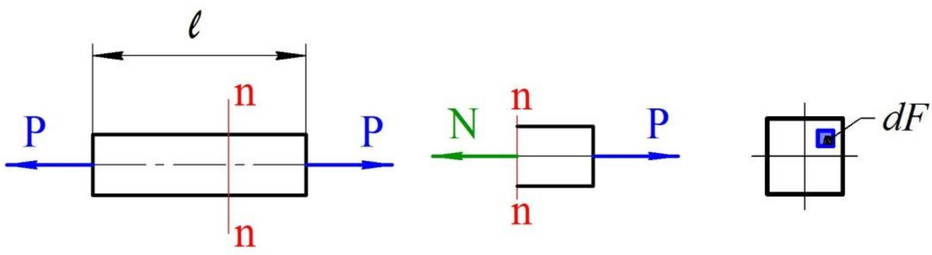


Figure 4.1 – Stretching the rod

To obtain the formula for normal stresses, follow the sequence given in Section 3.1.

Let us cut the rod with an arbitrary cross section $n - n$.

The static aspect of the problem is expressed by equation (3.4)

$$N = \int_F \sigma dF. \qquad (4.1)$$

From this equation, the stress σ cannot be determined, since the law of its distribution at the cross-sectional points is unknown.

In **the geometrical aspect** of the problem, we come to **the flat cross-section hypothesis: the cross-sections of the rod, which are flat before deformation, remain flat after it, moving gradually along the rod axis.**

Based on the flat cross-section hypothesis, it can be concluded that all fibres elongate by the same amount and their relative elongations are the same.

$$\varepsilon = \frac{\Delta\ell}{\ell} = const \tag{4.2}$$

- is **an analytical expression of the geometric aspect of the problem.**

The physical aspect of the problem is to determine the dependence of deformations on stresses. For elastic deformations, this relationship is linear, and is known as **Hooke's law (1.1)**

$$\varepsilon = \frac{\sigma}{E} \ \ or \ \sigma = E \cdot \varepsilon, \tag{4.3}$$

where E – the proportionality constant, also called **the longitudinal elastic modulus, first order modulus or Young's modulus.**

The modulus of elasticity – one of the physical constants of a material. It is expressed in units of stress, (Pa, MPa). Taking into account the constancy of the elastic modulus E for a homogeneous and isotropic material, as well as the constancy of ε, we find that

$$\sigma = E \cdot \varepsilon = const. \tag{4.4}$$

Substituting this into the original expression (4.1), we have

$$N = \int_F E \cdot \varepsilon dF = E\varepsilon \int_F dF = E\varepsilon F = \sigma F, \tag{4.5}$$

from which

$$\sigma = \frac{N}{F}. \tag{4.6}$$

The sign of the stress depends on the sign of the longitudinal force in the cross-section being studied. In the case of compression, the stress is considered negative.

Determination of rod deformations. From the previous expression (4.5), we can find the relative elongation

$$\varepsilon = \frac{N}{EF}. \tag{4.7}$$

Within a prismatic section of a rod of length ℓ, made of a homogeneous ($E = const$) material and with the same forces N, acting in its cross-sections, the elongations of each unit length are the same, so the absolute elongation

$$\Delta\ell = \varepsilon\ell = \frac{N\ell}{EF} \tag{4.8}$$

Formula (4.8) – is **Hooke's law for absolute elongation.**

The product EF in the denominator is called **the stiffness of the rod's cross-section in stretching and compression**, and has the dimension of force (N). The greater the stiffness of the rod, the less deformation it will experience for the same length. Stiffness characterises both the physical properties of the material and the geometric dimensions of the section. Sometimes it is convenient to use the concept of relative stiffness, which is equal to $\frac{EF}{\ell}$.

If the longitudinal force and cross-section are variable in the section under consideration, then for an element of infinitesimal length dz based on formula (4.8), we can write

$$\Delta(dz) = \frac{N(z)dx}{EF(z)}.$$

The total elongation of a segment of length ℓ is the sum of the elongations of all infinitesimal segments

$$\Delta\ell = \int_0^\ell \frac{N(z)dz}{EF(z)}. \tag{4.9}$$

Stretching and compression are accompanied by changes in the transverse dimensions of the rod (Figure 4.2).

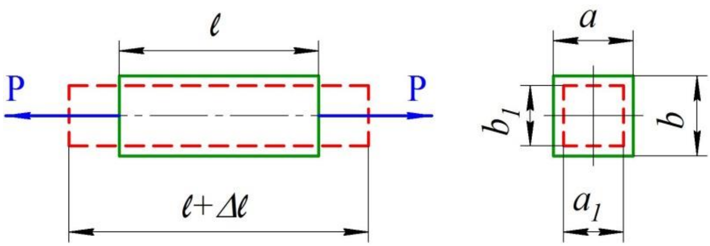


Figure 4.2 - Transverse deformation of a rod in stretching

By analogy with longitudinal deformation, the difference between the appropriate dimensions after and before it is called **absolute transverse deformation.**

$$\Delta a = a_1 - a; \;\; \Delta b = b_1 - b. \qquad (4.10)$$

Absolute transverse deformations are negative in stretching and positive in compression.

By dividing the absolute transverse deformation by the corresponding initial size, we obtain **the relative transverse deformation**

$$\varepsilon' = \frac{\Delta a}{a} = \frac{\Delta b}{b}. \qquad (4.11)$$

There is a constant ratio between the transverse ε and longitudinal ε' relative deformations in simple stretching and compression within the scope of Hooke's law. The absolute value of this ratio is called **the Poisson's ratio**

$$\boldsymbol{\mu} = \left|\frac{\boldsymbol{\varepsilon}'}{\boldsymbol{\varepsilon}}\right| \qquad (4.12)$$

– is **a dimensionless quantity.**

Taking into account that longitudinal and transverse deformations always have opposite signs, we have:

$$\varepsilon' = -\mu\varepsilon, \tag{4.13}$$

or, considering the preceding

$$\varepsilon' = -\mu\frac{\sigma}{E}\,. \tag{4.14}$$

The values of μ are in the range of 0 – 0,5 and are given together with E in the reference tables.

4.2 Condition of strength and stiffness. Permissible stresses and deformations

The main task of material strength is to ensure that the parts are reliably dimensioned so that they do not fracture.

The risk of beginning a fracture is not only characterised by internal forces and moments, but also by normal and tangential stresses and their combinations. Therefore, **the highest stress values required to ensure reliable operation of the part must be limited to certain permissible values, which are called permissible tensions.**

$[\sigma]_P$ – at a stretching;

$[\sigma]_{CT}$ – in compression;

$[\tau]$ – in case of shear.

In this way, dangerous cross-sections are found in which the stresses reach their highest modulus values and **the strength condition** is written down for these cross-sections:

$$\sigma_{max} = \left|\frac{N_{max}}{F}\right| \le [\sigma]. \tag{4.15}$$

The strength condition can be used to solve three types of problems:

1. Find the reliable dimensions of the rod from the known load (design calculation);
2. Check whether the part can withstand the specified load based on the known dimensions and material of the part (verification calculation);
3. Determine the permissible loads based on the known dimensions of the part, material and load pattern.

The stiffness condition must also be ensured:

$$\Delta\ell = \int_{\ell} \frac{N(z)dx}{EF(z)} \leq [\Delta\ell], \qquad (4.16)$$

where $\Delta\ell$ – change of length of the part;

$[\Delta\ell]$ – the permissible values of this change (permissible deformation).

Thus, in order to find a dangerous cross-section, it is necessary to build an epure of internal forces, i.e. the change in N_z along the length of the rod and determine the point with the maximum value of N_z indicating a dangerous cross-section.

4.3 Consideration of own weight in stretching and compression

In mechanical engineering, as a rule, the effect of own weight is not taken into account, because the parts are small, and the external load has a much greater impact on stress than own weight. However, when calculating mine hoist ropes, drill rods, bridge piers, building walls, and dams, the effect of its own weight should be taken into account.

Suppose that a straight rod of great length ℓ with a constant cross-section is fixed at its upper end and loaded with a force P ($Fig.$ 4.3). Let us determine how the longitudinal forces and stresses in the cross-sections of the rod change with and without taking into account its own weight.

Using the cross-sectional method, we build epures N_z, σ, $\Delta\ell$:

$$\sum P_{iz} = 0; \;\; N_z - P - Fz\gamma = 0;$$

from which

$$N_z = P + Fz\gamma, \qquad (4.17)$$

where γ – is the specific gravity of the material (for steel $78 \frac{кН}{м^3}$); F – is the cross-sectional area. At $= 0$ $N_z = P$; at $z = \ell$ $N_z = P + F\ell\gamma$.

The longitudinal force epure is shown in Fig. 4.3, b.

Normal stresses in the section

$$\sigma = \frac{N_z}{F} = \frac{P}{F} + \gamma z. \tag{4.18}$$

Normal stress reaches its maximum value in the upper section (at the rod fixation) at $z = \ell$:

$$\sigma_{max} = \frac{P}{F} + \gamma \ell, \tag{4.19}$$

where the first term is the stress due to the force P, and the second term is the stress due to its own weight. The normal stress epure is shown in Fig. 4.4, c).

Determination of the displacements of the rod sections. The minimum elongation of the rod will be at the fixation $\Delta\ell = 0$. In the cross-section of the rod at the greatest distance ξ from the free end (Fig. 4.4, a), we have $N_z = P + \gamma F \xi$. According to formula (4.9), $\Delta\ell = \int_0^\ell \frac{N_z dz}{EF}$ at $F = const$, we find:

$$\Delta\ell = \int_z^\ell \frac{N_z(\xi) d\xi}{EF} = \int_z^\ell \frac{(P + \gamma \mathrm{F} \xi) d\xi}{EF}\,. \tag{4.20}$$

The elongation $\Delta\ell$ of the rod is determined from (4.20), taking $z = 0$:

$$\Delta\ell = \frac{P\ell}{EF} + \frac{\gamma \ell^2}{2EF}. \tag{4.21}$$

The first term is the elongation of the rod due to the force P, and the second term is due to its **own weight**. The epure of cross-sectional displacements is shown in Fig. 4.4, d). Taking into account that the total weight of the rod is $Q = \gamma F \ell$ (mass of the cylinder), instead of formula (4.21) we have

$$\Delta\ell = \frac{P\ell}{EF} + \frac{Q\ell}{2EF}. \tag{4.22}$$

Therefore**, the absolute elongation of the rod** due to its own weight is the same as the elongation due to a concentrated force Q, equal to the weight of the rod and applied at the centre of gravity.

Strength condition for a dangerous cross-section in which σ is the largest

$$\sigma = \frac{P}{F} + \gamma\ell \leq [\sigma] \,. \tag{4.23}$$

from which

$$F = \frac{P}{[\sigma] - \gamma\ell}. \tag{4.24}$$

If there is no concentrated load, **the strength condition** is

$$\sigma = \gamma\ell \leq [\sigma]. \tag{4.25}$$

The length of the rod can be used to find the length at which the stress from its own weight alone reaches the permissible limit and the rod cannot carry a useful load.

$$\ell_{\text{гр}} = \frac{[\sigma]}{\gamma}. \tag{4.26}$$

Rules for building longitudinal force epures:

The longitudinal force epures are characterised by certain regularities, the knowledge of which allows us to assess the correctness of the built epures:

1) **epures *N* are always straightforward;**

2) in the area where there is no distributed load, the epure N – is a straight line parallel to the axis; and in the area under distributed load, it is a sloping line;

3) under the point of application of an external concentrated force, the epure must have a jump (step) in the magnitude of this force;

4) positive values are placed (in the selected scale) above the epure axis, negative values - below the axis.

4.4 Control questions

1. State Hooke's law; how is it expressed mathematically?
2. What characterises the modulus of elasticity of the first kind E?
3. What is the dimension of the modulus of elasticity?
4. Hooke's law for absolute elongations.
5. What is the stiffness of a rod in stretching and compression?
6. What is the Poisson's ratio?
7. Define the permissible stress.
8. Rules for building longitudinal force epures.
9. Conditions of strength and stiffness in stretching (compression).
10. Conditions of strength and stiffness in stretching (compression) taking into account its own weight.
11. Stresses and deformations in stretching (compression).
12. Stresses and deformations in stretching (compression) taking into account its own weight.

4. MECHANICAL CHARACTERISTICS OF MATERIALS

Basic characteristics of metals. When calculating the strength of machine details, it is necessary to know the mechanical characteristics of materials: strength, elasticity, which are characterises by the first-order modulus of elasticity E and Poisson's ratio μ, hardness (the ability of a given body to resist the penetration of another body by elastic or plastic deformation or by the destruction of a part of the surface of the body), and ductility (which characterises the ability of a material to give residual deformation). Therefore, materials are tested for tension, compression, shear, torsion, bending and stiffness. Detailed descriptions of all types of mechanical testing, as well as the machines and instruments used, are provided in special courses and laboratory manuals on the strength of materials.

5.1 Stretching tests of materials. Types of specimens

Most mechanical properties can be determined directly or indirectly by stretching tests performed on metals in accordance with GOST 1497-84.

Stretching tests. Specimens (Fig. 5.1) with a working part of length ℓ, and heads designed to be fixed in the grips of the testing machine are tested for stretching.

The elongation is determined at the initial calculated length ℓ_0 of the specimen. The most commonly used specimens are cylindrical. The ratio of the calculated length to the initial diameter ${}^{\ell_0}/_{d_0}$ for "long" specimens is 10, for "short" specimens ${}^{\ell_0}/_{d_0} = 5$.

Rectangular specimens are also used. Their calculated length ℓ_0 is related to the initial cross-sectional plane F_0 by the same dependencies as for cylindrical specimens: $\ell_0 = 11{,}3\sqrt{F_0}$ – for long specimens and $\ell_0 = 5{,}65\sqrt{F_0}$ – for short specimens.

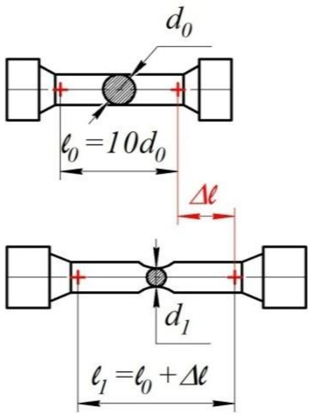


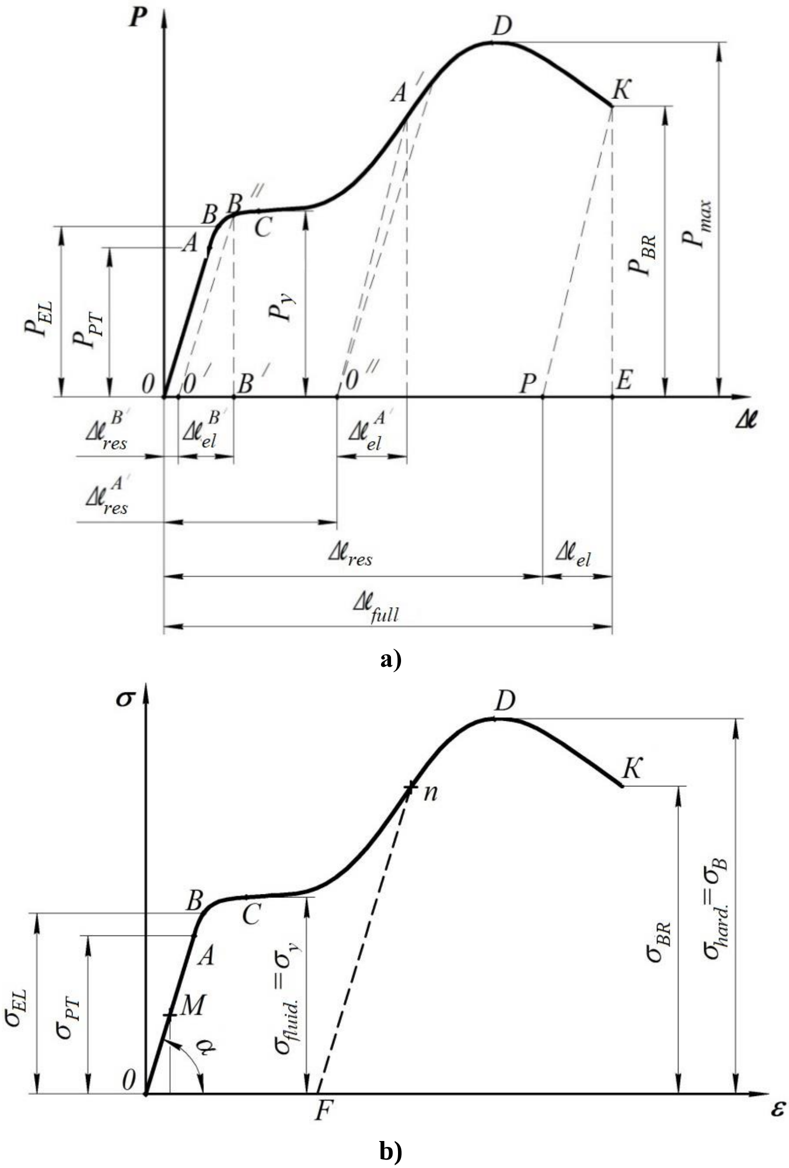
Figure 5.1 - Specimen for stretching test:
a) before the test; b) after the test

Cylindrical specimens with a diameter of $d_0 = 10$ мм; are used as the main specimens; the working length $\ell_0 = 100$ мм.

Welded joints are tested in accordance with GOST 6996-66.

5.2 Stretching diagrams ($P - \Delta\ell, \sigma - \varepsilon$)

For stretching tests, tensile machines (IMASH -22-71 P-10, P-20, P100), are used, which allow to determine the forces and corresponding deformations of the samples during the test. Based on these data, a **tensile diagram** is drawn, in which the forces are placed on the ordinate axis and the corresponding elongations $\Delta\ell$ on the abscissa axis. The nature of the tensile diagram depends on the properties of the material. A typical view of such a diagram for low-carbon steel is shown in Fig. 5.2(a).



a) Dependence of elongation on force;
b) Dependence of relative deformation on stress.
Figure 5.2 - Stretching diagrams for low-carbon steel

The form of the stretching diagram in the $P - \Delta\ell$ coordinates depends not only on the material properties, but also on the dimensions of the test specimen.

To obtain a diagram that characterises only the mechanical properties of the material, the original stretching diagram is rebuilt in the $\sigma - \varepsilon$ coordinates. The ordinates of this diagram are obtained by dividing the stretching force by the initial cross-sectional area of the sample ($\sigma = P/F_0$), and the abscissae by dividing the absolute elongation of the sample by its initial length ($\varepsilon = \Delta\ell/\ell_0$). The diagram in the $\sigma - \varepsilon$ coordinates, corresponding to the original diagram (Fig. 5.2, a) is shown in Fig. 5.2, b) points O, A, B, C, D, F of the original diagram correspond to points O, a, b, c, d, e, f of the $\sigma - \varepsilon$ diagram.

The diagrams have a number of characteristic points. At OA there is a direct proportional dependence between the elongation of the specimen and the force. At this stage, the stretching occurs in accordance with Hooke's law.

Let's denote the force at which the law of proportionality is no longer valid as P_{pt}. This value of force corresponds to point A on the diagram. The stress caused by the force P_{pt} is called the limit of proportionality and is calculated by the formula:

$$\sigma_{pt} = {}^{P_{pt}}\!/_{F_0} \ . \tag{5.1}$$

The limit of proportionality is the stress after which Hooke's law is violated. The deformation that is achieved is called elastic. It disappears completely after unloading.

Until the force P reaches a certain value, the deformation caused by it will disappear during unloading. The process of unloading is represented in the diagram by the same line as loading.

Let's denote by P_{el} the largest value of force at which the sample does not yet produce residual deformation when unloaded. In the diagram, this value corresponds to point B, and the elastic stage of the sample stretching is represented by the area OB.

The largest stress up to which no residual deformation is detected during unloading is called **the elastic limit.** This stress is caused by the force $P_{пр}$ and is determined by the formula

$$\sigma_{el} = {}^{P_{el}}/_{F_0}. \tag{5.2}$$

Points A and B, therefore the values of stress σ_{pt} and σ_{el} are close to each other, and the difference between them is usually neglected.

After point A when the specimen is further stretched, the stretching curve becomes curved and gradually rises to point C, where a transition to the horizontal section CD called **the yield area**, is observed.

At this stage of stress, the elongation of the specimen increases at a constant value of force, denoted by P_y. his process is called **material fluidity.** It is accompanied by residual (plastic) elongation that does not disappear after unloading.

Thus, **the yield strength σ_y is the lowest stress at which the sample deforms under a constant stretching force.** The yield strength is determined by the formula

$$\sigma_y = {}^{P_{\text{y}}}/_{F_0}. \tag{5.3}$$

After the yielding stage, the material again gains the ability to increase its strength to further deformation and can withstand a force that increases up to a certain limit. This phenomenon corresponds to the DE section (Fig. 5.2, a) of the stretching curve, which is called the **hardening section.** The point E corresponds to the greatest force P_{max}, that the sample can withstand.

The stress corresponding to the maximum force P_{max}, is called the temporary strength, or tensile strength, and is denoted by σ_{B}. It is determined by the formula

$$\sigma_{\text{B}} = {}^{P_{max}}/_{F_0}. \tag{5.4}$$

Up to this point, the elongations were distributed equally along the entire length ℓ_0 of the specimen. The cross-sectional areas of the specimen changed insignificantly and evenly along the entire length. Therefore, to calculate σ_{pt}, σ_{el}, σ_y and σ_{B} the initial values of the area F_0 were included in the formulas.

After reaching the force P_{max} the deformation of the specimen during tensile testing occurs over a short length. This sequentially leads to the formation of a local constriction in the form of a neck (Fig. 5.2, b), to a decrease in the force P, to an increasing deformation rate, and to the sample breaking.

Denoting the force at the moment of breaking by P_{br} we have:

$$\sigma_{br} = {}^{P_{br}}\!/_{F} \quad . \tag{5.5}$$

The tensile stress determined in this way is too conventional and cannot be used as a characteristic of the mechanical properties of steel. The conventionality lies in the fact that it is obtained by dividing P_{br} by the cross-sectional area of the specimen, which is much smaller than the initial one due to the formation of the neck.

The main elasticity and strength characteristics of materials used in the calculations are σ_{el} – elastic limit; σ_y – yield strength; σ_B – ultimate strength (temporary strength).

5.3 The concept of safety factor

Determining the value of **the permissible stress** is one of the most important tasks in strength calculations. As follows from the definition $[\sigma] = \sigma_B/n$ (n – **is the safety factor)**, the permissible stress is obtained as a part of the stress value σ_{dan}, which corresponds to the dangerous state of the material, σ_B being determined during mechanical testing. In order to avoid residual deformations, the yield strength σ_y is taken as the value of σ_B for plastic materials. In this case, the safety factor is called **the yield strength factor** n_y. Therefore, the formula becomes

$$[\sigma] = \frac{\sigma_y}{n_y}.$$

The choice of the safety factor depends on the state of plasticity of the material, the nature of the load application (static, dynamic, variable) and general factors:

– material heterogeneity (different mechanical properties in small samples and in details, hidden manufacturing defects, etc;)

– inaccuracy of setting external loads (direction of forces, small power factors are not taken into account);

– approximation of calculation schemes and certain approximation of calculation formulas (absolutely smooth surfaces and frictionless joints, concentrated weight force, imperial formulas);

–area of operation of structures (bridges, transport, building, aircraft, etc.);

These factors must be taken into account when selecting the safety factor n, which is sometimes called the basic factor.

For brittle materials (cast iron, high-carbon hardened steels, etc.), which do not have a zone of plastic deformation ("platform") in the stretching diagram, the tensile strength σ_{dan} is taken as the ultimate strength σ_B, and n is understood as the safety factor beyond the ultimate strength, so $[\sigma] = \sigma_\text{B}/n_\text{B}$.

5.4 Definition of material hardness

Sometimes an indirect method, such as hardness measurement, can be used to estimate the temporary strength.

The hardness of a material is its ability to resist mechanical penetration into its surface by another harder body called an indenter. The indenter, after being pressed in, leaves an imprint of a certain shape. The size of the indentation is used to determine the hardness of the test material.

Brinell method. A hardened steel ball of diameter D (Fig. 5.3) is pressed into the specimen (product) under the action of a load P, applied for a certain time (10 – 0 s). The minimum thickness of the test specimen should be at least ten times the depth of the imprint. After loading, the diameter of the imprint remaining on the sample is measured. The Brinell hardness number is expressed as the ratio of the load to the surface area of the spherical imprint F (mm^2)(DSTU ISO 6506-1:2007).

$$HB = \frac{2P}{\pi D(D - \sqrt{D^2 - d^2})}, \qquad (4.32)$$

where P – the applied load (kPa);
D – the diameter of the ball, mm;
d – diameter of the imprint, mm.

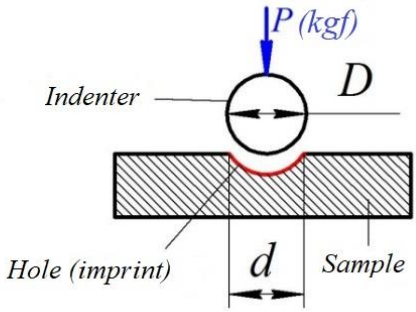


Figure 5.3 - Brinell's method.

Table 5.1 - Hardness values of various materials

Material	Hardness
Soft wood, pine	1,6 HBS 10/100
Solid wood	від 2,6 до 7,0 HBS 10/100
Aluminium	15 HB
Copper	35 HB
Duralumin	70 HB
Mild steel	120 HB
Stainless steel	250 HB
Glass	500 HB
Tool steel	650-700 HB

According to DSTU, the ball diameters are – 10, 5, 2,5 and 1 mm, and the load is – 187,5 kgf, 250 kgf, 500 kgf, 1000 kgf і 3000 kgf.

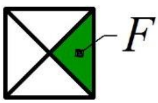
Rockwell method. According to the Rockwell method, the hardness is determined by pressing a hard alloy or a tip with a diamond cone with an angle of 120° at the apex or with a steel ball with a diameter of 1,85 mm.

The ball is used for testing mild steels (up to 220 HB) at a load of 981 H (100 kgf). (DSTU ISO 6508-1:2013). Hardness must be measured at least in three points. The average of the results of the second and third measurements is used for calculation.

The standards define 11 Rockwell hardness scales (A; B; C; D; E; F; G; H; K; N; T); these scales differ in the type of indenter, test load and constants in the formula for calculating the hardness based on the measurement results. The two most commonly used scales are B and C. Scale B uses a steel ball as an indenter. Scale C is used for harder materials (indenter is a diamond cone). To indicate the hardness determined by the Rockwell method, the symbol HR, is used, which is followed by a letter indicating the scale on which the test was performed (HRB, HRC etc.).

For the diamond cone test, the maximum penetration depth is 0.2 mm, and for the ball test, it is 0.26 mm.

Vickers method. The method is based on determining the hardness of an imprint left by a diamond indenter in the shape of a quadrangular pyramid, which is pressed into a surface under a load applied over a certain period of time. The hardness is calculated as the ratio of the force P applied to the tip to the area of the inclined surface of the imprint F; the unit of hardness is MPa.



The hardness determined by this method is denoted by HV (DSTU ISO 6507-1:2007).

5.5 The concept of concentration of stress

If the cross-sectional area varies smoothly along the length of the detail or specimen, the longitudinal stretching stress will be a function of the cross-sectional area.

Nominal stresses are those determined on the basis of the assumption that there is no stress concentration.

Stress concentration is an increase in stress in a solid body in places of shape change or discontinuity in the material.

Consider sudden changes in the cross-sectional area that lead to an uneven distribution of stresses and cause **stress concentration** (Figure 5.4).

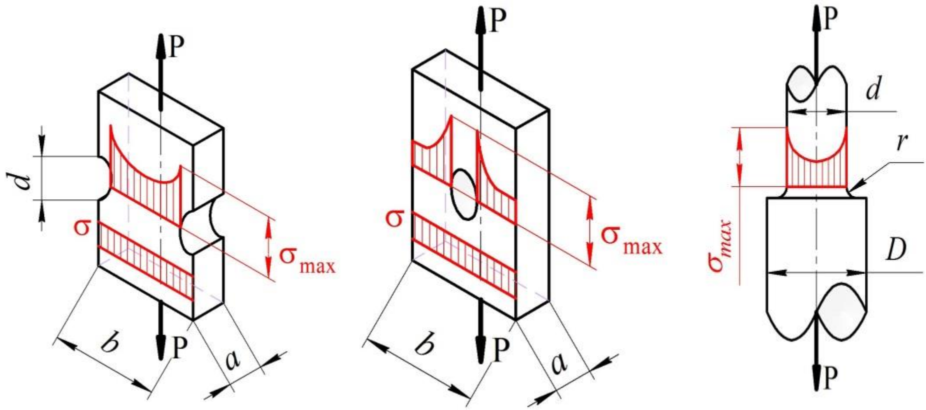


Figure 5.4 - Distribution of normal stress in the cross-section depending on the shape of the concentrator

A quantitative characteristic of stress concentration is the **theoretical concentration coefficient** $\alpha = \frac{\sigma_{max}}{\sigma_{min}}$.

The coefficient α depends on both the shape of the concentrator and the material properties. Therefore, a distinction is made between the **theoretical concentration coefficient** $\boldsymbol{\alpha}$, which is determined by the methods of elasticity theory and depends on the type of concentrator, and the **actual (effective) concentration coefficient** $\boldsymbol{k}$, which takes into account the combined effect of geometry and material properties.

The effective concentration coefficient is affected by the nature of the load.

It is the case with static loading:

$$k = \frac{P_I}{P_{II}},$$

where P_I – destructive loading of the specimen without a stress concentrator;

P_{II} – destructive loading of a specimen with a stress concentrator.

Other factors that cause stress **concentration (stress concentrators)** include holes, hollows, cracks, grooves, notches, corners, protrusions, sharp edges, threads, as well as surface irregularities and defects (risks, scratches, marks, welds, etc.). Stresses in the area of stress concentrators are determined by methods of elasticity theory, numerical methods (finite element method) or experimentally by strain gauge methods (methods: photoelasticity, holographic interferometry, using strain gauges).

5.6 Control questions

1. Dimensions of long and short specimens.
2. Stretching diagrams, basic principles of construction.
3. How to determine the stress within the limits of proportionality, elasticity, yield, strength?
4. How can the value of the elastic modulus E be determined from a stretching diagram?
5. Why does the stretching diagram show the stress, at which the specimen fractures, below the tensile strength?
6. What stress is used as the basis for selecting the allowable stress for a brittle material?
7. What stress is used to determine the allowable stress for a yielding material?
8. In what cases is stress concentration neglected when selecting a permissible stress?
9. Safety factor. Safety factor for contact stresses.
10. Basic methods for determining the hardness of materials.
11. The concept of stresses concentration, types of concentrators.
12. Theoretical and actual concentration coefficients. How to determine the Brinell, Rockwell and Vickers hardness number?

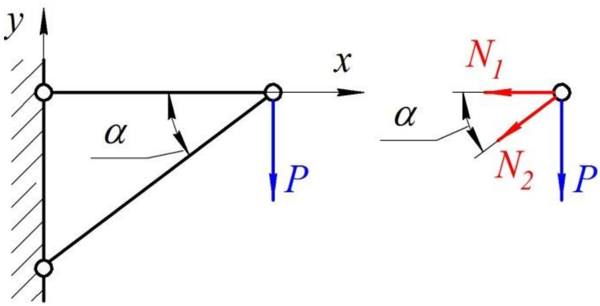
6. STATICALLY INDETERMINATE CONSTRUCTIONS

6.1 Statically indeterminate problems

Statically indeterminate structures are those whose forces in the elements cannot be determined from the equations of statics alone. In addition to the equations of statics, equations containing deformations of structural elements must be used to calculate such constructions.

The difference between the number of unknowns and the number of statistical equations is called the degree of static uncertainty of the construction.

For example:



a) b)

Figure 6.1 - Example of a statically defined construction element

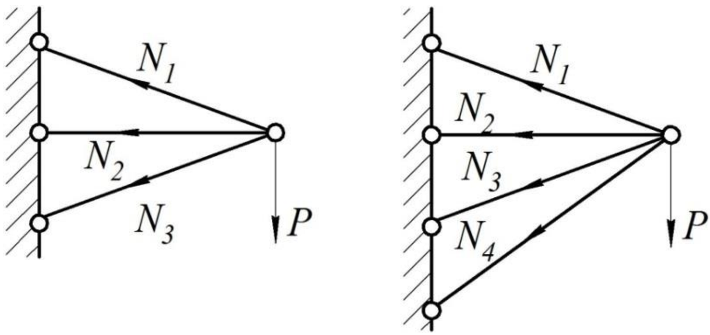
The construction is statistically determined, since two statistical equations can be written and two unknowns can be determined (N_1, N_2).

$$\sum_{1}^{n} P_{ix} = -N_1 - N_2 cos\alpha = 0; \sum_{1}^{n} P_{iy} = -P - N_2 sin\alpha = 0;$$

$$N_2 = -\frac{P}{sin\alpha}; \;\; -N_1 + \frac{P}{sin\alpha} cos\alpha = 0; \;\; N_1 = Pctg\alpha.$$

In the case of the load shown in Fig. 6.2, a), the system is statically indeterminate once, since we can write two statics equations $\sum P_{iX} = 0$ and

$\sum P_{iy} = 0$ and have three unknowns $N_1;\ N_2;\ N_3$ and the structure shown in Fig. 6.2, b) is twice statically indeterminate (we have four unknowns$N_1;\ N_2;\ N_3; N_4$.



a) **b)**

Figure 6.2 - Statically indeterminate systems

6.2 Solving statically indeterminate problems

To solve the problems of statically indeterminate systems that work in stretching or compression, the equations obtained as a result of considering the static, geometric and physical aspects of the problem are considered together. They should be considered in the following sequence:

– the static aspect of the problem. We draw up equations of equilibrium for individual elements of the construction with unknown forces.

– the geometric aspect of the problem. We consider the system in a deformed state, which makes it possible to establish relationships between deformations or displacements of individual construction elements. The equations obtained are called the deformation compatibility (continuity) equations.

– the physical aspect of the problem. Based on Hooke's law, we express the displacements or deformations of construction elements through unknown forces acting on them. In the case of temperature loads, they must be taken into account in the physical aspect.

– synthesis. Solving static, geometric and physical equations together, we find the unknown forces.

6.3 Features of statically indeterminate systems

Based on the examples discussed, the following features of statically indeterminate systems can be noted:

– the distribution of forces between elements depends on the stiffness of these elements. If you increase the stiffness of one of them, it will absorb more force. By changing the ratio of the stiffness of construction elements, you can change the distribution of forces in them in any way you want.

– in statically indeterminate constructions, when the temperature of their elements changes compared to the temperature at which the construction was made, forces and stresses occur.

– in elements of statically indeterminate constructions, forces and stresses may exist. When there is no external load, they are called initial (or installation). Forces arise either due to manufacturing inaccuracies or for an explicit purpose (e.g. tightening bolts, press fit, etc.).

– in statically indeterminate constructions, it is generally not possible to achieve stresses equal to the permissible stresses in all elements at the same time. This should be borne in mind when designing such constructions.

6.4 Control questions

1. What structures are called statically indeterminate?
2. The degree of static indeterminacy.
3. Features of statically indeterminate systems.
4. Formulate the static, geometric and physical aspects of the problem.
5. Show examples of once and twice statically indeterminate problems.
6. How to determine the elongation of a rod Δl_t when the temperature changes from t_1 to t_2 degrees?
7. How are the stresses in the elements of statically indeterminate constructions determined?

7. FUNDAMENTALS OF STRESS AND STRAIN THEORY

Stresses arise as a result of the interaction of body particles under load. External forces try to change the relative position of the particles, and stresses prevent this displacement.

7.1 Stress at a point

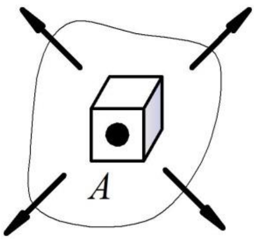


Figure 7.1 – Stressed state at point A

When studying the stress state of a body at point A, a volume element is usually distinguished in the form of an infinitesimal parallelepiped (Fig. 7.1), which is shown in an enlarged scale in Fig. 7.2, where the beginning of the coordinates is aligned with point A.

Due to the smallness of the selected element, it can be assumed that the stress on its edges is uniformly distributed. The full stress on the edges is represented as normal and tangential components (projections of the full stress on the coordinate axes (Fig. 7.2).

The normal stresses σ_x, σ_y, σ_z (directed along the normal to the section) shown in the figure are **positive because they stretch the parallelepiped. They cause linear deformation.**

The tangential stresses are denoted by τ with two indices: **the first** corresponds to the direction of the normal to the plane, and **the second** to the direction of the stress itself. In other words: **the first** index shows perpendicular to which axis the stress is directed, **the second** one parallel to which axis the stress is directed (Fig. 7.2, c).

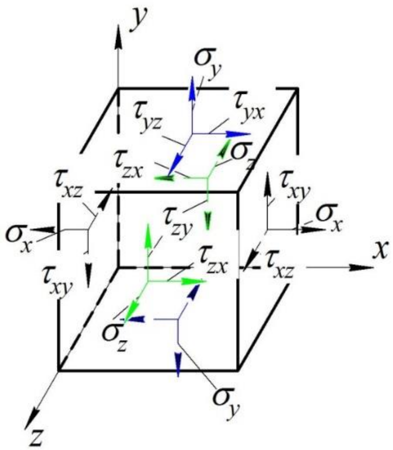


Figure 7.2 - Total stress on the element edges

Thus, nine components of stresses act on the edges of an elementary parallelepiped. This set can be written in the form of a square matrix or **stress tensor** T_σ.

$$T_\sigma = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zy} & \tau_{zy} & \sigma_z \end{vmatrix},$$

where each line shows the stress components corresponding to the areas perpendicular to the x, y, z axes. It will be shown below that if the stress tensor is known, the stress at any site of the element can be determined.

7.2 The law of parity of tangential stresses. Principal sites and principal stresses

Not all nine components of stress are independent. This can be easily verified by drawing up the equilibrium conditions for the element with respect to its rotation (Figure 7.2). To do this, we equate to zero the sums of the moments of all forces about the x, y, z axes:

$$\sum M_{kx} = 0; \quad \sum M_{ky} = 0; \quad \sum M_{kz} = 0.$$

Write the sum of moments about the O_z axis.

Forces parallel to this axis and crossing it are not included in the equation. The moments of the forces σ_y on the two edges perpendicular to the z-axis, are balanced, as are the moments of the forces σ_y on the upper and lower edges of the element.

Thus, we have $\tau_{xy} dydzdx - \tau_{yx} dxdzdy = 0$

Therefore, we find $\tau_{xy} = \tau_{yx}$.

Analogously, from the other two equations we find: $\tau_{yz} = \tau_{zy}$; $\tau_{xz} = \tau_{zx}$.

So we have the equations:

$$\tau_{xy=}\tau_{yx}; \quad \tau_{yz=}\tau_{zy}; \quad \tau_{xz=}\tau_{zx}, \tag{7.1}$$

that express **the law of parity of tangential stresses: the tangential stresses on any two but mutually perpendicular sites, which are directed perpendicular to the line of intersection of the sites, are the same in modulus.** At the same time, **they try to rotate the element in different directions.**

The principal stresses are denoted by $\sigma_1, \sigma_2, \sigma_3$. The indexes should be set so that the inequality

$$\sigma_1 > \sigma_2 > \sigma_3. \tag{7.2}$$

This inequality is written down taking into account the signs of the stresses.

Depending on the number of principal stresses acting, **three types of stress state** are distinguished.

1 – **uniaxial or linear.** A stress state in which there is one principal stress. The other two are zero.

2 – **biaxial or flat.** A state in which two principal stresses are not zero, but one is zero**.**

3 – **triaxial or volumetric stress state.** A state in which all three principal stresses are not zero.

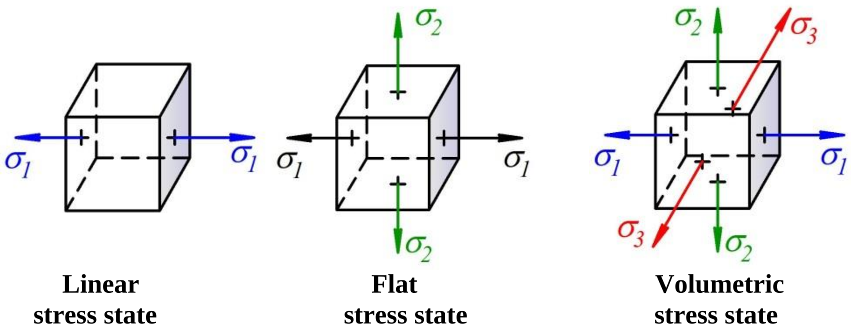


Figure 7.3 - Types of stress state

In addition, the stress state can be homogeneous and inhomogeneous. In a homogeneous stress state, the stress is the same at every point in any section and in all parallel sections. In the inhomogeneous state, the element should be considered infinitesimal. In this case, the assumption of equality of stress distribution on its edges is fulfilled to the nearest second-order small.

7.3 Linear stress state

Let us look at a prismatic rod (Fig. 7.4) subjected to simple stress by a force P. In the cross-sections of the rod, sufficiently distant from the points of application of external concentrated forces, the stresses are distributed evenly.

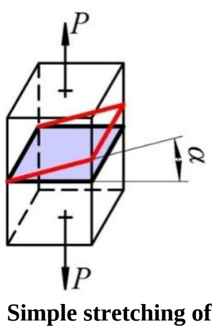


Figure 7.4 - Simple stretching of a rod

In cross-sections perpendicular to the force P, the normal stress is:

$$\sigma_x = \frac{N}{F_0} = \frac{P}{F_0}.$$

The tangential stress here is zero. Therefore, these sections are the principal sites.

Now let's look at the stress on a non-principal site rotated counterclockwise by an angle α (Figure 7.5). We will call this site the -site, and the stress on it:

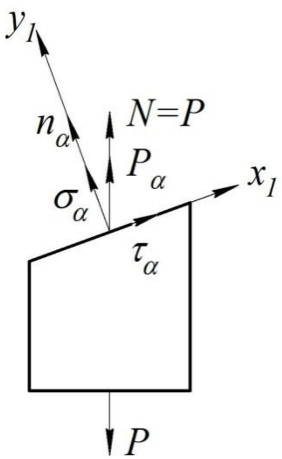


Figure 7.5 - Stress on a tilted platform

$P_\alpha -$ the full stress operating on this site;

$\sigma_\alpha -$ normal stress;

$\tau_\alpha -$ tangential stress;

$F_0 -$ area of the principal site;

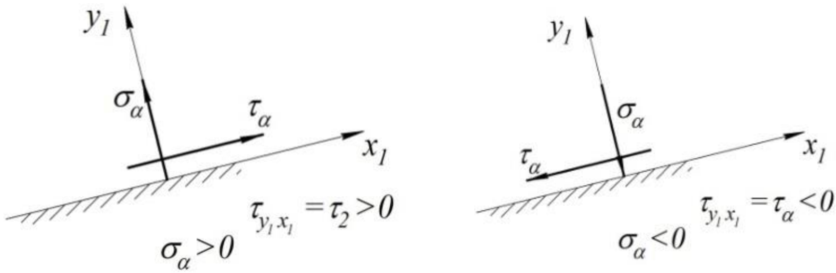
$F_\alpha -$ the area of the tilted platform.

The axial force in the cross-section $N = P$ is equivalent to the total stress P_α, acting on the site F_α, the area of which is

$$F_\alpha = \frac{F_0}{\cos\alpha}.$$

So,, $P_\alpha \cdot F_\alpha = N$. Whence

$$P_\alpha = \frac{N}{F_\alpha} = \frac{N}{F_0} cos\alpha = \sigma_0 cos\alpha. \qquad (7.3)$$



a) b)

Figure 7.6 - Stress on a site rotated by an angle α

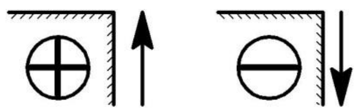
By projecting P_α onto the normal n_α and onto the cross-sectional area F_α, we get the formulas for determining the normal and tangential stresses acting on the tilted platform (Fig. 7.6, a):

$$\sigma_\alpha = P_\alpha cos\alpha = \sigma_0 cos^2\alpha; \tag{7.4}$$

$$\tau_\alpha = P_\alpha sin\alpha = \sigma_0 sin\alpha \cdot cos\alpha = \frac{\sigma_0}{2} sin2\alpha, \tag{7.5}$$

where $2sin\alpha \cdot cos\alpha = sin2\alpha$.

The rule for determining the signs of stress in relation to the tilted axes x_1, y_1: stress is considered positive if it tries to rotate the element counterclockwise on the right edge.



As you can see from the formulas, when the site is rotated, the stress changes in the following way (taking into account that $cos^2 45° = \frac{1}{2}$, $\sin(2 \cdot 45°) = \sin 90° = 1$):

α	σ_α	τ_α
0°	σ_0	0
45°	$\frac{\sigma_0}{2}$	$\frac{\sigma_0}{2}$
90°	0	0

So:

– in simple stretching

$$\sigma_1 = \sigma_0 = \frac{N}{F_0}, \qquad \sigma_2 = \sigma_3 = 0;$$

– in compression

$$\sigma_1 = \sigma_2 = 0, \quad \sigma_3 = -\sigma_0.$$

The tangential stress reaches its highest value at $\alpha = \pm 45°$, and

$$\tau_{max} = \frac{\sigma_1}{2}.$$

7.4 Plane stress state

Studying the stress state of structural elements, we often have to deal with the plane (biaxial) stress state. It occurs in torsion, bending and complex strength. All the definitions and rules introduced in the previous subsection remain valid for the plane stress state.

Let's look at a rectangular parallelepiped (Fig. 7.7), whose edges are the principal sites. The principal stresses act on them σ_1, σ_2, which stretch, and the third principal stress $\sigma_3 = 0$ (the principal direction corresponding to σ_3, is perpendicular to the drawing area).

Let's choose a cross-section whose normal makes an angle α_1 with the direction I (Fig. 7.7). With direction II the same normal makes an angle α_2. In the same cross-section, there are normal σ_α and tangential τ_α stresses that depend on σ_1, and σ_2.

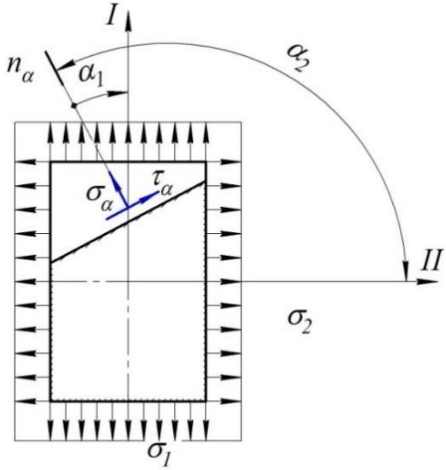


Figure 7.7 - Plane stress state

Their magnitude is obtained by considering the effects of σ_1, and σ_2 separately and adding the results (using the principle of superposition). The effect of the normal stress caused by σ_1, is expressed by formula (7.4) and is equal to $\sigma_1 cos^2\alpha_1$; the second part of the stress σ_α, which is caused by σ_2, is equal to $\sigma_2 cos^2\alpha_2$, де $\alpha_2 = 90° + \alpha_1$. Total normal stress:

$$\sigma_\alpha = \sigma_1 cos^2\alpha_1 + \sigma_2 cos^2\alpha_2 = \sigma_1 cos^2\alpha_1 + \sigma_2 cos^2(\alpha_1 + 90°)$$

or

$$\sigma_\alpha = \sigma_1 cos^2\alpha_1 + \sigma_2 sin^2\alpha_1 \, . \qquad (7.6)$$

Using the same considerations, the values of tangential stresses are determined by formula (7.5) for the selected site:

$$\tau_\alpha = \frac{1}{2}[\sigma_1 \sin 2\alpha_1 + \sigma_2 \sin 2\alpha_2] =$$
$$= \frac{1}{2}[\sigma_1 \sin 2\alpha_1 + \sigma_2 \sin 2(\alpha_1 + 90°)]$$

or

$$\tau_\alpha = \frac{\sigma_1 - \sigma_2}{2} 2 sin\alpha_1. \qquad (7.7)$$

In the following formulas for σ_α and τ_α the angle α_1 will be denoted by α and **will always be measured from the direction of the largest principal stress in the counterclockwise direction.**

Using formulas (7.6) and (7.7) for the stress at the $a - a$ site (Fig. 7.8), it is easy to find the stress at the $b - b$ site, which is perpendicular to it and has a normal n_β, that makes an angle $\beta = \alpha + 90°$ with the direction of the largest principal stress:

$$\sigma_\beta = \sigma_1 cos^2\beta + \sigma_2 sin^2\beta =$$
$$= \sigma_1 cos^2(\alpha + 90°) + \sigma_2 sin^2(\alpha + 90°);$$

$$\sigma_\beta = \sigma_1 sin^2\alpha + \sigma_2 \, cos^2\alpha; \qquad (7.6')$$

$$\tau_\beta = -\frac{\sigma_1 - \sigma_2}{2} sin^2\beta = \frac{\sigma_1 - \sigma_2}{2}\sin(2\alpha + 180°),$$

$$\tau_\beta = -\frac{\sigma_1 - \sigma_2}{2} sin^2\alpha. \qquad (7.7')$$

Adding the formulas (7.6) and (7.6′) , we get

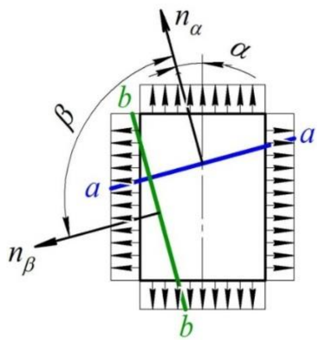


Figure 7.8 - Plane stress state (turn to 90°)

$$\sigma_\alpha + \sigma_\beta = \sigma_1 + \sigma_2 = const. \qquad (7.8)$$

The property of normal stress (invariance): **the sum of normal stresses on two mutually perpendicular sites is invariant with respect to the inclination of these sites.**

For tangential stress, comparing formulas (7.7) and (7.7′), we obtain:

$$\tau_\beta = -\tau_\alpha. \qquad (7.9)$$

Thus, **the tangential stresses on two mutually perpendicular platforms are equal in magnitude and opposite in sign (the law of parity of tangential stresses).**

To find the largest value of the normal stress, it is necessary to study expression (7.6) to the maximum. Let's equate the derivative of α with σ_α and get:

$$\frac{d\sigma_\alpha}{d\alpha} = -2\sigma_1 cos\alpha + 2\sigma_2 \sin\alpha\, cos\alpha$$

or

$$\frac{d\sigma_\alpha}{d\alpha} = -(\sigma_1 - \sigma_2)sin\alpha = 0. \qquad (7.10)$$

Comparing (7.10) and (7.7), we see that the condition of maximum for σ_α coincides with the condition of zero tangential stress at the corresponding sites.

Consequently, σ_α will have the highest value either at $\alpha = 0°$, or at $\alpha = 90°$. Since $\sigma_1 > \sigma_2$, then

$$\sigma_{\alpha\, max} = \sigma_1 \text{ (when } \alpha = 0°),$$
$$\sigma_{\alpha\, min} = \sigma_2 \text{ (when } \alpha = 90°),$$

i.e., the largest and smallest normal stresses at a given point are the principal stresses σ_1 and σ_2, which act along mutually perpendicular sites free of tangential stresses.

The largest value of tangential stresses, as can be seen from formula (7.7), will be

$$\tau_{\alpha\, max} = \frac{\sigma_1 + \sigma_2}{2} \text{(with } sin2\,\alpha = 1,\ \ \alpha = 45°). \quad (7.11)$$

The sites parallel to σ_2, will have the highest tangential stress:

$$\tau_{max} = \frac{\sigma_1}{2}. \qquad (7.11')$$

7.5 Direct problem in plane stress state. The circle of stresses (Mohr’s circle)

In the theory of stressed state, two main problems can be distinguished: direct and inverse.

Direct problem. At a point, **the positions of the principal pads and the principal stresses corresponding to them are known:** it is necessary to find the normal and tangential stresses acting on the pads that are inclined at a given angle α to the principal pads.

The inverse problem. At a point, **the normal and tangential stresses** acting in two mutually perpendicular sites passing through the point **are known, and the principal sites and principal stresses are to be found.** The analytical solution of the direct problem is given by formulas (7.4) – (7.7′).

The dependence of stresses on the angle of inclination of the site on which they act has a simple geometric interpretation in the form of a diagram called **Mohr's stress circle**. Carl Kuhlmann was the first to create a graphical representation of the longitudinal and transverse stresses of a horizontal beam in bending.

Mohr's contribution is to use this approach for flat and volumetric stress states and to define a strength criteria based on the stress circle diagram.

Let's mark it:

$$\alpha = \sigma_2 + \frac{\sigma_1 - \sigma_2}{2}; \quad R = \frac{\sigma_1 - \sigma_2}{2}. \qquad (7.12)$$

Then the stresses on inclined sites can be represented as:

$$\sigma_\alpha = \alpha + R\cos 2\alpha; \quad \tau_\alpha = R\sin 2\alpha.$$

These equations represent a circle in parametric form. They are equivalent to the equation:

$$(\sigma_\alpha - \alpha)^2 + \tau_\alpha^2 = R^2.$$

This method makes it possible to determine tangential and normal stresses at any point in the structure.

Let's analyse the stress state using a simple graphic construction. Let's take a rectangular coordinate system σ-τ, i.e., the abscissa axis is where we put the values of principal σ_1, σ_2, and normal σ_α, σ_β stresses, and the ordinate axis is where we put the values of τ_α, τ_β (Fig. 7.9). It is convenient to direct the σ axis parallel to the largest principal stress σ. The procedure for graphically solving the problem will be shown on a specific example of the stress state shown in Fig. 7.9.

Having chosen a new scale for the stresses, we draw segments along the σ axis : $OA = \sigma_1; OB = \sigma_2$. On the segment *AB* as a diameter, we draw a circle with the centre at point *C*. Thus, we have constructed **a stress circle or Mohr's circle. Radius of Mohr's circle:**

$$R = AC = BC = CD = \frac{AB}{2} = \frac{OA - OB}{2} = \frac{\sigma_1 - \sigma_2}{2}. \quad (7.13)$$

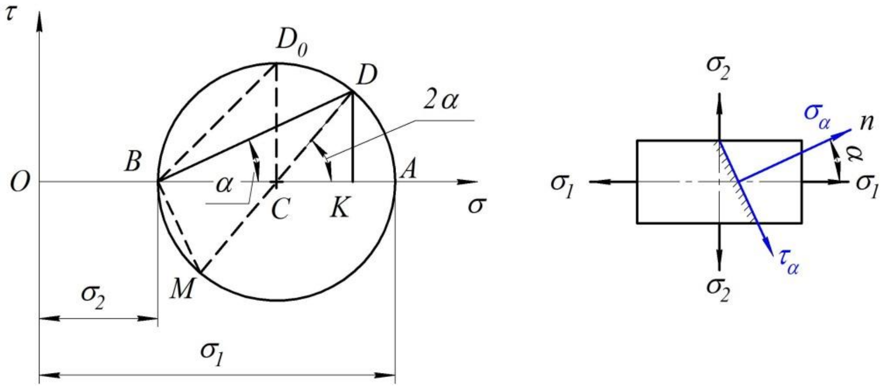
The point *D* of Mohr's circle will correspond to the selected site; its coordinates *OK* and *DK* are equal to σ_α and τ_α respectively. Indeed, from the triangle *KCD*, we have:

$$KD = Rsin2\alpha = \frac{\sigma_1 - \sigma_2}{2} sin2\alpha = \tau_\alpha;$$

$$OK = OB + BC + KC = \sigma_2 + \frac{\sigma_1 - \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} cos^2\alpha =$$

$$= \sigma_2 + \frac{\sigma_1 - \sigma_2}{2}(1 + cos^2\alpha) = \sigma_2 + \frac{\sigma_1 - \sigma_2}{2} cos^2\alpha =$$

$$= \sigma_2 + \sigma_1 cos^2\alpha - \sigma_2 cos^2\alpha = \sigma_1 cos^2\alpha + \sigma_2 sin^2\alpha = \sigma_\alpha.$$



а) **б)**

Figure 7.9 – Mohr's Circle

Having determined $\sigma_\alpha, \tau_\alpha$ by constructing a Mohr’s circle, we depict them in the figure of the selected element, taking into account the signs of these stresses (Fig. 7.9). Let's combine the line of action of the largest principal stress σ_1 with the σ axis on the circle; then the *BD* line is inclined to the axis at an angle of αwill be parallel to the normal to the site under consideration, and therefore parallel to σ_α; the *BM* line will be parallel τ_α.

As can be seen from Fig. 7.10, the largest value of tangential stresses is equal to CD_0, i.e., the radius of the circle:

$$max\,\tau_\alpha = \frac{\sigma_1 - \sigma_2}{2}; \qquad (7.13)$$

respectively, angle $2\alpha = 90°$ and angle $\alpha = 45°$. In the circle, the value of $\tau_{\alpha\,max}$ is represented by the ordinate CD_0, the abscissa for which is

$$OC = \frac{\sigma_1 - \sigma_2}{2},$$

on the same site, where $\tau_\alpha = \tau_{max}$ the normal stress is the average.

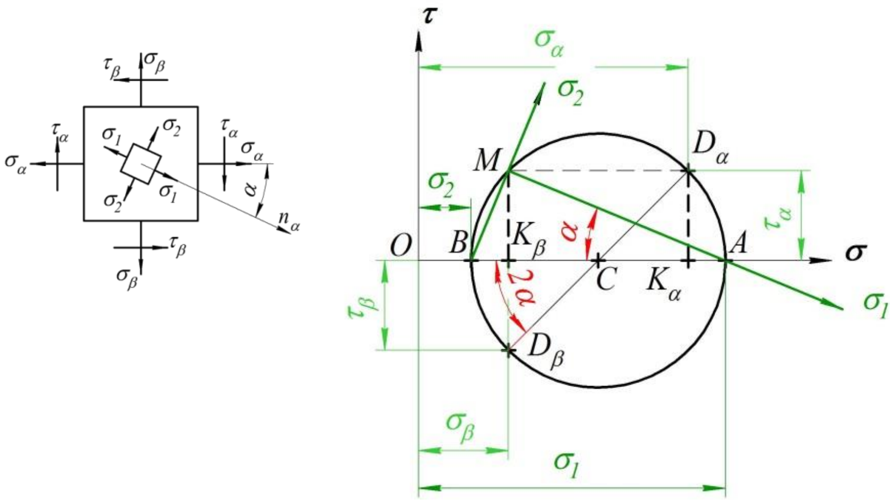
7.6 Inverse problem in the plane stress state

Sometimes it is necessary to solve the inverse problem of the one discussed in the previous subsection, i.e., from the known stresses σ_α, τ_α, σ_β, τ_β it is necessary to find the principal stresses.

Let's solve this problem using the stress circle. Let's assume that the stresses σ_α , τ_α , σ_β , τ_β on the mutually perpendicular areas of the selected element are known (Fig. 7.10, a) and $\sigma_\alpha > \sigma_\beta$, a $\tau_\alpha > 0$. In the rectangular coordinate system $\sigma \sim \tau$ select the point D_α with coordinates σ_α , τ_α and the point D_β with coordinates σ_β , τ_β (Fig. 7.10, b).

As noted in the direct problem, the points D_α and D_β lie at the ends of the same diameter of the Mohr's circle. Next, we find the centre of the circle - point C and the radius CD_α = CD_β. We draw a circle with this radius. The abscissas of the points of intersection of the circle with the $\sigma - axis$ – segments OA and OB – give the values of the principal stresses σ_1 and σ_2.

To determine the position of the main sites, let's find the pole and use its properties.



a) b)
Figure 7.10 - The inverse problem

For this purpose, from point D_α draw a line parallel to the line of action σ_α The point M where this line intersects the circle will be the pole. Connect the pole M with points A and B, and we get the direction of the principal stresses σ_1 and σ_2.

The principal sites are perpendicular to the found directions of principal stresses. In Fig. 7.10(a), inside the original element, we remove the element bounded by the principal sites and show the principal stresses σ_1 and σ_2.

Using Mohr's circle, we determine the analytical expressions of the principal stresses σ_1 and σ_2.

$$\sigma_1 = OA = OC + CA;$$
$$\sigma_2 = OB = OC - CB. \tag{7.14}$$

So,

$$OC = \frac{\sigma_\alpha + \sigma_\beta}{2}; \tag{7.15}$$

$$AC = CB = CD_\alpha = \sqrt{CK_\alpha^2 + D_\alpha K_\alpha^2} = \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_\alpha^2}\,. \quad (7.16)$$

Substituting (7.16) and (7.15) into (7.14), we get:

$$\sigma_1 = \left(\frac{\sigma_\alpha + \sigma_\beta}{2}\right) + \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_\alpha^2} \;;$$

$$\sigma_1 = \left(\frac{\sigma_\alpha + \sigma_\beta}{2}\right) - \sqrt{\left(\frac{\sigma_\alpha - \sigma_\beta}{2}\right)^2 + \tau_\alpha^2} \;; \qquad (7.17)$$

or

$$\sigma_1 = \frac{1}{2}\left[\sigma_\alpha + \sigma_\beta + \sqrt{(\sigma_\alpha - \sigma_\beta)^2 + 4\tau_\alpha^2}\right];$$

$$\sigma_2 = \frac{1}{2}\left[\sigma_\alpha + \sigma_\beta - \sqrt{(\sigma_\alpha - \sigma_\beta)^2 + 4\tau_\alpha^2}\right]. \qquad (7.17)$$

Taking into account the accepted rule of signs, we can find the formula for the tangent of the angle of inclination of the principal stress relative to the σ axis. From Fig. 7.12, b), it can be seen that

$$tg\alpha = -\frac{MK_\beta}{AK_\beta} = -\frac{MK_\beta}{OA - OK_\beta} = \frac{-\tau_\beta}{\sigma_1 - \sigma_\beta}.$$

So,

$$tg\alpha = \frac{-\tau_\beta}{\sigma_1 - \sigma_\beta}. \qquad (7.18)$$

This formula determines the single value of the angle α, by which the normal n_α must be rotated to find the direction of the algebraically larger principal stress.

7.7 Volumetric stress state (v.s.s)

In calculation practice, the v.s.s. is rarely encountered, because it is always possible to neglect the lower stress and reduce it to a flat or even linear stress state. However, if all three principal stresses are of the same order of magnitude, then

$$\sigma_\alpha = \sigma_1 cos^2\alpha_1 + \sigma_2 cos^2\alpha_2 + \sigma_3 cos^2\alpha_{3,}$$

$$\tau_\alpha = \sqrt{\sigma_1{}^2 cos^2\alpha_1 + \sigma_2{}^2 cos^2\alpha_2 + \sigma_3{}^2 cos^2\alpha_3 - \sigma_\alpha{}^2},$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Sites that are equidistant from the principal axes are called "octahedral", and the stresses acting on these sites are called **octahedral stresses** (Figure 7.11).

Characteristics of these sites:

$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$; $cos^2\alpha_1 + cos^2\alpha_2 + cos^2\alpha_3 = 1$;

$$cos^2\alpha = \frac{1}{3}; \qquad \sigma_{\text{окт}} = \frac{\sigma_1 + \sigma_2 + \sigma_2}{3};$$

$$\tau_{\text{окт}} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

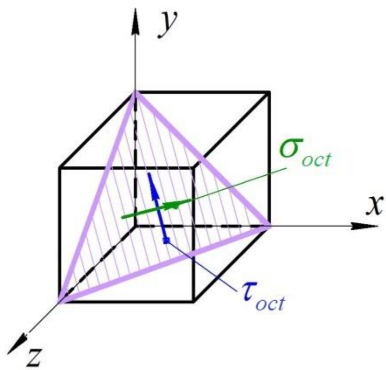


Figure 7.11 - Octahedral stresses

7.8 Deformations under volumetric stress. Generalized Hooke's law

When studying deformations and considering the issues in the volumetric and flat states, we will assume, in addition to the basic hypotheses and assumptions, that the material follows Hooke's law and the deformations are small.

Simple stretching-compression tests have shown that **the relative longitudinal strain**

$$\varepsilon = {}^{\sigma}/_{\mathrm{E}}, \tag{7.19}$$

and **the relative transverse strain**

$$\varepsilon' = -\mu\, {}^{\sigma}/_{\mathrm{E}}, \tag{7.20}$$

where $\mu = \frac{\varepsilon\prime}{\varepsilon}$ – **is the dimensionless Poisson's ratio.**

For an absolutely brittle material, the Poisson's ratio is 0, and for an absolutely elastic material, it is 0.5. For most steels, this coefficient is approximately 0.3, and for rubber it is approximately 0.49. The Poisson's ratio can be negative, although this is an exotic situation. This means that the transverse dimensions of the body increase when stretched. Materials with such properties are called "auxetics", they are polymers.

Let us establish the relationship between deformations and stresses in the general case of a volumetric stress state.

Looking at the deformation of an element with dimensions $a \times b \times c$ in the general case of v.s.s. from the principal stresses $\sigma_1, \sigma_2, \sigma_3$, (Fig. 7.12), applying Hooke's law and the principle of superposition, we obtain expressions for **the main relative elongations.**

As a result of deformation, the edges of the element change their length and are equal to $(a + \Delta a)$; $(b + \Delta b)$; $(c + \Delta c)$.

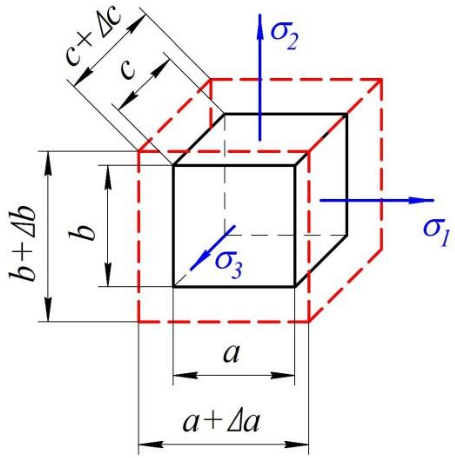


Figure 7.12 - Principal stresses and principal elongations

Values

$$\varepsilon_1 = \frac{\Delta a}{a}; \varepsilon_2 = \frac{\Delta b}{b}; \varepsilon_3 = \frac{\Delta c}{c}$$

are called **principal elongations** and **are the relative elongations in the principal directions.**

Using the principle of superposition, we can write

$$\varepsilon_1 = \varepsilon_1' + \varepsilon_1'' + \varepsilon_1''',$$

where ε_1' – is the relative elongation in the direction σ_1 caused by stress $\sigma_1\ alone\ (\sigma_2 = \sigma_3 = 0)$; ε_1'' – is the relative elongation in the same direction caused by the action of σ_2 alone; ε_1'''– is the elongation caused by the action of σ_3.

Since the direction of σ_2 for the stress σ_1 itself is longitudinal (it stretches the face a), and σ_2 and σ_3 are transverse (they narrow the edge a), then, applying formulas (7.19) and (7.20), we find that

$$\varepsilon_1' = \frac{\sigma_1}{E}; \quad \varepsilon_1'' = -\mu\frac{\sigma_2}{E}; \quad \varepsilon_1''' = -\mu\frac{\sigma_3}{E}.$$

Adding up these values, we have:

$$\varepsilon_1 = \varepsilon_1' + \varepsilon_1'' + \varepsilon_1''' = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E} = \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right].$$

Similarly, we obtain the formulas for the other two elongations σ_2 and σ_3. Then

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_1)\right]\ ; \\ \varepsilon_2 &= \frac{1}{E}\left[\sigma_2 - \mu(\sigma_1 + \sigma_3)\right]\ ; \\ \varepsilon_3 &= \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1 + \sigma_2)\right]\ . \end{aligned}\right\} \tag{7.21}$$

These formulas represent **the generalised Hooke's law** for an isotropic body, i.e. **the relationship between deformations and principal stresses in the case of a triaxial stress state.**

Note that compressive stresses are substituted into these formulas with a minus sign. From (7.21), it is easy to derive Hooke's formulas for the plane stress state. For example, when $\sigma_2 = 0$:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}[\sigma_1 - \mu\sigma_3]; \\ \varepsilon_2 &= -\frac{\mu}{E}[\sigma_1 + \sigma_3]; \\ \varepsilon_3 &= -\frac{1}{E}[\sigma_3 - \mu\sigma_1]. \end{aligned}\right\} \tag{7.22}$$

Volumetric strain

Let's determine the dependence of the change in volume ε_V on the principal stresses. Before strain, the element occupied a volume $V_0 = abc$. In the deformed state, its volume is

$$\begin{gathered} V = (a + \Delta a)\cdot(b + \Delta b)\cdot(c + \Delta c) = \\ = abc\left(1 + \frac{\Delta a}{a}\right)\left(1 + \frac{\Delta b}{b}\right)\left(1 + \frac{\Delta c}{c}\right) = \\ = V_0(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) = \end{gathered}$$

$$= V_0(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1 \cdot \varepsilon_2 + \varepsilon_2 \cdot \varepsilon_3 + \varepsilon_3 \cdot \varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3).$$

Taking into account the smallness of the relative linear deformations, the last four terms can be neglected. Then the relative change in volume will be:

$$\varepsilon_V = \frac{V - V_0}{V_0} = \frac{V_0(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3)}{V_0} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Expressing the principal elongations through the principal stresses according to (7.21), we obtain

$$\varepsilon_V = \frac{1 - 2\mu}{E}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \quad (7.23)$$

Particularly in the case of homogeneous all-round compression, when

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -\mathrm{P},$$

$$\varepsilon_V = \frac{-\mathrm{P}}{\mathrm{K}},$$

where

$$\mathrm{K} = \frac{\mathrm{E}}{3(1 - 2\mu)}. \quad (7.24)$$

The value of K is called **the modulus of volume strain.**

Formula (7.23) shows that when the Poisson's ratio $\mu = 0{,}5$ (for example, rubber), the volume of a body does not change.

7.9 Potential strain energy

The potential strain energy is the energy accumulated in a body during its elastic deformation and is denoted by U. The value of the potential strain energy per unit volume (1 см^3) of a body is called the specific potential strain energy and is denoted by u.

The potential strain energy is numerically equal to the work of external forces, which is consumed during elastic deformation, i.e.

$$U = A_P. \tag{7.25}$$

Within the limits of elasticity, the total work of deformation is determined by the area of the triangle *OAK* (Figure 7.13):

$$A_P = \frac{P \cdot \Delta \ell}{2}. \tag{7.25′}$$

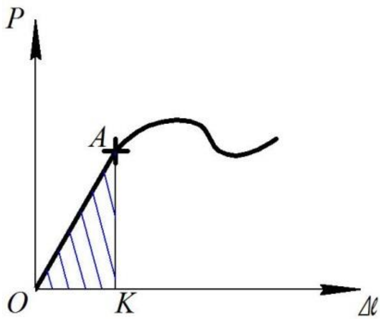


Figure 7.13 - Stretch diagram

In simple stretching (compression):

$$U = \frac{P \cdot \Delta \ell}{2}. \tag{7.26}$$

Specific potential energy:

$$u = \frac{P \cdot \Delta \ell}{2F \ell} = \frac{\sigma\varepsilon}{2}. \tag{7.27}$$

Considering that

$$\varepsilon = \frac{\sigma}{E},$$

we have

$$u = \frac{\sigma^2}{2E}. \tag{7.28}$$

If we take a cube with an edge length equal to 1, on which the principal stresses $\sigma_1, \sigma_2, \sigma_3$ act, then these forces produce displacements $\varepsilon_1, \varepsilon_2, \varepsilon_3$ since the edges have a unit length. Based on formula (7.27), we have:

$$u = \frac{\sigma_1 \varepsilon_1}{2} + \frac{\sigma_2 \varepsilon_2}{2} + \frac{\sigma_3 \varepsilon_3}{2}. \tag{7.29}$$

Substituting the values $\varepsilon_1, \varepsilon_2, \varepsilon_3$ from Hooke's general law (7.21) into these formulas, we find:

$$u = \frac{1}{2E}[\sigma_1 + \sigma_2 + \sigma_3 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]. \tag{7.30}$$

When an element is deformed, both its volume and shape change (from a cube to a parallelepiped). Then the total specific potential energy of deformation:

$$u = u_V + u_s, \tag{7.31}$$

where u_V – the specific potential energy of volume change, u_s – is the energy accumulated due to the change in the shape of the element:

$$u_V = \frac{1 - 2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2; \tag{7.32}$$

$$u_s = \frac{1 + \mu}{3E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1) = \\ = \frac{1 + \mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_1)^2 + (\sigma_3 - \sigma_1)^2]. \tag{7.33}$$

7.10 Control questions

1. What causes tension in the body?
2. What are the values of the indices for stresses?
3. What stresses are called the main?
4. How are principal stresses marked?
5. What is the law of parity of tangential stresses?
6. Three types of stress state.
7. The rule of signs for determining stresses.
8. In what section of the bar in stretching do the maximum normal stresses occur and in what section do the maximum tangential stresses occur?
9. What is the state of the material called linear, plane and volumetric stress state?
10. Mohr's circle. Principle of construction and possibilities of use.
11. What sites in the body are called the main ones?
12. Generalised Hooke's law.
13. The potential energy of deformation.
14. Poisson's ratio.
15. Direct problem in plane stress state.
16. Inverse problem in a plane stress state.
17. Write the stress tensor.

8. STRENGTH CRITERIA

8.1 Strength criteria

The most important task of engineering calculation is to assess the strength of machine and construction elements based on a known stress state.

Dangerous stresses are those at which **residual deformations** appear (plastic deformation occurs), i.e., fracture begins.

From the dangerous stresses (with a known safety factor), the permissible stresses in stretching $[\sigma_s]$ or compression $[\sigma_c]$ in a linear stress state are determined. This issue is more difficult to solve in the plane and volumetric stress state (see Section 7).

In these cases, studies have shown that for the same material, a dangerous state can occur at different limit values of the principal stresses σ_{1st}, σ_{2st}, σ_{3st} depending on their ratio.

Therefore, based on theoretical and practical studies of the behaviour of various materials, a hypothesis is introduced about the predominant influence of a certain factor on the strength. It is believed that the failure of a material under any stress condition will occur only when this factor reaches a certain limit value. The limit value of a factor is found from simple stretching or compression tests, and sometimes torsional tests. Thus, to solve the problems of plane and volumetric stress state, the concept of **strength criteria** is introduced. It makes it possible to reduce a complex stress state to a simple one and find an equivalent (calculated) stress that will give the same safety factor in both cases. **The safety factor** is understood as the number n, which shows how many times all components of the stress state σ_1, σ_2, σ_3, must be increased simultaneously to make it the ultimate stress state:

$$\sigma_{1st} = n\sigma_1; \quad \sigma_{2st} = n\sigma_2; \quad \sigma_{3st} = n\sigma_3. \tag{8.1}$$

The hypothesis chosen in this way is called **the mechanical theory of strength**, or simply **the theory of strength.**

There are four classical theories of strength.

I theory of strength – the theory of maximum normal stress. According to this theory, material failure will occur when the greatest normal stress reaches a dangerous value for a given material. The strength condition with a safety factor n is as follows

$$\sigma_1 = \sigma_{max} \leq [\sigma] = \frac{\sigma_{st}}{n}, \tag{8.2}$$

where n **– safety factor**;

σ_{st} – determined by simple stretching (compression).

Experimental verification shows that this theory gives satisfactory results only for very brittle materials (e.g. stone, brick, ceramics, tool steel).

II theory – the theory of the largest linear deformations.

According to this theory, the highest linear deformation in absolute value is considered the strength criteria. In other words, fracture occurs when the largest linear relative elongation ε_{max} reaches the dangerous value ε_{H}. The last one is determined by simple stretching (compression) of samples of a given material. Thus, **the fracture condition** is as follows

$$\varepsilon_{max} = \varepsilon_H, \tag{8.3}$$

and **the strength condition**:

$$\varepsilon_{max} \leq [\varepsilon] = \frac{\varepsilon_H}{n}. \tag{8.4}$$

Using Hooke's generalised law, the strength condition is expressed in terms of stress $[\sigma] = [\varepsilon] \cdot E, or\ [\varepsilon] = \frac{[\sigma]}{E}$. If the greatest relative elongation is ε_1, and the permissible stress is $[\sigma]$, then from (8.4) we have:

$$\varepsilon_{max} = \varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)],$$

or

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) \leq [\sigma]. \tag{8.5}$$

As can be seen from the strength condition (8.5), for the second theory, it is not one of the three principal stresses that should be compared with the permissible stress, but their combination. **The equivalent stress** in this case is

$$\sigma_{equ}^{\text{II}} = \sigma_1 - \mu(\sigma_2 + \sigma_3). \tag{8.6}$$

Experimental testing of this theory has shown consistent results only for brittle material conditions (e.g. alloy cast iron or high-strength low-temperature steels).

III theory - the theory of the greatest tangential stresses.

According to this theory, material fracture will occur when the largest tangential stress τ_{max} reaches the dangerous value $\tau_{\text{н}}$, which is determined when the ultimate state is reached in the case of simple stretching of samples of a given material.

The fracture condition is as follows

$$\tau_{max} = \tau_{\text{н}}, \tag{8.7}$$

and **the strength condition**

$$\tau_{max} \leq [\tau] = \frac{\tau_H}{n}. \tag{8.8}$$

According to formulas (8.7, 8.8), $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$, and $\tau_{\text{н}} = \frac{1}{2}\sigma_{\text{н}}$, then conditions (8.7) and (8.8) can be expressed in terms of principal stresses.

The condition of fracture:

$$\sigma_1 - \sigma_3 = \sigma_n; \tag{8.9}$$

Strength condition:

$$\sigma_1 - \sigma_3 = [\sigma]. \tag{8.10}$$

Thus, **the equivalent stress according to the third theory** is

$$\sigma_{equ}^{\text{III}} = \sigma_1 - \sigma_3. \tag{8.11}$$

The third theory of strength is well supported by research for plastic materials, in which the permissible stresses in stretching and compression are the same. **The disadvantage of this theory is that it does not take into account the intermediate principal stress σ_2.**

IV theory – is the criteria for the specific potential energy of deformation of a shape change.

According to this theory, a dangerous state (yielding) in the general case of a stressed state occurs when the last one can be determined by simple stretching at the moment of yielding.

The condition for the onset of fluidity:

$$U_{sh} = U_{sh.y.}. \tag{8.12}$$

Strength condition:

$$U_{sh} \leq [U_{sh}]. \tag{8.13}$$

Assuming that Hooke's law is valid up to the limit state, the specific potential energy of the shape change can be written as:

$$U_{sh} = \frac{1+\mu}{3\mathrm{E}}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]. \tag{8.14}$$

At simple stretching at the moment of yielding ($\sigma_1 = \sigma_y; \sigma_2 = \sigma_3 = 0$) we have

$$U_{sh.y.} = \frac{1+\mu}{3\mathrm{E}}\sigma_y^2 . \tag{8.15}$$

Substituting (8.14) and (8.15) into condition (8.12), we obtain

$$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]} = \sigma_y. \tag{8.16}$$

The strength condition is as follows

$$\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]} \leq \frac{\sigma_y}{n} = [\sigma], \tag{8.17}$$

and **the equivalent stress according to the fourth theory**

$$\sigma_{equ}^{IV} = \sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]}.$$

Studies have provided good support for the fourth theory for plastic materials in stretching and compression. For these materials, the calculations based on the fourth theory are more accurate than those based on the third theory.

8.2 The concept of Mohr's theory

The Mohr's criterion is based on the assumption that the strength of materials in the general case of a stress state depends mainly on the value and sign of the largest σ_1 and smallest σ_3 principal stresses. The modulus-average principal stress σ_2 has only a minor effect on strength. Experiments with copper, nickel and cast iron tubes have shown that the error due to the fact that σ_2 is not taken into account is no higher than 12-15%. Based on this assumption, any stress state can be represented by a Mohr's circle based on the stresses σ_1 i σ_3.

Strength condition according to Mohr's theory:

$$\sigma_{br}^{\mathrm{M}} = \sigma_1 - \alpha\sigma_3 \leq [\sigma], \tag{8.19}$$

where

$$\alpha = \frac{[\sigma]_s}{[\sigma]_{co}}.$$

This theory is in good agreement for both brittle and ductile materials. If $\alpha = 1$, we obtain the formula for III theory

Since the first and second theories of strength have significant drawbacks, the opinion that they are inappropriate for use has recently been expressed. Therefore, for practical calculations, the fourth or third theory of strength should be recommended for materials with equal strength to stretching and compression, and **Mohr's theory for materials with different strength to stretching and compression, i.e. brittle materials.**

The drawbacks of the considered theories, as well as the emergence of new materials (plastics, composites, etc.), led to the development of new strength theories (Y.I. Yagn, G.S. Pisarenko, Y.B. Fridman, etc.). Most of them are based on the choice of a boundary surface shape that allows to take into account the strength features of a given class of materials under a complex stress state. All this is discussed in special courses.

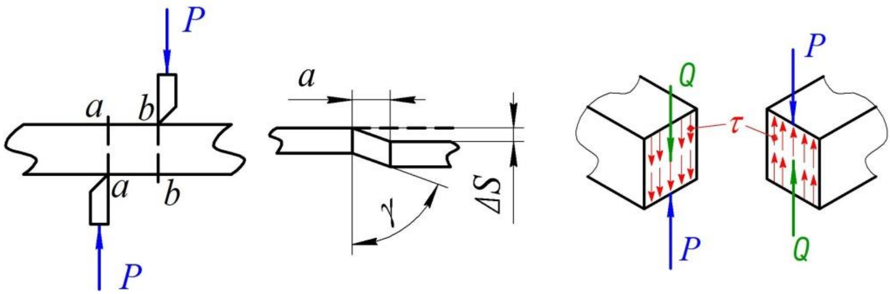
8.3 Control questions

1. What are the assumptions of the first, second, third and fourth theories of strength?
2. What are the strength theories used for?
3. For which materials should Mohr's theory be used?
4. What is meant by the safety factor?
5. The condition of strength according to the first theory.
6. For what materials does the first theory of strength give satisfactory results?
7. Strength condition and equivalent stress according to the II theory of strength.
8. For which materials does the second theory of strength give satisfactory results?
9. Strength condition and equivalent stress according to the III theory of strength.
10. For which materials does the III theory of strength give satisfactory results?
11. Strength condition and equivalent stress according to the fourth theory of strength
12. For what materials does the fourth theory of strength give satisfactory results?
13. What is the reason for the error and what value does it reach when using Mohr's criteria?

9. SHEAR

9.1 Calculation for a cut

Shear deformation is possible when an element is subjected to two identical oppositely directed forces that are close to each other and act perpendicular to the axis of the rod (Fig. 9.1, a). Parts of various connections (riveted, bolted, welded) operate under shear conditions.



a) **b)** **c)**

Figure 9.1 - Shear strain

The absolute displacement of the cross-sections aa and bb is denoted by ΔS and is called **the absolute shear** (Fig. 9.1, b). The angle γ is called **the angle of shear** or **relative shear.** From Fig. 9.1, b, we have $tg\gamma = \frac{\Delta S}{\alpha}$ and, taking into account the smallness of the angle γ, we can assume that

$$\gamma = \frac{\Delta S}{\alpha}. \tag{9.1}$$

Determination of tangential shear stresses

Using the method of cross-sections, we find that the transverse force $Q = P$ on the segment ab. Let's find the connection between the transverse force and the stress. From formula (3.4), we have $\int_F \tau dF = Q$.

Considering that the tangential stresses τ are equally distributed over the cross-sectional area F (Fig. 9.1, c) ($\tau = const$), we have $Q = P = \tau F$, whence

$$\tau = \frac{Q}{F} = \frac{P}{F}. \tag{9.2}$$

Shear strength condition

$$\tau_{max} = \frac{Q_{max}}{F_{shear}} \leq [\tau]. \tag{9.3}$$

where F_{shear} – is the total area to be sheared; Q_{max} – is the maximum transverse force; $[\tau]$ – is the permissible shear stress.

9.2 Pure shear

The stress state when only tangential stresses act on the four faces of a rectangular element is called **pure shear** (Fig. 9.2).

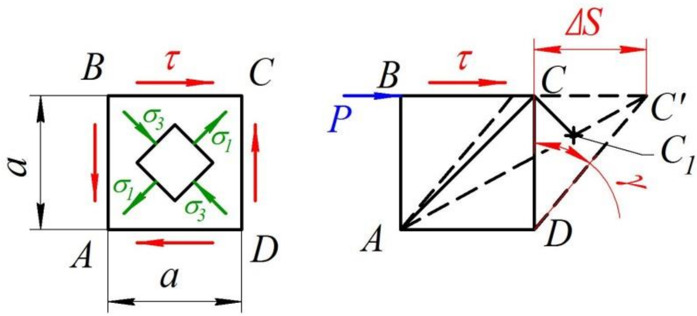


Figure 9.2 - Scheme of pure shear

In this case, the principal sites are inclined to the edges of the element at an angle of 45°, and the principal stresses are $\sigma_1 = -\sigma_3 = \tau$.

In pure shear, the diagonal ac, which coincides with the direction of σ_1, lengthens, and the diagonal bd, which coincides with the direction of the compressive stress σ_3, shortens. As a result, the square becomes a rhombus. **The shear angle** (relative shear) is determined by formula (9.1).

Hooke's law for pure shear

The dependence between load and shear strain can be seen in the so-called shear diagram. It can be obtained by twisting a thin-walled pipe (Figure 9.3). The diagram shows the strength characteristics: the proportionality limit $\tau_{пц}$, the yield limit $\tau_{Т}$ and the ultimate strength $\tau_{В}$. The selected element of the pipe wall is under pure shear conditions.

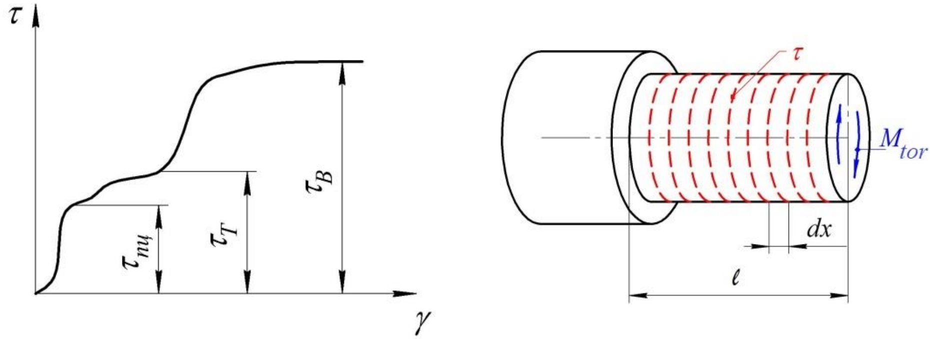


Figure 9.3 - Torsion of a thin-walled pipe

Looking at the deformation of this element, we can determine that there is a linear dependence between the relative shear γ and the tangential stress τ which can be expressed by the formula (Hooke's law for pure shear):

$$\gamma = \frac{\tau}{G} \ \ or \ \ \tau = G \cdot \gamma, \qquad (9.4)$$

where G – **is the proportionality ratio, which is called the shear modulus of elasticity, or the second-order modulus of elasticity,** and is expressed in megapascals (MPa). For isotropic materials, there is a certain dependence between the modulus of elasticity G in shear and the modulus of elasticity E in stretching. To determine it, consider the strain of the element under pure shear conditions (see Fig. 9.2). The length of the diagonal AC, before strain, $\ell = a\sqrt{2}$.

Let's find the elongation $\Delta\ell$ of the diagonal AC.

Looking at the geometric picture of the strain, we have

$$\Delta\ell = \mathrm{C_1C} \approx \mathrm{C'C_1}\cos 45° = \frac{\Delta S}{\sqrt{2}}.$$

$$\varepsilon_1 = \frac{\Delta\ell}{\ell} = \frac{\Delta S}{a\sqrt{2}\sqrt{2}} = \frac{\gamma}{2},$$

where

$$\gamma = \frac{\Delta S}{a}; \Delta S = a\gamma.$$

Since the principal stress σ_1 acts in the direction of the diagonal AC, its relative elongation

$$\varepsilon = \varepsilon_1 = \frac{1}{\mathrm{E}}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] =$$
$$= \frac{1}{\mathrm{E}}[\tau - \mu(0 - \tau)] = \frac{\tau(1+\mu)}{\mathrm{E}}. \qquad (9.5)$$

According to Hooke's law for pure shear:

$$\gamma = \frac{\tau}{G}$$

$$\varepsilon_1 = \frac{\tau}{2\sigma} = \frac{\tau(1+\mu)}{E},$$

whence

$$G = \frac{E}{2(1+\mu)}. \qquad (9.6)$$

when

$$\mu = \frac{1}{3}...\frac{1}{4}$$

will be $G = (0{,}375\ldots0{,}4)\,E$, and for steel $G = 8 \cdot 10^4$ MPa.

Potential strain energy

Let's write an expression for the displacement of one edge relative to the other (i.e., the absolute shear expression ΔS) in pure shear. Denoting the area of a edge by F, the equivalent shear force by $Q = F\tau$ and the distance between the sheared edges by a (Figure 9.2), we have:

$$\Delta S = \gamma \cdot a = \frac{\tau}{G} \cdot a = \frac{Q \cdot a}{GF}, \tag{9.7}$$

i.e.

$$\Delta S = \frac{Qa}{GF}. \tag{9.8}$$

Formula (9.8) expresses Hooke's law for absolute shear.

Specific potential energy of strain of an element in pure shear:

$$u = \frac{U}{V} = \frac{Q^2 a}{2GF \cdot aF} = \frac{Q^2}{2F^2 G}, \tag{9.9}$$

or

$$U = \frac{Q^2 a}{2GF}, \quad V = aF, \qquad u = \frac{\tau^2}{2G}, \tag{9.10}$$

where U - is the total potential strain energy in pure shear;

V - the volume of the selected element.

Permissible shear stresses

Let's check the strength of the element under pure shear deformation. The tangential stresses on the edges of the element are equal to τ,

is the permissible stretching stress for the material $[\sigma]$.

As already mentioned, the principal stresses in pure shear are $\sigma_1 = \tau$; $\sigma_2 = 0$; $\sigma_3 = -\tau$. he strength condition depends on the choice of strength theory.

According to the second theory of strength:

$$\sigma_1 - \mu\sigma_3 \leq [\sigma].$$

Substituting the values of the principal stresses, we find

$$\tau \leq \frac{[\sigma]}{1 + \mu}.$$

It means that **the permissible stress at pure shear**

$$[\tau] = \frac{[\sigma]}{1 + \mu}. \qquad (9.11)$$

For metals $\mu = 0{,}25 \dots 0{,}42$. So, according to the second theory of strength

$$[\tau] = (0{,}7 \dots 0{,}8)[\sigma]. \qquad (9.12)$$

According to the third theory of strength:

$$\sigma_1 - \sigma_3 \leq [\tau] \text{ or } \tau - (-\tau) \leq [\sigma],$$

whence

$$\tau \leq \frac{[\sigma]}{2} = [\tau],$$

i.e.

$$[\tau] = 0{,}5[\sigma]. \qquad (9.13)$$

According to the fourth theory of strength:

$$\sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} \leq [\sigma],$$

or

$$\tau \leq \frac{[\sigma]}{\sqrt{3}}.$$

So,

$$[\tau] = \frac{[\sigma]}{\sqrt{3}} \approx 0{,}6[\sigma]. \qquad (9.14)$$

The shear strength condition (section) can be written in the usual form:

$$\tau_{max} = \frac{Q_{max}}{F} \leq [\sigma]. \qquad (9.15)$$

It should be noted that the permissible shear stress for a section $[\tau]$ depends on material properties, nature of loading and type of structural elements. Specific values of $[\tau]$ for some materials in relation to riveted and welded joints are given in Table 9.1.

Table 9.1 - Permissible shear stresses for riveted and welded joints

Connection type	Stress for a cross-section, MPa
Connection with a rivet:	
basic elements made of steel 20………………….	100
rivet in drilled holes (class B)….	140
rivet in punched holes (classC)…..	100
Welded connection:	
manual welding, thinly coated electrodes ……...	80
--//--, thickly coated electrodes………..	110
automatic welding……………………..	110

9.3 Calculation of welded joints

Welded joints can be made with the following **types of seams**: butt; end; flange.

Butt seams are tested for strength based on stretching strength. In butt seams with a thickness of parts δ not exceeding 8 mm, the edges are not treated: a) with a thickness of 8...20 mm, the edges are beveled on one side (V-shaped weld); b) with a thickness of more than 20 mm, the edges are beveled on both sides (X-shaped weld); c) if the direction of the angular seam is perpendicular to the direction of force, the seam is called a frontal or end seam.

In butt joints (Fig. 9.4, a), **the strength condition** for a seam of length ℓ **in stretching**:

$$\sigma_{max} = \frac{P}{F_{seam}} = \frac{P}{\ell \cdot \delta} \leq [\sigma'_s], \qquad (9.16)$$

in compression

$$\sigma_{max} = \frac{P}{\ell \cdot \delta} \leq [\sigma'_{com}]. \qquad (9.17)$$

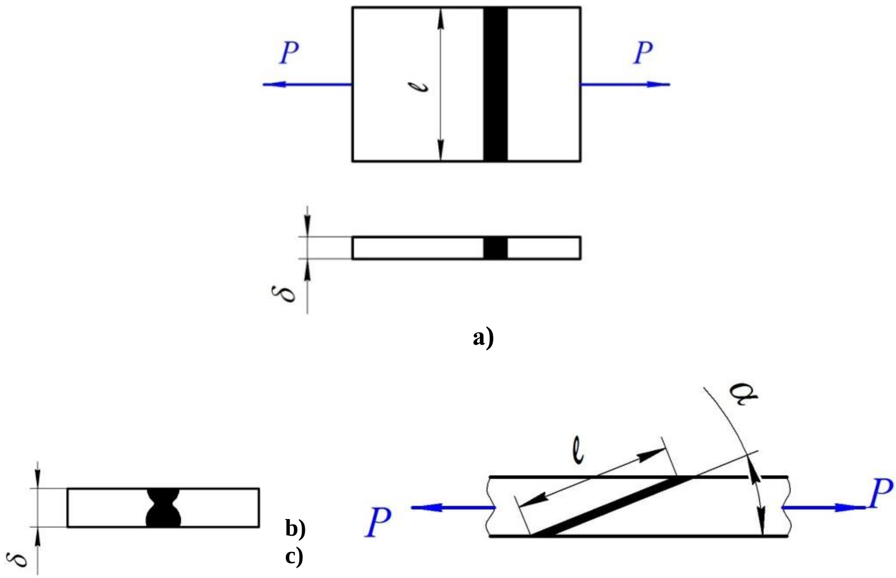


Figure 9.4 - Calculation of butt joints

If the seam is oriented at an angle α (Fig. 9.5, c) (usually $\alpha = 45°$) to the direction of the force P, the strength condition is determined by the formula

$$\tau = \frac{P}{\ell \cdot \delta} \leq [\tau'\ ]. \qquad (9.18)$$

In the given dependences: σ and τ – normal and tangential stresses in the seam materials; P - acting force; δ – thickness of the elements to be joined; $[\sigma_s']$, $[\sigma_{com}']$ – permissible normal stresses of the seam material; $[\tau'\ ]$ – permissible tangential stress of the seam material.

The strength condition for fly tying in straight (frontal) (Fig. 9.5) and flank (Fig. 9.6) **seams** is as follows:

$$\tau = \frac{P}{F_{seam}} \leq [\tau'\ ] \ or \ \ \tau = \frac{P}{2 \cdot 0{,}7 \cdot k \cdot \ell_T} \leq \ [\tau'\ ], \qquad (9.19)$$

where $F_{seam} = 2 \cdot 0{,}7\ell_T$ – seam area;

k – the length of the seam leg;

$\ell_T = \ell - 10\,mm$ the estimated length of the seam (10 mm is taken for lack of fusion along the edge of the part);

$[\tau']$ – for electric arc welding is selected depending on the type of welding and the type of electrode (manual or automatic, type of electrode and coating, etc.) and is described in specialised literature.

$F_{seam} = 0{,}7\delta$ – is the length of the seam leg.

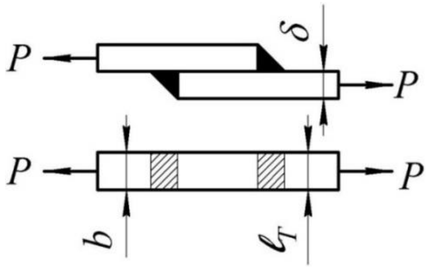


Figure 9.5 - Straight (frontal) seam

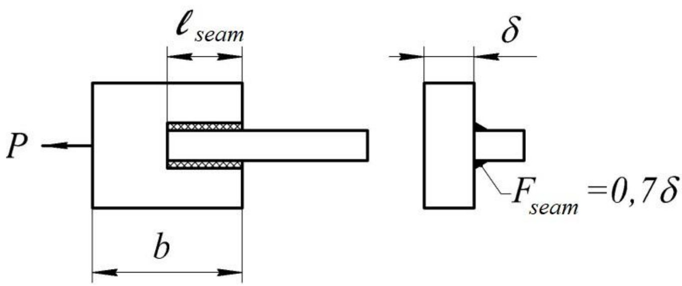


Figure 9.6 - Flank seam

9.4 Control questions

1. What is shear strain?
2. What is called absolute and relative shear? What is their dimension?
3. What stress state is called pure shear?
4. How is Hooke's law expressed in pure shear?
5. Under what conditions does shear occur?
6. What is the formula for determining tangential shear stresses?
7. What is the relationship between the moduli of elasticity of the first and second kind?
8. What is called a crush?
9. What are the types of welded joints?
10. The condition of strength for shear.
11. Strength condition for welded joints (for different types of welds).
12. Total potential strain energy in pure shear.
13. Allowable stress in pure shear.
14. Condition of bolt shear strength.
15. Condition of the bolt's crushing strength.

10 TORSION

10.1 Epures of torque moments

Torsion is a type of deformation in which only one of the six forces, **the internal torsional moment** ($M_z = M_{tor} \neq 0$), occurs in the cross-sections of a structural element, and all other forces are zero.

Transmission shafts operate under torsional conditions.

The theory of **torsion** is based on the following **three hypotheses** (assumptions):

1. The cross-section of a rod that is flat before twisting remains flat during twisting. This assumption is called **the flat section hypothesis.**
2. The radii of the cross-sections of a bar remain straight during twisting.
3. The distance between the cross-sections does not change.

These hypotheses allow us to determine the stresses that occur in a round cross-sectional bar.

Torques vary in the cross-sections of the shaft. Torque epures are created using the cross-sectional method. The epure is used to determine the position of the dangerous section of the rod. The dangerous section is the one in which the largest modulus torque is applied.

The internal torsional moment is equal to the algebraic sum of all external moments about the z-axis acting on one side of the section being examined

$$M_{tor} + \sum m_{kz} = 0.$$

Rule of signs for torques.

Looking from the direction of the axis**, a positive torque is the one that acts in a counterclockwise direction** (Figure 10.1).

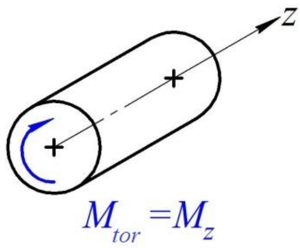


Figure 10.1 - Torsional deformation

External moments can be concentrated and distributed along the shaft length where: m – intensity $\left[\frac{\text{Нм}}{\text{м}}\right]$.

The rule for constructing epures is the same as for constructing longitudinal force epures. That is:

– mentally cut the shaft into sections (a section is the length of the shaft between two concentrated moments, or where the law of distributed load is the same);

– - using the method of cross-sections, draw up the equilibrium equation

$$\sum m_{kz} = 0.$$

Based on the obtained values of the moments at each site, we build an epure at an arbitrarily chosen scale (Fig. 10.2, d).

Rules for checking the construction of epures

1. In the area where there is no distributed load, the line of the epure is parallel to the axis of the rod.

2. In the area where the distributed load is applied, the torque epure is a straight line inclined to the axis of the rod.

3. In the section where the concentrated moment acts, the torque epure will have a gap (jump) equal to the magnitude of this moment.

10.2 Torsional stress and strain. Strength and stiffness conditions

10.2.1 Torsional stress. To determine the stresses in cross-sections, consider a rod that is twisted by a pair of forces (Fig. 10.3).

d_0– diameter of the rod;

ℓ – length of the rod;

AB – forming;

φ – torsion angle;

BB' – is the absolute displacement of one cross-section relative to the other;

γ – angle of relative displacement on the surface, where $\gamma = \text{const}$.

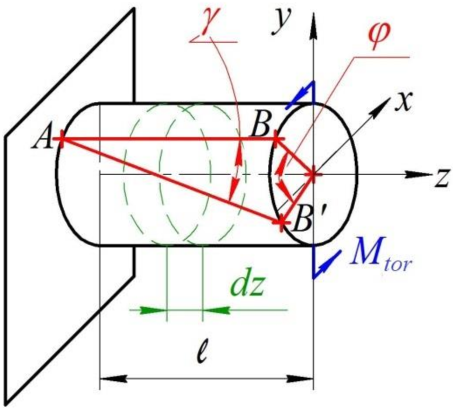


Figure 10.3 - Torsional strain

To solve the problem of determining torsional stress, we consider three aspects of the problem.

Static aspect of the problem:

Since M_{tor} is the only internal force factor in torsion, it is reasonable to assume that only tangential stresses act in the cross–sections. In this case, the five equations (3.4) are identically transformed to zero, and (3.5) takes the form

$$M_{tor} = \int_F \rho \tau dF, \qquad (10.1)$$

where ρ is a variable radius,

$$0 \le \rho \le \frac{d_0}{2};$$

τ – is the tangential stress acting on the elementary site dF.

The nature of the stress distribution across the cross-section will be clarified by considering the geometric pattern of torsional strain of the shaft.

Geometric aspect of the problem:

Let's consider a shaft element of length dz. The shaft is twisted by an external torque M_{tor}, which causes internal torsional moments M_{tor} in the cross-sections.

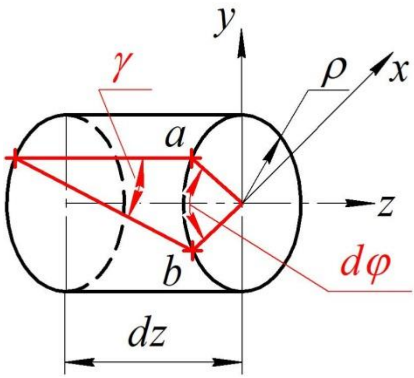


Figure 10.4 - Geometric aspect of the problem

The twisting angle of a shaft segment of length dz (Fig. 10.4) is $d\varphi$. Absolute displacement of one section relative to another

$$ab = \rho d\varphi = \gamma dz, \text{ where } \gamma = \rho\frac{d\varphi}{dz},$$

$$\gamma = \rho\theta, \tag{10.2}$$

$$\theta = \frac{d\varphi}{dz},$$

where θ – is the relative twist angle, cm^{-1}.

The physical aspect of the problem. Since the shaft element at any point of the cross-section, at a distance of ρ from the centre of the cross-section, undergoes a pure shear, we have

$$\tau = \gamma G. \tag{10.3}$$

Formula (10.3) is **Hooke's law of torsion**.

Synthesis. Substituting (10.2) into (10.3), and then into equation (10.1), we determine the stress acting in the cross-section of the shaft in torsion:

$$M_{tor} = \int\limits_F \rho \cdot \theta \cdot G \cdot \rho \cdot dF = \theta G \int\limits_F \rho^2 dF = \theta G \mathcal{J}_\rho.$$

From this, we derive **the formula for determining the relative twist angle of a round rod:**

$$\theta = \frac{d\varphi}{dz} = \frac{M_{tor}}{G\mathcal{J}_\rho}\,, \tag{10.4}$$

where $G\mathcal{J}_\rho$ – **is the torsional stiffness of the cross-section of the rod,** Н·м^2; $\mathcal{J}_\rho$ **is the polar moment of inertia of a round rod.**

On the basis of (10.4), a formula can be written to determine the mutual torsion angle of two cross-sections located at a distance ℓ

$$\varphi = \int\limits_0^\ell \frac{M_{tor}}{G\mathcal{J}_\rho} dz. \tag{10.5}$$

Using this formula, we can draw an epure φ.

If the torques in the cross-sections do not change within a rod segment of length ℓ, then

$$\varphi = \theta \cdot \ell = \frac{M_{tor} \cdot \ell}{G\mathcal{J}_\rho}. \tag{10.6}$$

This formula expresses **Hooke's law of torsion.**

The tangential stress τ at any point of the shaft cross-section

$$\tau = \frac{M_{tor} \cdot \rho}{\mathcal{J}_\rho}. \tag{10.7}$$

Maximum tangential stress (on the outer layer of the rod material at $\rho = r$ and $\frac{\rho}{\mathcal{J}_\rho} = \frac{1}{W_\rho}$)

$$\tau_{max} = \frac{M_{tor\,max} \cdot r}{\mathcal{J}_\rho} = \frac{|M_{tor\,max}|}{W_\rho}, \tag{10.8}$$

where $W_\rho = \frac{\mathcal{J}_\rho}{r} = \frac{\mathcal{J}_\rho}{\rho}$ – **is the polar torque of torsional strength, cm3.**

Let's write the equations for torsional strength and stiffness. According to (10.8), the strength condition is written as follows:

$$\tau_{max} = \frac{M_{tor\,max}}{W_\rho} \le [\tau], \tag{10.9}$$

where $[\tau]$ – **permissible tangential torsional stress,**

$[\tau] = 0{,}5 - 0{,}6[\sigma]$.

$[\sigma]$ – permissible stress determined by simple stretching.

The stiffness condition, according to (10.4)

$$\theta_{max} = \frac{M_{tor}}{G\mathcal{J}_\rho} \le [\theta], \tag{10.10}$$

where $[\theta]$ – permissible relative twist angle.

Based on formulas (10.9) and (10.10), we have:

$$W_\rho \ge \frac{M_{tor}}{[\tau]} \quad \text{and} \quad \mathcal{J}_\rho \ge \frac{M_{tor}}{G[\theta]}. \tag{10.11}$$

Thus, to determine the dangerous cross-section (to find the cross-section where $|M_{tor\,max}|$ acts) it is necessary to draw an epure M_{tor}.

The strength condition can be used to solve three types of problems:

1. Using the known load and material, find the diameter of the rod at which the strength condition is satisfied

$$\tau_{max} = \frac{M_{tor\,max}}{\frac{\pi d^3}{16}} \le [\tau], \qquad d = \sqrt[3]{\frac{16 M_{tor\,max}}{\pi[\tau]}}.$$

2. Determine the permissible load based on the known dimensions of the part and material.

$$M_{tor\,max} = \frac{\pi d^3}{16} \cdot [\tau] - \text{ design calculation.}$$

3) When the load, dimensions and material of the part are known, check whether **the strength condition** is met

$$\tau_{max} = \frac{M_{tor\,max}}{W_\rho} \le [\tau].$$

10.2.2 Calculation of the tubular section (hollow shaft).

Shafts that transmit high power are made tubular. As you can see from the epure (Fig. 10.5), the shaft material is lightly loaded in the middle. For example, a solid shaft that transmits power should have a diameter of 300 mm. If we take a tubular shaft with an outer diameter of *D* = 350 mm and an inner diameter of *d* = 275 mm, it will transmit the same power, but its weight will be 2 times less.

$$\tau_{max} = \frac{M_{tor\,max}}{W_\rho} = \frac{16 M_{tor\,max}}{\pi D^3 (1-\alpha^4)} \le [\tau],$$

where $W_\rho = \frac{\pi D^3 (1-\alpha^4)}{16}$, – is the polar moment of strength for tubular cross-section,

$\alpha = \frac{d}{D}$ – the ratio of the internal d to the external diameter D.

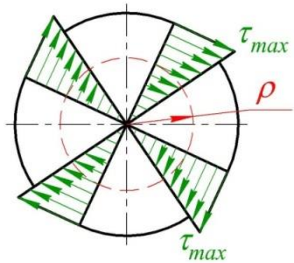


Figure 10.5 - Variation of stresses τ in the shaft cross-section

If the torques are applied unevenly along the length of the shaft, the shaft is stepped to save material and reduce weight. Such a shaft is called an equal-strength shaft.

10.2.3 Determination of the twist angle of a circular shaft.
We have already shown that the torsional angle is equal:

$$\varphi = \int_0^{\ell} \frac{M_{tor}}{G\mathcal{I}_\rho} dz \ .$$

If the torque does not change within the section and the diameter d-is constant, then

$$\varphi = \frac{M_z \cdot \ell}{G\mathcal{I}_\rho}. \qquad (10.12)$$

If the shaft has several force sections or if the torque epure crosses the zero line, the torsion angle of one section relative to the other is equal to the algebraic sum of the torsion angles in each section (i.e., taking into account the direction of this torsion).

$$\varphi = \sum_{i=1}^{n} \frac{M_{zi} \cdot \ell_i}{G\mathcal{I}_\rho}. \qquad (10.12')$$

10.3 Statically undefined torsional problems

Example 10.3. Consider a circular shaft, fixed at both ends of the sides and loaded with a torque M in the shaft section at point C (Fig. 10.6). The moment M, dimensions a and b.

Draw an epure of torques along the length of the shaft.

Under such a load, reactive moments M_{A} and M_B will occur at the shaft fixing points in the planes perpendicular to the shaft axis. Let's consider three aspects of the problem:

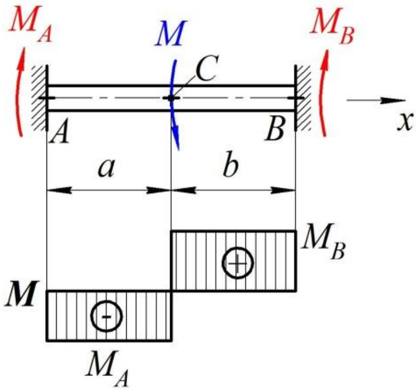


Figure 10.6 - Statically indeterminate system

The static aspect of the problem.

$$\sum M_{kz} = 0 \qquad - M_A - M_B + M = 0. \tag{10.13}$$

The geometric aspect of the problem. Since both ends of the shaft are fixed, the angle of rotation of the sections B relative to A is zero:

$$\varphi_{\mathrm{B-A}} = \varphi_{\mathrm{B-C}} + \varphi_{\mathrm{C-A}} = 0. \tag{10.14}$$

The physical aspect of the problem. Using formula (10.5), write the expressions for the torsion angles:

$$\varphi_{B-C} = -\frac{M_B \cdot b}{G\mathfrak{I}_\rho}; \qquad \varphi_{C-A} = \frac{(M - M_B) \cdot a}{G\mathfrak{I}_\rho}. \tag{10.14$'$}$$

Synthesis. Substituting (10.14′) into (10.14), we get

$$-\frac{M_B \cdot b}{G\mathfrak{I}_\rho} + \frac{(M - M_B) \cdot a}{G\mathfrak{I}_\rho} = 0.$$

Hence, taking into account (10.13), we find that

$$M_B = \frac{M \cdot a}{a + b}, \qquad M_A = \frac{M \cdot b}{a + b}.$$

The torque epure is shown in Fig. 10.6.

10.4 Torsion of rods with non-circular cross-section

In engineering practice, rods with non-circular cross-sections (rectangular, triangular, etc.) are often subjected to torsion. In these cases, **the hypothesis of flat cross-sections cannot be applied**, since the cross-sections are curved (deplaned). Therefore, the study of the VAT of such rods cannot be performed by the methods of material strength. To solve this problem, the methods of elasticity theory are used. The research results showed that within the limits of Hooke's law, the largest tangential stresses are absent and the total and relative torsional angles can be determined by the following formulas:

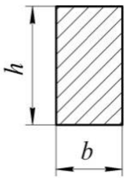
$$\tau_{max} = \frac{M_{tor\,max}}{W_t}; \tag{10.15}$$

$$\varphi = \frac{M_{tor} \cdot \ell}{G\mathcal{J}_t}; \tag{10.16}$$

$$\theta = \frac{M_{tor}}{G\mathcal{J}_t}\ , \tag{10.17}$$

where $\mathcal{J}_{\mathrm{t}}$ and W_t – geometric characteristics: **moment of inertia (cm4) and moment of strength (cm3) in torsion.** Formulas for determining $\mathcal{J}_t$ and W_t for some sections are given in Table 10.1

Table 10.1 Geometrical characteristics of some torsional sections

<table>
<tr><th>Cross-sectional shape</th><th>Torsional moment of inertia, $\mathcal{J}_t$, cm^4</th><th>Torsional moment of strength, W_t, cm^3</th><th>Points with the highest tangential stresses $\tau_{max} = \frac{M_{tor}}{W_t}$</th><th colspan="4">Notes</th></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td colspan="4">5</td></tr>
<tr><td rowspan="12"></td><td rowspan="12">$\mathcal{J}_t = \beta h b^3$</td><td rowspan="12">$W_t = \alpha h b^2$</td><td rowspan="12">In the middle of the long sides $\tau_{max} = \frac{M_{tor}}{W_t}$; in the middle of the short sides $\tau = \gamma\tau_{max}$; In the angles $\tau = 0$.</td><td>h/b</td><td>α</td><td>β</td><td>γ</td></tr>
<tr><td>1</td><td>0,208</td><td>0,141</td><td>1</td></tr>
<tr><td>1,5</td><td>0,231</td><td>0,196</td><td>0,859</td></tr>
<tr><td>1,75</td><td>0,239</td><td>0,214</td><td>-</td></tr>
<tr><td>2,0</td><td>0,246</td><td>0,229</td><td>0,795</td></tr>
<tr><td>2,5</td><td>0,256</td><td>0,249</td><td>-</td></tr>
<tr><td>3,0</td><td>0,267</td><td>0,263</td><td>0,753</td></tr>
<tr><td>4,0</td><td>0,282</td><td>0,281</td><td>0,745</td></tr>
<tr><td>6,0</td><td>0,299</td><td>0,299</td><td>0,743</td></tr>
<tr><td>8,0</td><td>0,307</td><td>0,307</td><td>0,743</td></tr>
<tr><td>1,0</td><td>0,313</td><td>0,313</td><td>0,743</td></tr>
<tr><td>∞</td><td>0,333</td><td>0,333</td><td>0,743</td></tr>
</table>

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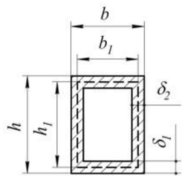
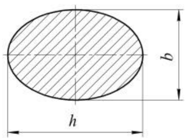
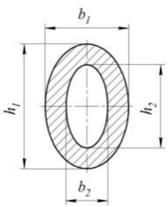
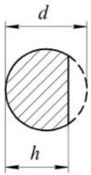
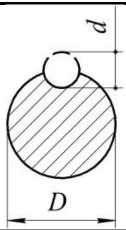
Continuation of Table 10.1

1	2	3	4	5
	$\mathcal{J}_t =$ $= \dfrac{h_0^2 b_0^2 \delta_1 \delta_2}{h\delta_2 + b\delta_1 - \delta_1^2 - \delta}$	$W_{t1} = 2h_0 b_0$ $W_{t2} = 2h_0 b_0$	In the middle of the long sides $\tau_1 = \dfrac{M_{tor}}{W_{t1}}$; in the middle of the short sides $\tau_2 = \dfrac{M_{tor}}{W_{t2}}$.	In the inner corners, there is a high concentration of stresses that reach the yield strength of the material. If there is a rounding of the radius r, the concentration factor $\alpha_c = 1,74\sqrt[3]{\dfrac{\sigma_{max}}{r}}$.
	$\mathcal{J}_t =$ $= \dfrac{h_0^2 b_0^2 \delta_1 \delta_2}{h\delta_2 + b\delta_1 - \delta_1^2 - \delta}$	$W_t = \dfrac{\pi b^2 h}{16}$	At the outer points of small semi-axes $\tau_{max} = \dfrac{M_{tor}}{W_t}$; at the end of large semi-axes $\tau_1 = \dfrac{\tau_{max}}{m}$.	$\dfrac{h}{b} = m > 1$

Continuation of Table 10.1

<table>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr>
<td></td>
<td>$$\mathcal{J}_t = \frac{\pi m^3 b_1^4 (1-\alpha^4)}{16(m^2+1)}$$</td>
<td>$$W_t = \frac{\pi b_1^3}{16} \times$$ $$\times\,(1-\alpha^4)m$$</td>
<td>At the end of small semi-axes $\tau_{max} = \frac{M_{tor}}{W_t}$; at the end of large semi-axes $\tau_1 = \frac{\tau_{max}}{m}$.</td>
<td>$$\frac{h_1}{b_1} = \frac{h_2}{b_2} = m > 1$$ $$\frac{h_2}{h_1} = \frac{b_2}{b_1} = \alpha < 1$$</td>
</tr>
<tr>
<td></td>
<td>$$\mathcal{J}_t = \frac{d^4}{16}\left(2,6\frac{h}{d} - 1\right)$$</td>
<td>$$W_t = $$ $$= \frac{d^3}{8}\,\frac{2,6\frac{h}{d}-1}{\left(0,3\frac{h}{d}+0,7\right)}$$</td>
<td>$$\tau_{max} = \frac{M_{tor}}{W_t}$$</td>
<td>$$\frac{h}{d} > 0,5$$</td>
</tr>
<tr>
<td></td>
<td>$$\mathcal{J}_t = \alpha\frac{D^4}{16}$$</td>
<td>$$W_t = \beta\frac{D^3}{8}$$</td>
<td>At the bottom of the groove $$\tau_{max} = \frac{M_{tor}}{W_t}$$</td>
<td>
<table>
<tr><th>d/D</th><th>α</th><th>β</th><th>d/D</th><th>α</th><th>β</th></tr>
<tr><td>0,00</td><td>1,57</td><td>1,57</td><td>0,40</td><td>0,76</td><td>1,22</td></tr>
<tr><td>0,05</td><td>0,80</td><td>1,56</td><td>0,60</td><td>0,66</td><td>0,92</td></tr>
<tr><td>0,10</td><td>0,81</td><td>1,56</td><td>0,80</td><td>0,52</td><td>0,63</td></tr>
<tr><td>0,20</td><td>0,82</td><td>1,46</td><td>1,00</td><td>0,38</td><td>0,38</td></tr>
</table>
</td>
</tr>
</table>

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10.4.1 Rectangular cross-section rods.

In practice, rods of rectangular cross-section are most often used. In this case, the distribution of tangential stresses is as shown in Fig. 10.7.

The greatest stresses occur near the surface, in the middle of the long sides of the rectangular section at points C and D.

They are determined by formula (10.14), where

$$W_t = \alpha h b^2; \qquad (10.18)$$

h and b – sides of the cross-section;

α – a coefficient that depends on the ratio of h and b.

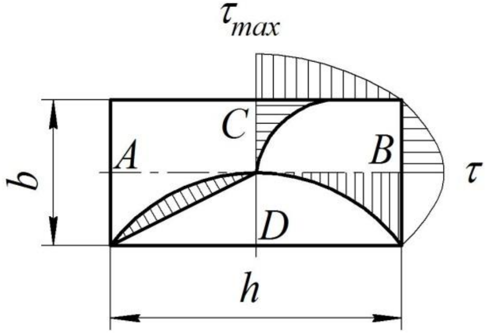


Figure 10.7 - Stress epures in a rectangular torsional section

The stresses that occur in the middle of the short sides (at points A and B), are determined as follows

$$\tau = \gamma \tau_{max}. \qquad (10.19)$$

Relative and total torsional angles by formulas (10.16) and (10.17), where

$$\mathcal{J}_t = \beta h b^3. \qquad (10.20)$$

The coefficients $\alpha,\ \beta,\ \gamma$ depend on the aspect ratio $h\ and\ b$.

h/b	α	β	γ
1	0,208	0,141	1
10	0,313	0,313	0,743

The torsional strength and stiffness conditions for rods of rectangular cross-section are calculated using the following formulas:

$$\tau_{max} = \frac{M_{tor\,max}}{\alpha h b^2} \leq [\tau]; \qquad (10.21)$$

$$\theta = \frac{M_{tor}}{G\beta h b^3} \leq [\theta]\,. \qquad (10.22)$$

10.4.2 Open composite rods. Rods with cross-sections in the form of angles, I-beams or channels can be calculated using the same formulas as rods with rectangular cross-sections. In this case, the cross-sections are conditionally divided into rectangular parts, and the geometric characteristic $\mathcal{I}_t$ is determined as a sum:

$$\mathcal{I}_t = \frac{1}{3}\eta \sum b_{\mathrm{i}}^3 h_{\mathrm{i}}\,. \qquad (10.23)$$

When the aspect ratio is $^{h}/_{b} > 10$ the coefficients α and β in formulas (10.20) and (20.21) are equal to $\alpha + \beta = \frac{1}{3}$. The coefficient η in formula (10.23) will vary depending on the actual section. For example, **for an angle η=1,0, a channel η=1,12, a T-beam η=1,15, an I-beam η=1,2**.

The geometric strength characteristic **(torsional moment) W_t** is determined by the formula:

$$W_t = \frac{\mathcal{I}_t}{\delta_{max}}, \qquad (10.24)$$

where δ_{max} – is the maximum thickness of the rectangles included in the expanded complex section.

The torsion angle and the largest tangential stress are determined as follows:

$$\varphi = \frac{M_{tor} \cdot \ell}{G\beta hb^3}; \tag{10.25}$$

$$\tau_{max} = \frac{M_{tor}\delta_{max}}{\mathcal{J}_t}, \tag{10.26}$$

where according to formula (10.24)

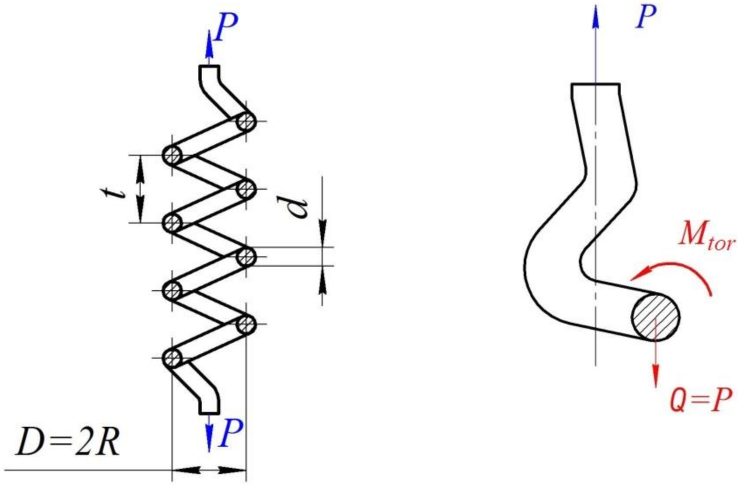
$$\delta_{max} = \left(\frac{\mathcal{J}_{ti}}{W_{ti}}\right)_{max}.$$

10.5 Calculation of coiled cylindrical springs

Accurate strength calculations for coiled coil springs are difficult because the spring wire can be subjected to complex loads of torsion, shear and bending. However, at small angles of inclination of the coils, the effect of bending can be neglected.

Let's consider a cylindrical helical spring with an average diameter of $D = 2R$, which has n coils and is made of wire with a diameter of d. The spring is stretched by a central force P (Fig. 10.9, a).

Let's divide the spring by the diameter area and consider the equilibrium of the upper part. To ensure equilibrium, we apply forces P and M_{tor} at the cross-sectional point (Fig. 10.9, b). Thus, the spring wire undergoes shear and torsional strains. Due to shear, the tangential stresses will be evenly distributed across the section (Fig. 10.10, a).



a) b)

Figure 10.9 - Calculation of helical springs

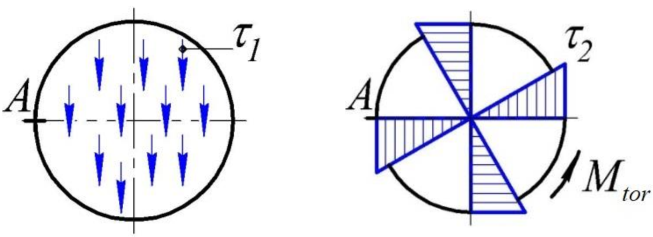
From torsion, the maximum stress τ_2 occurs on the surface (Fig. 10.10, b):

$$\tau_1 = \frac{P}{F} = \frac{4P}{\pi d^2}, \qquad \text{where } F = \frac{\pi d^2}{4},$$

τ_1 – tangential stresses in the wire due to shear;

$$\tau_2 = \frac{M_{tor}}{W_\rho} = \frac{16 M_{tor}}{\pi d^3}, \qquad \text{where } W_\rho = \frac{\pi d^3}{16},$$

τ_2 – tangential torsional stresses in the wire.



a) b)

Figure 10.10 - Shear (a) and torsional (b) stresses in the wire

The maximum stress will be at point A (Fig. 10.10, a)

$$\tau_{max} = \tau_1 + \tau_2 = \frac{16PD}{\pi d^3}\left(1 + \frac{d}{4D}\right). \qquad (10.27)$$

Ignoring the expression in parentheses, we get

$$\tau_{max} = \frac{16PD}{\pi d^3}.$$

When calculating high-powered springs (railway transport), the expression in brackets must be taken into account, as shear stresses are significant due to the large value of $\frac{d}{R}$.

Fracture in this case begins from the inside of the coil, where the largest total tangential stresses τ_1 and τ_2. The formula τ_{max} takes the following form:

$$\tau_{max} = \frac{16PD}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0{,}615}{m}\right), \qquad (10.28)$$

where $m = \dfrac{D}{d}$.

Determination of λ spring extension

Let us consider the torsional strains of a selected spring element of length dS (Fig. 10.11).

Under the action of the torque M_{tor} the cross-section will rotate relative to the left cross-section by a certain angle $d\varphi$. Due to this, point O will move to point O_1.

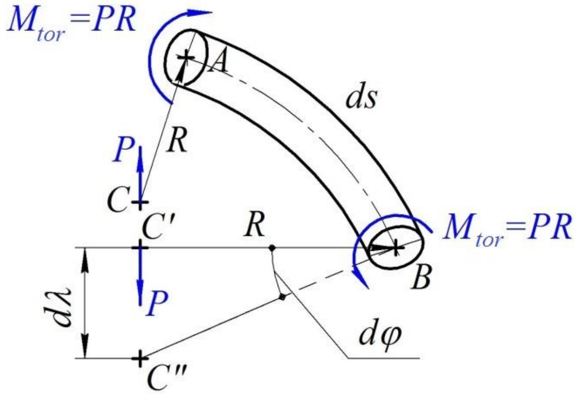


Figure 10.11 - Determining the tension of a λ spring

$d\lambda$ – **spring draft** of the element length dS.

The end of the spring will drop by $d\lambda = Rd\varphi$; where

$$d\varphi = \frac{M_{tor} dS}{G\mathcal{I}_\rho}; \;\; M_{tor} = P \cdot R; \;\; \mathcal{I}_\rho = \frac{\pi d^4}{32}.$$

$$d\lambda = \frac{P \cdot R \cdot R \cdot dS \cdot 32}{G\pi d^4}.$$

The full lowering of the lower end of the spring, i.e. its elongation, is determined by the formula:

$$\lambda = \int_0^{2\pi Rn} \frac{32P \cdot R^2 dS}{G\pi d^4} = \frac{64P \cdot R^3 \pi n}{G\pi d^4} = \frac{64P \cdot R^3 n}{Gd^4}$$

$$\lambda = \frac{8P \cdot D^3 n}{Gd^4}, \qquad (10.29)$$

where P – is the force that stretches (or compresses) the spring;

D – spring diameter;

n – number of spring coils;

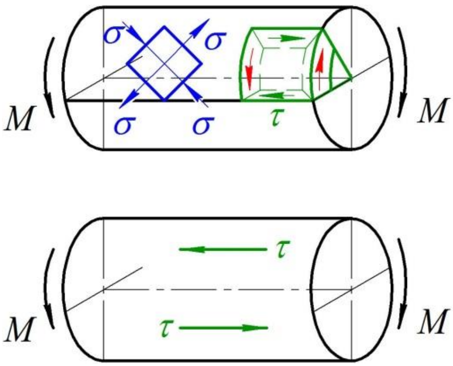
d – wire diameter;

G – modulus of elasticity of the wire material.

10.6 Principal torsional stresses

Analytical calculations and experiments show that when shafts are torsionally stressed, both principal normal and principal tangential stresses occur in them. This nature of torsional stress distribution is confirmed by the appearance of torsional fractures of various materials.

Wood (some plastics) fracture in torsion due to tangential stresses acting along the fibres.



Ductile materials (steels, alloys, etc.) fracture under tangential stresses acting in transverse directions.

Brittle materials (cast irons) fracture under normal stresses by fracturing at an angle of 45°.

Figure 10.12 - Scheme of distribution of different types of torsional stresses in different materials

10.7 Control questions

1. What type of deformation is called torsion?
2. What is the sign rule for torques?
3. Rules for drawing torsion epure.
4. What angle is called the full angle of torsion?
5. How is the formula for the total angle of torsion written?
6. What value is called torsional stiffness?
7. Hooke's law of torsion.
8. What is the torsional moment of strength? What is its dimension?
9. What is the formula for determining the deflection of a cylindrical coil spring?
10. The condition of torsional strength.
11. The condition of stiffness for torsion.

11. BENDING

11.1 The most common types of beams and supporting structures

Beams are rectilinear rods placed on supports and working in bending. In the field of material strength, from the point of view of calculations for strength, stiffness and stability, a beam is not only a building beam, but also other structural elements on supports: a shaft, a bolt, a car axle, a gear tooth, etc.

In the calculation schemes, the beam is usually represented by its axis, and the cross section is shown separately. In this case, all loads should be reduced to the axis of the beam, and the force plane will coincide with the plane of the drawing.

Actual fixing points for structural elements are diverse, but in material strength they are reduced to conventional supports: clamping and hinge supports.

A hinged movable support is a support that is placed on rollers that do not prevent the beam from moving along the supporting surface. In a movable hinged support (Figure 11.1, a), only one reaction R_y occurs - a force in the direction of movement restriction (perpendicular to the direction of movement).

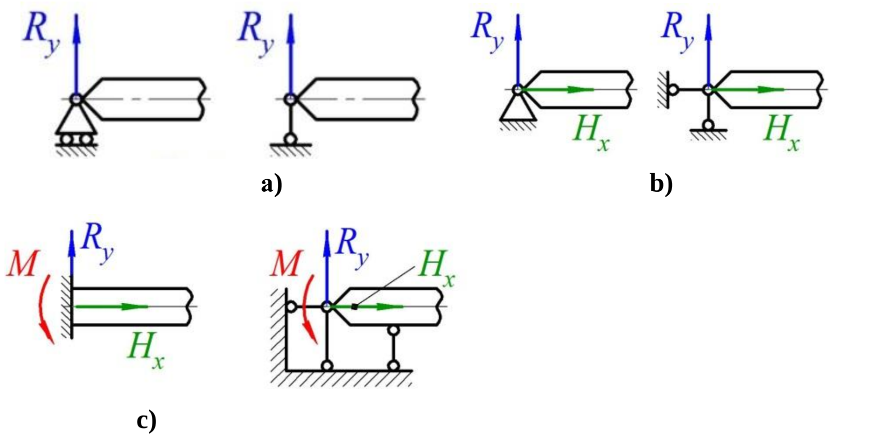


Figure 11.1 - Types of supports and their reactions

Immovable hinged support is a support that prevents progressive movement of the beam along the coordinate axes, but allows it to rotate about the hinge axis. In an immovable hinged support (Fig. 11.1, b), two components can occur - the vertical reaction R_y and the horizontal reaction H_x. The support allows the beam cross-section to rotate around the hinge, but does not allow progressive movement.

Rigid clamping or **pinching** is a support that does not allow the structure to move. From the support side, the beam **(cantilever)** is subjected to a force and a reactive moment of a pair of forces, which are taken into account as three components: vertical reaction R_y, horizontal reaction H_x, and support moment M (Fig. 11.1, c). The support does not allow any movement of the fixed end of the beam.

The reactions of beam supports are referred to as external loads.

A **cantilever** is an unsupported part of a beam.

In real life, the number of spans is always more than one. Such beams are called **multi-span beams.**

A beam that covers one span and has two supports is called a **split beam.** A beam without intermediate joints that covers several spans and has several supports is called an **undivided multi-span beam** (statically indeterminate).

Types of loads.

All existing loads can be summarised as follows:

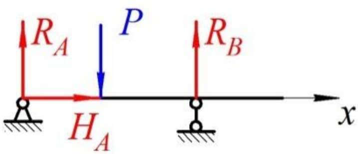
1. Concentrated force P, Н (Fig. 11.2).

2. Concentrated moment M, Н·м (Fig.11.2, b, c).

3. Distributed load with intensity q, Н/м (Fig. 11.3).

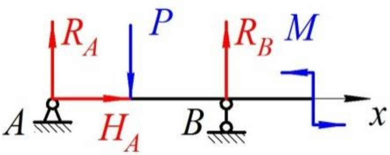
Intensity is the amount of load applied to a certain length.

Depending on the types of supports used in the scheme, the following **types of beams** are distinguished:



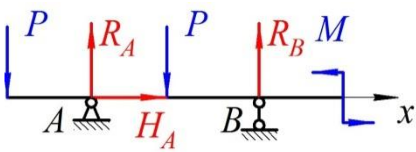
– **simple single span beam** on two supports, the distance ℓ between the two supports is called the **span**;

a)



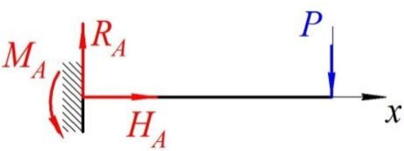
– a double-supported (single-span) beam with one cantilever (right cantilever);

b)



– double-supported single-span beam with two cantilevers (left and right);

c)



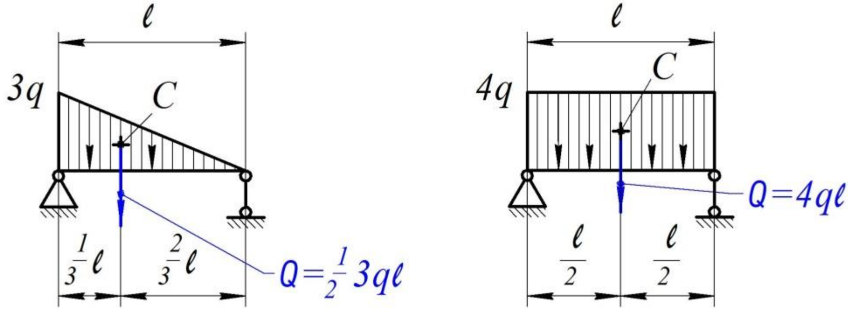
- rigidly clamped cantilever beam.

d)

Figure 11.2 - Types of beams

To solve the problem, the distributed load is replaced by an equal-acting force equal to the area of the load epure and applied at the centre of gravity of the figure (Fig. 11.3).

Example 11.1



a) **b)**

Figure 11.3 - Equivalent applied load

The equivalent $=\frac{1}{2}3q\ell$, where C is the centre of gravity of the triangle, which is located on $\frac{1}{3}\ell$ from the left support (Fig. 11.3, a).

11.2 Differential dependences of D.I. Zhuravsky in bending

These differential dependencies express the relationship between bending moment and load and transverse force

Let us select an element of length dx from the beam. The selected element must be in equilibrium under the action of external loads and internal forces (Figure 11.4).

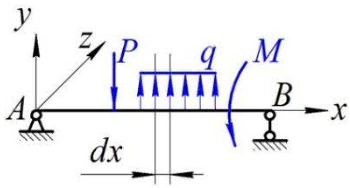
$$\sum P_{ky} = 0;$$

$$-dQ_y + qdx = 0; \quad \frac{dQ_y}{dx} = q. \qquad (11.1')$$

Thus, **the intensity of the distributed load in this section is equal to the first derivative of the transverse force along dx.**

Let's write the equation of the sum of moments of forces relative to point O_2 (Fig.11.4, b):

$$\sum M_{kO_2} = 0; \; -M_z + (M_z + dM_z) - Q_y dx - \frac{q(dx)^2}{2} = 0.$$



a)
b)

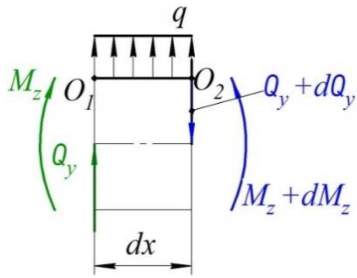


Figure 11.4 - Random loading of a beam

Neglecting the component $\frac{q(dx)^2}{2}$ and taking into account (11.1′), which is a term of the second order of smallness, we have:

$$\frac{dM_z}{dx} = Q_y, \quad \frac{d^2M_z}{dx^2} = q. \tag{11.1}$$

Equation (11.1) is **the differential bending dependence**.

Based on the differential dependencies in bending, a number of rules have been derived that allow controlling **the correctness of the bending epures Q_y and M_z.**

1. In areas where there is no distributed load epure Q_y is straight, where there is a distributed load – epure is inclined to the x-axis.
2. In areas where $Q_y > 0$ epure M_z increases.
3. Leaps in the epure where there are concentrated loads.

11.3 Plane bending. Construction of internal force epure in plane bending

Bending strain consists of a curvature of the axis of a straight rod or a change in the curvature of a curved rod. In straight rods, displacements of points δ, that are perpendicular to the initial position of the axis are called deflections. The axles of railway cars, springs, gear teeth, floor beams, levers, etc. are bent.

As you know, bending can be pure and transverse.

Bending is called **plane bending** if the cross-section of the beam has axes of symmetry and external loads act in the symmetry plane, the axis of the beam bends but remains in the symmetry plane.

Flat, pure bending is defined as bending when there is only one internal force acting in the cross-section, the bending moment M_x.

If two internal forces act in the cross-section - bending moment M_x and transverse force Q_y , then such a bending is called **plane transverse bending.**

In plane bending, the initial straight axis of the beam is curved, but lies in the longitudinal plane of symmetry, remaining a flat curve.

11.3.1 Drawing epures of internal bending forces.

The procedure for drawing internal force epure is illustrated by examples.

The sign rule for transverse force and bending moment.

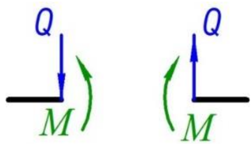


Figure 11.5 - Bending sign rule

The direction of the force is considered positive when:

- the transverse force Q_y rotates the cut part of the beam clockwise;
- the bending moment M_z acts so that the upper fibres of the beam are shortened and the lower fibres are lengthened

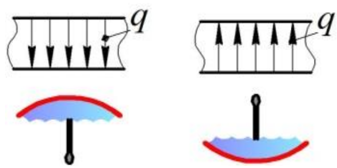


Figure 11.6- Rule for determining the bend of a parabola

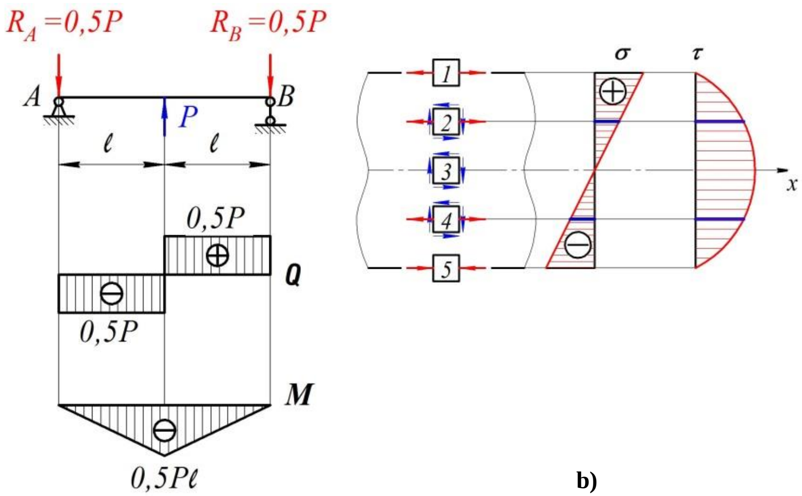
11.4 Calculation of bending strength and stiffness

Based on the nature of the distribution of the stresses acting in transverse bending, we conclude that the stress state in the cross-sections of the rods is not homogeneous, and this should be taken into account in strength calculations. Let's consider a two-supported beam on hinged supports (Fig. 11.7).

In an arbitrary cross-section of the beam, in addition to the support sections A and B, transverse forces and bending moments simultaneously act, the epures of which are shown in Fig. 11.7 (a), showing the distribution of normal and tangential stresses along the height of the section. Let us select a number of points in the section of the rod and analyse the stress state in them.

Point 1: This is the point furthest from the neutral layer.
Here

$$\sigma = \sigma_{max} = \frac{M_z}{W_z};\ \tau = 0.$$



a) **b)**

a) – beam loading scheme and force epure;

b) – stress state at points along the height of the beam section.

Figure 11.7 - Before analysing the stress state of a beam in transverse bending

We have a linear stress state, and the strength condition for this point is written as follows:

$$\sigma_{max} = \frac{M_z}{W_z} \le [\sigma]. \qquad (11.1)$$

Point 2. Here

$$\sigma = \frac{M_z}{\mathcal{J}_z} y; \qquad = \frac{QS_z(y)}{b(y)\mathcal{J}_z}.$$

A plane stress state is observed. To check point 2 for strength, use the appropriate strength criterion **depending on the material of the beam.**

Determine the principal stresses at this point.

$$\sigma_1 = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\tau^2}\right); \quad \sigma_2 = 0; \quad \sigma_3 = \frac{1}{2}\left(\sigma - \sqrt{\sigma^2 + 4\tau^2}\right).$$

If the material of the beam is **brittle**, the criterion of the highest normal stresses (first theory of strength) should be used. Strength condition according to this theory:

$$\sigma_{\text{P}}^{\text{I}} = \sigma_1 = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\tau^2}\right). \qquad (11.2)$$

If the material of the beam is **ductile**, then we use the criterion of the highest tangential stress (the third theory of strength) or the criterion of the highest potential energy of deformation (the fourth theory of strength). According to these theories, the calculated stresses are equal to

$$\sigma_{\text{P}}^{\text{III}} = \sigma_1 - \sigma_3;$$
$$\sigma_{\text{P}}^{\text{I}V} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}.$$

Substituting the found expressions for the principal stresses, we obtain the strength conditions in the following form:

$$\sigma_{\text{P}}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} \le [\sigma]; \qquad (11.3)$$

$$\sigma_{\mathrm{P}}^{\mathrm{IV}} = \sqrt{\sigma^2 + 3\tau^2} \leq [\sigma]. \qquad (11.4)$$

If the beam material **resists tension and compression differently,** the Mohr's criterion (the fifth theory of strength) should be used: $\sigma_{\mathrm{P}}^{V} = \sigma_1 - \frac{[\sigma]_s}{[\sigma]_{com}}\sigma_3$.

Marking

$$\frac{[\sigma]_s}{[\sigma]_{com}} = \alpha,$$

(see Formula 8.19), we obtain **the strength condition:**

$$\sigma_P^V = \frac{1-\alpha}{2}\sigma + \frac{1+\alpha}{2}\sqrt{\sigma^2 + 4\tau^2} \leq [\sigma]. \qquad (11.5)$$

Point 3: This point belongs to the neutral layer, where $\sigma = 0$; $\tau = \tau_{max}$.

Strength condition:

$$\tau_{max} = \frac{Q_{max} \cdot S_{max}}{b\mathcal{J}_z} \leq [\tau]. \qquad (11.6)$$

Point 4. Here, the strength conditions are similar to those for point 2.

Point 5. As in point 1, a linear stress state occurs here. If the distance from the neutral layer to point 5 is the same as to point 1, i.e. the section is symmetrical about the *z*-axis, then the strength condition for point 5 is written similarly to condition (11.8). Since in our example compressive stresses are acting at this point, we obtain the following strength condition:

$$|\sigma_{max}| = \frac{M}{W} \leq [\sigma_{com}]. \qquad (11.7)$$

If this point is closer to the neutral layer than point 1, the strength condition should be written as follows:

$$\sigma'_{max} = \frac{M_z}{J_z} y_5 \leq [\sigma_{com}]. \qquad (11.8)$$

Here σ'_{max} – the stresses at point 5, and they are the largest compressive stresses in the section; y_5 – is the distance from the neutral layer to point 5.

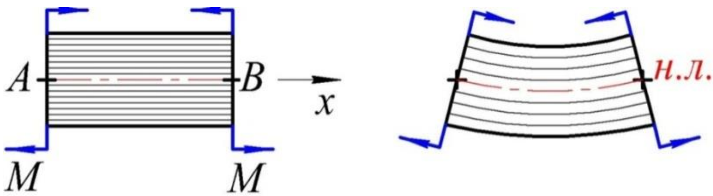
Note. If the rod material resists stretching and compressing equally, then only points 1, 2 and 3 should be checked.

11.4.1 Determination of normal stresses in a beam in pure bending. Consider the case of pure plane bending. AB is a fibre on the neutral layer (Fig. 11.8a).

A set of points distributed on a single line is called a fibre.

In bending, **the flat section hypothesis or Bernoulli hypothesis** takes place, **according to which flat sections before deformation remain flat and perpendicular to the neutral line after deformation, rotating one relative to the other by a certain angle.**

a)



b)

Figure 11.8- Pure bending

The set of fibres that do not change their length when bent is called the central layer or neutral line (Figure 11.8, b).

The part of the fibres above the neutral line shortens, i.e. undergoes compression deformation. The part of the fibres below the neutral line lengthens, i.e. undergoes stretching deformation.

Let's select an element of length $dx = 1$ from the beam with two parallel sections (Fig. 11.9) and consider its deformed state.

θ– **the relative angle of rotation of the section along the length** ℓ or the curvature of the neutral line.

$$\theta = \frac{1}{\rho}, \quad dx = \rho d\varphi; \quad \frac{1}{\rho} = \frac{d\varphi}{dx} = \theta, \qquad (11.8')$$

where ρ – **the radius from the centre of curvature to the neutral line.**

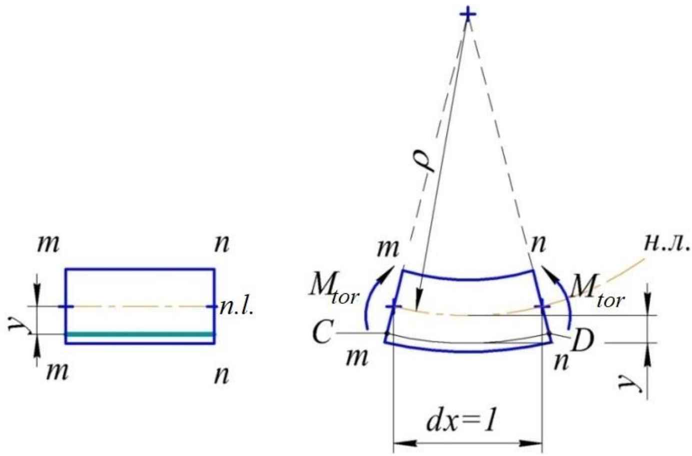


Figure 11.9 - Pure bending of element dx

Let's consider the case of pure bending of a beam (Fig. 11.9). Of the six internal force factors that can act in cross-sections in the general case of bending, only the bending moment $M$is not zero in pure bending.

Let us draw a cross section $m - m$ at a distance x_1 from the beginning of the coordinates. Select the element of area dF з with coordinates y and z.

In pure bending, all the forces Q and moments M_{tor}, associated with tangential stresses are zero. Thus, only three of the equilibrium conditions (3.4) remain.

To determine the normal stresses in the bending cross-section, consider 3 aspects of the problem:

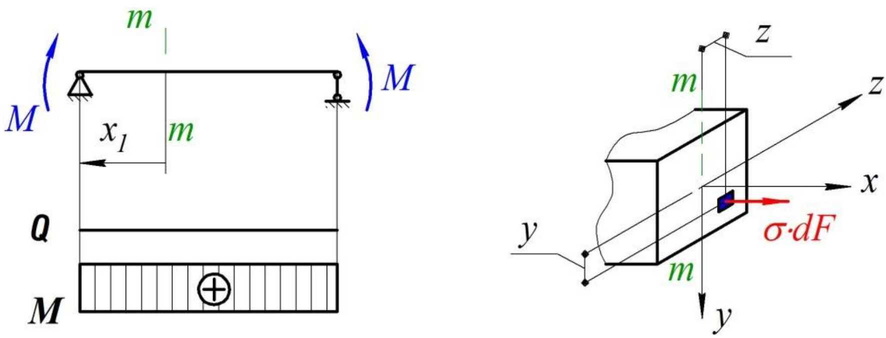


Figure 11.10 - Pure plane bending

The static aspect of the problem.
In general, the following acts in the cross-section

$$N = \int_F \sigma dF; \quad M_y = \int_F \sigma z dF; \quad M_z = \int_F \sigma y dF.$$

But in pure bending, only M_z acts in the cross section. That is,

$$N = 0; \;\; M_y = 0; \;\; M_z = \int_F \sigma y dF. \qquad (11.9)$$

Geometric aspect of the problem.
Relative deformation of the fiber (Fig.11.13, b)

$$CD = \varepsilon_x = \theta \cdot y, \qquad (11.10)$$

or considering (11.8'), we obtain

$$\varepsilon_x = \frac{y}{\rho}, \qquad (11.10')$$

where y – the distance from the neutral line to the point where the stress is determined σ.

Thus, solving the geometric aspect of the problem showed that **the relative longitudinal strain is proportional to the distance of the fiber from the neutral line.**

The physical aspect of the problem:

Since there are no tangential stresses at the elementary site dF of the cross-section, the material fibers along the beam do not press on each other and the stresses between them are zero, the fiber CD (Fig. 11.13, b) is in a linear stress state (simple stretching or compression). Therefore, Hooke's law for it should be written as follows:

$$\varepsilon_x = \frac{\sigma}{\mathrm{E}} \;\; or \;\; \sigma = \varepsilon_x \cdot \mathrm{E}. \qquad (11.11)$$

Synthesis. Exclude ε_{x} from the dependencies (11.10′) and (11.11):

$$\sigma = \frac{E}{\rho}\mathrm{y}.$$

Substituting (11.11) into (11.5), we obtain

$$M_z = \int_F \varepsilon_z \, EydF, \qquad (11.12)$$

and then substituting (11.10′) into (11.12) we have:

$$M_z = \int_F \theta E y^2 dF = \theta E \, \mathcal{I}_z, \qquad (11.13)$$

where (considering that $\int_F \; y^2 dF = \mathcal{I}_z$,

$$\theta = \frac{M_Z}{E\mathcal{I}_z} \quad \text{and} \quad \frac{1}{\rho} = \frac{M_Z}{E\mathcal{I}_z}\,. \qquad (11.14)$$

Then

$$\varepsilon_x = \frac{M_Z y}{E\mathcal{J}_z}, \qquad (11.15)$$

and therefore

$$\sigma = \frac{M_Z y}{\mathcal{J}_z}\,. \qquad (11.16)$$

Here, at $y = 0$; $\sigma = 0$; $y = y_{max}$; $\sigma = \sigma_{max}$.

Formula (11.16), which was first derived by the French scientist **C. Navier,** allows determining the normal stresses in pure bending of a beam σ at any point of its cross section.

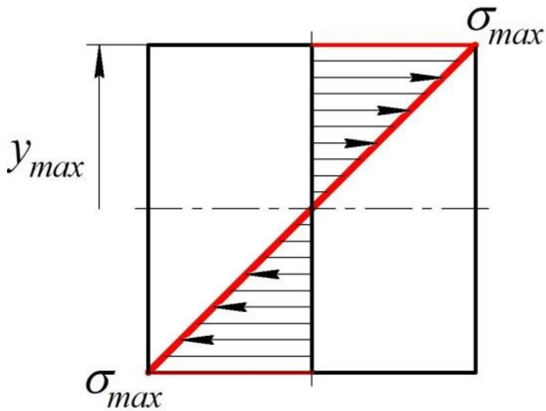


Figure 11.11 - Distribution of normal stresses according to Navier's formula along the height of the beam in bending

The distribution of normal stresses along the height of the beam in bending is shown in Fig. 11.11.

Considering that

$$\frac{\mathcal{J}_z}{y_{max}} = W_x,$$

the strength condition for normal bending stresses will be

$$\sigma_{max} = \frac{M_{zmax}}{W_z} \leq [\sigma].$$

The position of the neutral layer is determined from the equation

$$N = \int_F \sigma dF = 0; \quad \sigma = \varepsilon_x E = \theta y E.$$

Then

$$N = \theta E \cdot ydF = \theta E \int_F ydF = 0,$$

but $E\theta \neq 0$, so

$$\int_F ydF = S_x = 0.$$

Thus, **the neutral line in bending passes through the centre of gravity of the section.** And the y and x will be the main central axes. In this case, the force plane coincides with one of the main axes of inertia, and the second axis will be the neutral line.

11.4.2 Determination of tangential bending stresses (formula of D.I. Zhuravsky).

In plane transverse bending, when the cross-sectional forces $Q_y \neq 0$ and $M_z \neq 0$, occur, not only normal stresses σ, but also tangential stresses τ occur.

It is known from the theory of elasticity that in plane transverse bending, the hypothesis of plane sections does not hold. The cross-sections are curved (deplaned) during deformation. However, the effect of this factor on the magnitude and distribution of normal stresses depends on the size of the beam. If $\frac{h}{\ell} \leq \frac{1}{5}$ (h – the height of the cross-section,

ℓ – the length of the beam), then the calculation error does not exceed 1%.

Thus, it can be assumed with an acceptable error that in flat transverse bending, **for low long beams**, normal stresses are distributed along the height of the beam according to a linear law and can be calculated using the well-known **Navier's formula:**

$$\sigma_x = \frac{M_z}{J_z} \cdot y.$$

The tangential stresses τ in the plane of the cross-section will be determined through the tangential stresses acting in the longitudinal sections. The formula for determining τ will be studied on the example of a cantilever beam of rectangular cross section (Fig. 11.12) loaded with a concentrated force at the free end.

Let us consider a beam element dx, bounded by two adjacent planes. In both cross-sections I and II according to the epure Q, transverse forces of the same magnitude act. The bending moments in these sections are different: M and $M + dM$ respectively (Fig. 11.12, a). Under the action of these forces, normal and tangential stresses occur in the sections.

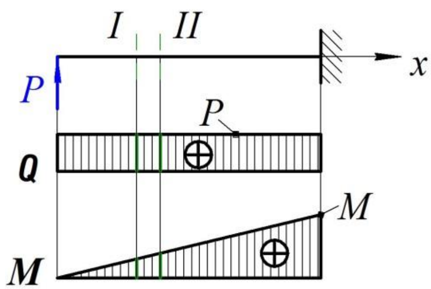


Figure 11.12 - Beam in transverse bending

The normal stresses in sections I and II are found using the Navier's formula (11.16). These stresses for an arbitrary layer of fibers will be equal, respectively:

$$\sigma' = \frac{M_z}{J_z} y; \qquad \sigma'' = \frac{M_z + dM_z}{J_z} y. \qquad (11.17)$$

The normal stress epures are shown in Fig. 11.13, a.

To find the tangential stresses, we formulate some assumptions about the nature of their distribution in the cross-section.

1. Tangential stresses τ in the cross section are parallel to the transverse force Q.

2. from the neutral layer, the tangential stresses are the same in magnitude over the entire width of the cross-section.

Note. These assumptions are valid only for sections with an aspect ratio $h/b > 2$, when the transverse force is parallel to the h side.

Next, cut off a part of the rod element in a plane parallel to the neutral layer of the beam at a distance y from it (Fig. 11.13, b). Consider the conditions of equilibrium of the elementary parallelepiped $A_1A_2B_1B_2C_1C_2D_1D_2$. To do this, we first analyse the forces acting in its edges.

The edges $A_1A_2B_1B_2$, $C_1C_2D_1D_2$ and $A_1A_2C_1C_2$ belong to the side surface of the rod that is not under load, so no forces act here.

The normal σ' and tangential τ' stresses act in the edge $A_1B_1C_1D_1$.

Let's find the equivalent of the normal stresses

$$N_1 = \int_F \sigma' dF.$$

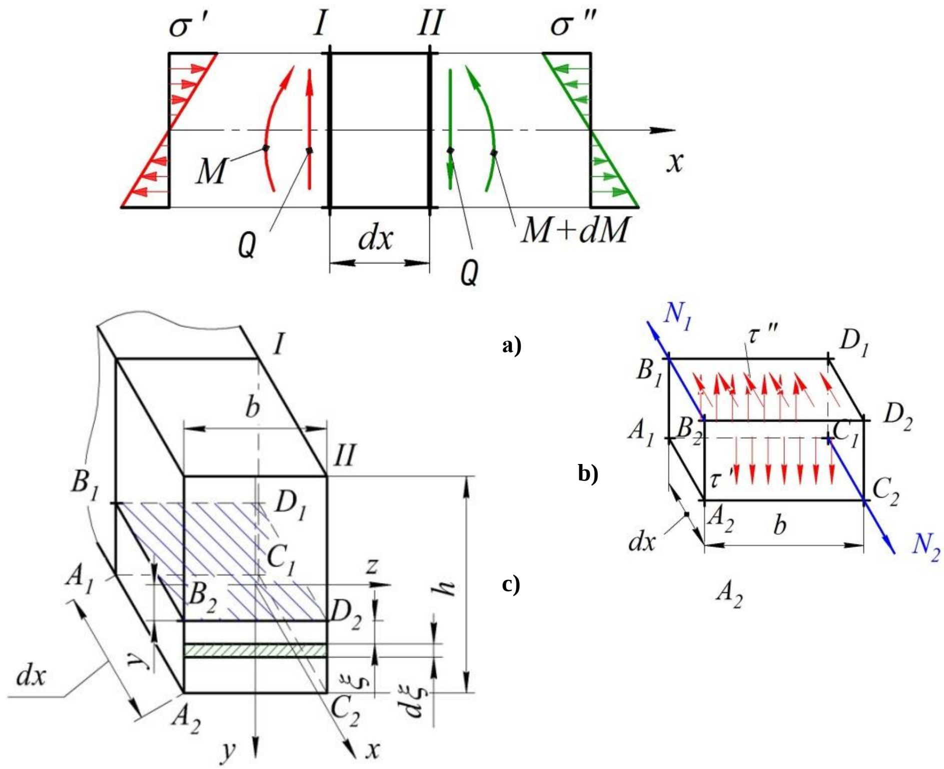
Here, the elementary site $dF = bdy$ is located at a distance $\boldsymbol{y}$ from the neutral line of the section (Fig. 11.17, b).

Then.

$$N_1 = \int_F \frac{M_z}{J_z} \mathrm{y} dF = \frac{M_z}{J_z} \int_F \mathrm{y} dF\,.$$

Here $\int_F \mathrm{y} dF = S_Z(y) -$ the static moment of the edge area $A_1B_1C_1D_1$ relative to $z - axis$, i.e. the part of the cross-sectional area located between the fibre layer at the **y**-level and the beam edge. Hence.

$$N_1 = \frac{M_z}{J_z} S_z(y). \tag{11.18}$$



a)

b)

c)

a) load scheme of the beam element;
b) scheme of separation of a part of a beam element;
c) loading scheme of a part of a beam element.
Figure 11.13 - Determination of tangential bending stresses

Similarly, let's find the equivalent N_2 on the edge $A_2B_2C_2D_2$:

$$N_2 = \frac{M_z + dM_z}{\mathcal{J}_z} S_z(y). \qquad (11.19)$$

Now let us consider the edge $B_1B_2D_1D_2$. We neglect the normal stresses on this edge, which arise due to the lateral pressure between the fibers when the beam is bent, because they are negligible. The tangential stresses τ'' arise here according to the law of parity of tangential stresses, since tangential stresses τ' act on orthogonal faces (Fig. 11.13, c). Due to the smallness of the edge $B_1B_2D_1D_2$ (one of its dimensions is) we consider the stresses τ'' to be uniformly distributed, and their equivalent

$$dT = \tau'' b dx = \tau b dx.$$

Let's write the equation of equilibrium of the element by projecting the forces on the x -axis:

$$\sum X_k = N_2 - N_1 - dT = 0$$

,

or

$$\frac{M_z + dM_z}{\mathcal{J}_z} S_z(y) - \frac{M_z}{\mathcal{J}_z} S_z(y) - \tau b dx = 0.$$

Hence

$$\tau b dx = \frac{dM_z}{\mathcal{J}_z} S_z(y).$$

Taking into account that $dMz/dx = Q$, we finally get:

$$\tau = \frac{QS_z(y)}{b\mathcal{J}_z}. \qquad (11.20)$$

Formula (11.20) was first derived by **D.I. Zhuravsky**.

For a rectangular section, the static moment is equal to

$$S_z(y) = b(0{,}5h - y)\left(y + \frac{0{,}5h - y}{2}\right) =$$
$$= b(0{,}5h - y)(0{,}5y + 0{,}25h) = \frac{bh^2}{8}\left(1 - \frac{4}{h^2}y^2\right).$$

Taking into account that for a rectangle

$$\mathcal{J}_z = \frac{bh^3}{12}$$

we get:

$$\tau = \frac{Q}{b}\frac{S_z(y)}{8}\left(1 - \frac{4}{h^2}y^2\right)\frac{12}{bh^3} = \frac{3}{2}\frac{Q}{bh}\left(1 - \frac{4}{h^2}y^2\right). \qquad (11.21)$$

From the above formula, it follows that the relationship between the tangential stresses in the cross-section under transverse bending and the position of the fiber layer relative to the neutral line is parabolic. At the extreme points of the section at $y = h/2$ $\tau = 0$. **The highest tangential stresses** will occur in the neutral layer when $y = 0$:

$$\tau_{max} = \frac{3}{2}\frac{Q}{bh} = \frac{3}{2}\frac{Q}{F}. \qquad (11.22)$$

The epure of the distribution of tangential stresses along the height of a rectangular section is shown in Fig. 11.14.

Although **Zhuravsky's formula** was derived for rectangular sections with a ratio of $h/b > 2$, it can be used in practice for sections of any shape, except for narrow rectangles arranged so that the force line is parallel to the smaller side of b. We will discuss such sections later.

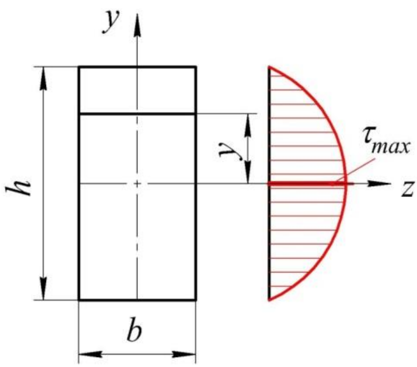


Figure 11.14 - Tangential stress epure for a rectangular section

Thus, for an arbitrary section, the **Zhuravsky formula** can be written in the following form:

$$\tau = \frac{QS_z(y)}{b(y)J_z}. \qquad (11.23)$$

Here $b(y)$ – the width of the section at the level where the tangential stresses are determined, and which will be a variable value for any section.

11.4.3 About the rational shape of the bending section

Analysing the stress epure, it can be seen that the normal stresses are zero along the longitudinal line, while the tangential stresses reach their maximum. On the contrary, in the outermost fibers, which are furthest from the longitudinal line, normal stresses reach the highest modulus values, and tangential stresses are zero. Calculation practice has shown that normal stresses are usually several times higher than tangential stresses. Therefore, it makes sense to design the section so that most of the material would be in the zone of high stresses. This requirement is met by cross-sections in the form of I-beams and channels rolling profiles, as well as various box and ring sections (Figure 11.15).

Figure 11.15 - Types of beam sections

It can be concluded that since the main strength to deflection is in the outer layers of the material, it is advisable to use beams with cross-sections in which the fibers of the base material are located at a distance from the neutral axis when bending.

The ratio $\varepsilon = \frac{W_z}{\sqrt{F^3}}$ can be called **the rationality factor of a bending section.** That is, the smaller the cross-sectional area (or weight) at the same strength, the more rational the cross-section will be.

This principle explains why machine levers, connecting rods and other parts, as well as rails, beams, machine bodies and frames in the plane of bending moment action, have a special cross-sectional shape in which the parts furthest from the neutral axis are reinforced. In general, these cross-sections resemble a board placed on an edge. Thus, in the case of steel beams, the industry has come to standardised cross-sections of rolled sections such as I-beams or channels.

Round hollow sections are also effective in bending applications. This explains the increasing use of these cross-sections for rotating parts that are also subject to bending forces (e.g. railway carriage axles).

11.5 Control questions

1. What kind of structures are called beams?
2. What are the three types of beam supports?
3. Types of beams.
4. What is called a console?
5. What is the intensity of the distributed load?
6. What is called plane pure bending?
7. What is called plane transverse bending?
8. What are the relationships between the intensity of the distributed load, bending moment and transverse force?
9. Rules of signs in bending.
10. Why are bending moments and transverse forces epures are drawn?
11. How does the neutral line pass in bending?
12. The principle of choosing a rational section.
13. Zhuravsky's formula.
14. Condition of strength in bending under normal stresses.
15. Condition of strength in bending under tangential stresses.

12. DETERMINATION OF BEAM DISPLACEMENT

12.1 Differential equation of the bent beam axis

In many cases, bending elements of building and engineering structures must be designed not only for strength but also for stiffness. In this case, it often happens that the required cross-sectional dimensions of a timber (beam), obtained from the stiffness calculation, are larger than those obtained from the strength condition.

By stiffness calculation, we mean the assessment of the elastic flexibility of a beam under the applied load and the selection of cross-sectional dimensions at which the displacement will not exceed the limits set by the standards.

The curved axis of the beam is called **the bent axis or elastic line**, and the displacement of the centre of gravity is called **the deflection of the beam** in a given section and is denoted by the letter $\boldsymbol{w}$.

In general**, the curvature of the bent axis of a beam** is (11.14):

$$\frac{1}{\rho_x} = \frac{M_x}{E\mathcal{J}_x}.$$

The following equation of curvature of a plane curve is known from the course of analytical geometry:

$$\frac{1}{\rho_x} = \pm \frac{\dfrac{d^2w}{dx^2}}{\left[1 \mp \left(\dfrac{dw}{dx}\right)^2\right]}.$$

Let's equate the right-hand sides of these equations. The value of the term $\left(\frac{dw}{dx}\right)^2$ is many times less than unity and can therefore be neglected. Then we obtain the basic **differential equation of the elastic line of the beam (for small deformations):**

$$\frac{dw^2}{dx^2} = \frac{M_x}{E\mathcal{I}_x}. \tag{12.1}$$

Equation (12.1) can be used to calculate linear and angular displacements in beams under any loading conditions.

Having integrated equation (12.1) for the first time, we have an expression for the angle of rotation θ_x:

$$\theta_x = \frac{dw}{dx} = \int \frac{M(x)}{E\mathcal{I}_x} dx + C, \tag{12.2}$$

which has one arbitrary constant C.

Integrating this equation for the second time, we find the expression for the deflection $w(x)$

$$w(x) = \int dx \int \frac{M(x)}{E\mathcal{I}_x} dx + C(x) + D, \tag{12.3}$$

where C, D – constant integration.

The values of these constants are determined from the conditions of beam anchorage:

a) if the beam has an anchorage at the end (Fig. 12.1), the deflection and angle of rotation at the anchorage are zero:

$$w_A = 0;\ \theta_A = 0;$$

b) for a beam on two hinged supports, the deflections on these supports are zero.

Note that if the distributed load $q(x)$ is given, the equation of the elastic line can be written in the form:

$$\frac{d^2}{dx^2}\left[E\mathcal{I}_x \frac{d^2 w(x)}{dx^2}\right] = q(x). \tag{12.4}$$

This method is useful when the beam has one section.

Note that if the beam has two or more sections (n), then it is necessary to write differential equations for each section and the number of constants will be twice or $2n$ times larger.

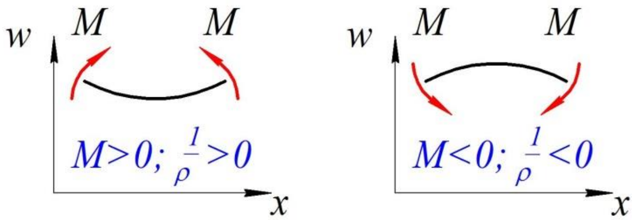


Figure 12.2 - Sign rule in a beam in bending

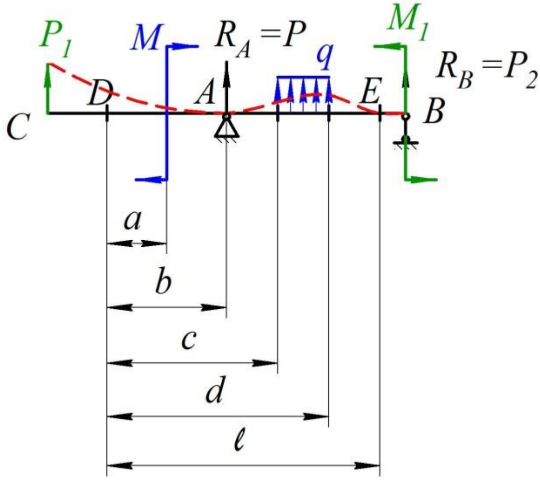
12.2 Determination of displacements in beams by the method of initial parameters

Let us look at a beam subjected to the main types of loads. The direction of the loads is chosen so that the bending moments in the cross-sections of the beam are positive (Fig. 12.3, a).

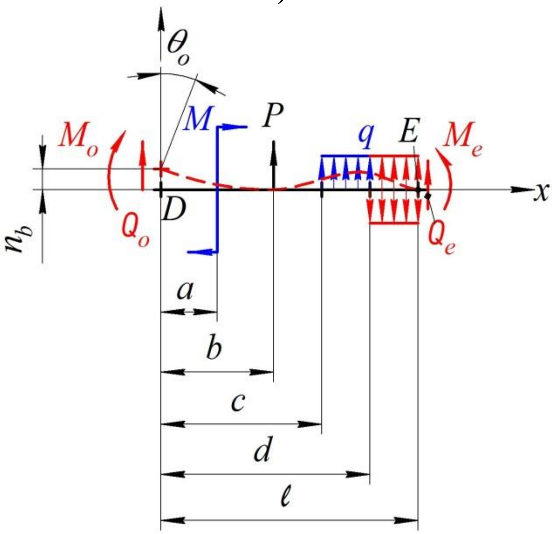
To reduce the number of integration constants to two, it is necessary that the constants in all sections of the beam are the same when integrating the elastic line equation. This is possible if the expressions for bending moments in each section contain all the terms that were included in the expressions in the previous section, and the additional terms that appear in this section disappear at the common boundary with the previous section. Such conditions can be ensured by writing the differential equations of the elastic line of the beam following a certain algorithm:

1. The beginning of the coordinate system is always chosen at the leftmost section on the beam axis. In this system, expressions for the bending moments in each section are made.

2. The expressions for bending moments are always obtained from the equilibrium conditions for the left side of the beam. That is, these expressions must include the loads applied to the beam to the left of the section.



a)



b)

Figure 12.3 - Beam loading scheme (a) and design scheme for the initial parameter method (b)

3. If a concentrated moment M, acts on the left side of the beam, then it should be represented in the equation as the product

$M(x-a)^0$, where $(x-a)^0 = 1$. Here a – the abscissa of the point of application of the moment M (see Fig. 12.3).

4. When the distributed load does not reach the end of the beam, it should be continued by applying a compensating distributed load of the opposite sign in this section. These additional distributed loads in Fig. 12.3, b) are highlighted in red.

5. The obtained differential equations of the elastic line are integrated over the beam sections without opening the brackets.

Let us look at a part of the beam of length ℓ, bounded by sections D and E (Fig. 12.3, b). Place the origin at point D and write the expressions for the bending moments in each section.

Section I:

$$M(x) = M_0 + Q_0 x.$$

Section II:

$$M(x) = M_0 + Q_0 x + M(x-a)^0.$$

Section III:

$$M(x) = M_0 + Q_0 x + M(x-a)^0 + P(x-b).$$

Section IV:

$$M(x) = M_0 + Q_0 x + M(x-a)^0 + P(x-b) + q\frac{(x-c)^2}{2}.$$

Section V:

$$M(x) = M_0 + Q_0 x + M(x-a)^0 + P(x-b) + \\ +q\frac{(x-c)^2}{2} - q\frac{(x-d)^2}{2}.$$

In each subsequent section, new effective loads are added to the moment expressions of the previous section. The expressions for the moments at the boundaries of each current and next section will be the same. For example, at the boundary of sections II i $(x = b)$, we get

$$M_{II} = M_{III} = M_0 + Q_0 x + M(x-a)^0.$$

Let's write the differential equations of the elastic line on each section, starting with the first one.

Section I:

$$\frac{d^2 w(x)}{dx^2} = \frac{1}{E\mathcal{J}_z}[M_0 + Q_0 x]. \qquad (12.5)$$

Integrating the equation twice, we get the following:

$$\theta(x) = \frac{dw(x)}{dx} = \frac{1}{E\mathcal{J}_z}\left[M_0 x + Q_0\frac{x^2}{2} + C_1\right]. \quad (12.6)$$

$$w(x) = E\mathcal{J}_z\frac{dw(x)}{dx} =$$
$$= \frac{1}{E\mathcal{J}_z}\left[M_0\frac{x^2}{2} + Q_0\frac{x^3}{6} + C_1 x + D_1\right]. \quad (12.7)$$

Section II:

$$\frac{d^2w(x)}{dx^2} = \frac{1}{E\mathcal{J}_z}[M_0 + Q_0 x + M(x-a)^0]. \quad (12.8)$$

$$\theta(x) = \frac{dw(x)}{dx} = \frac{1}{E\mathcal{J}_z}\left[M_0 x + Q_0\frac{x^2}{2} + M(x-a) + C_{II}\right]. \quad (12.9)$$

$$w(x) = \frac{1}{E\mathcal{J}_z}\left[M_0\frac{x^2}{2} + Q_0\frac{x^3}{6} + M\frac{(x-a)^2}{2} + C_{II}x + D_{II}\right]. (12.10)$$

Section III:

$$\frac{d^2w(x)}{dx^2} = \frac{1}{E\mathcal{J}_z}[M_0 + Q_0 x + M(x-a)^0 + F(x-b)]. \quad (12.11)$$

$$\theta(x) = \frac{dw(x)}{dx} = \frac{1}{E\mathcal{J}_z}[M_0 x + Q_0\frac{x^2}{2} + M(x-a) +$$

$$+P\frac{(x-b)^2}{2} + C_{III}]. \quad (12.12)$$

$$w(x) = \frac{1}{E\mathcal{J}_z}[M_0\frac{x^2}{2} + Q_0\frac{x^3}{6} +$$

$$+M\frac{(x-a)^2}{2} + F\frac{(x-b)^2}{2} + C_{III}x + D_{III}] \quad . \quad (12.13)$$

Section IV:

$$\frac{d^2w(x)}{dx^2} = \frac{1}{E\mathcal{J}_z}[M_0 + Q_0 x + M(x-a)^0 +$$

$$+P(x-b) + +q\frac{(x-c)^2}{2}]. \qquad (12.14)$$

$$\theta(x) = \frac{dw(x)}{dx} = \frac{1}{E\mathcal{J}_z}[M_0 x + Q_0\frac{x^2}{2} + M(x-a) +$$

$$+P\frac{(x-b)^2}{2} + q\frac{(x-c)^3}{6} + C_{IV}]. \qquad (12.15)$$

$$w(x) = \frac{1}{E\mathcal{J}_z}[M_0\frac{x^2}{2} + Q_0\frac{x^3}{6} +$$

$$+M\frac{(x-a)^2}{2} + P\frac{(x-b)^3}{6} + q\frac{(x-c)^4}{24} + C_{IV}x + D_{IV}] \;. \quad (12.16)$$

Putting $x = d$ for the neighbouring sections IV and V we get:

$$\theta_{IV}|_{x=d} = \frac{1}{E\mathcal{J}_z}[M_0 d + Q_0\frac{d^2}{2} + M(d-a) +$$

$$+P\frac{(d-b)^2}{2} + q\frac{(d-c)^3}{6} + C_{IV}] =$$

$$\theta_V|_{x=d} = \frac{1}{E\mathcal{J}_z}[M_0 d + Q_0\frac{d^2}{2} + M(d-a) +$$

$$+P\frac{(d-b)^2}{2} + q\frac{(d-c)^3}{6} - q\frac{(d-d)^3}{6} + C_V.$$

From this we get

$$C_{IV} = C_V. \qquad (12.17)$$

$$w_{IV}|_{x=d} = \frac{1}{E\mathcal{J}_z}[M_0\frac{d^2}{2} + Q_0\frac{d^3}{6} +$$

$$+M\frac{(d-a)^2}{2} + P\frac{(d-b)^3}{6} + q\frac{(d-c)^4}{24} + C_{IV}d + D_{IV}] =$$

$$w_V|_{x=d} = \frac{1}{E\mathcal{J}_z}[M_0\frac{d^2}{2} + Q_0\frac{d^3}{6} +$$

$$+M\frac{(d-a)^2}{2} + P\frac{(d-b)^3}{6} + q\frac{(d-c)^4}{24} + C_Vd + D_V].$$

Hence, taking into account (12.6),

$$D_{IV} = D_V. \qquad (12.18)$$

By performing similar operations for the boundaries of other sections, we are convinced that the corresponding arbitrary integration constants are equal in all sections:

$$C_I = C_{II} = C_{III} = C_{IV} = C_V \quad . \qquad (12.19)$$

$$D_I = D_{II} = D_{III} = D_{IV} = D_V \quad . \qquad (12.20)$$

The constants C and D can be found from equations (12.5) and (12.6) by setting : x=0:

$$\theta|_{x=0} = \theta_0 = \frac{C}{E\mathcal{J}_z}, \qquad w|_{x=0} = w_0 = \frac{D}{E\mathcal{J}_z}. \qquad (12.21)$$

In other words, the integration constants C and D are proportional to the rotation angle and deflection at the origin.

In general, **the equations for deflections and angles of rotation can be written as follows:**

$$w(x) = w_0 + \theta_0 x + \frac{1}{E\mathcal{J}_z}[M_0\frac{x^2}{2!} + Q_0\frac{x^3}{3!} +$$

$$+\sum M\frac{(x-a)^2}{2!}+\sum F\frac{(x-b)^3}{3!}+$$

$$+\sum q\frac{(x-c)^4}{4!}-\sum q\frac{(x-d)^4}{4!}\ ]. \qquad (12.22)$$

$$\theta(x)=\theta_0+\frac{1}{EJ_z}\Bigg[M_0\frac{x^2}{1!}+Q_0\frac{x^3}{2!}+\sum M\frac{(x-a)^2}{1!}+$$

$$+\sum F\frac{(x-b)^3}{2!}+\sum q\frac{(x-c)^4}{3!}-\sum q\frac{(x-d)^4}{3!}\Bigg]\ . \qquad (12.23)$$

Equation (12.22) is called the generalised (universal) equation of the elastic line of a beam, and equation (12.23) is called the generalised (universal) equation of the angles of rotation of the beam cross-sections.

The bending moment M_0 and transverse force Q_0, acting in the section of the beam coinciding with the beginning of the coordinates are called **static initial parameters**, and the deflection w_0 and the angle of rotation θ_0 in this cross-section are called geometric initial parameters.

The absolute value of the maximum deflection of the beam $|w_{max}|$ is denoted by f. Then the bending stiffness conditions take the form:

$$\theta \leq [\theta], \qquad (12.24)$$

$$f \leq [f], \qquad (12.25)$$

here $[\theta]$ – the permissible angle of rotation of the cross-section;

$[f]$ – permissible deflection arrow.

Remarks. In relation to the deflection epure as a graphical representation of a function, the epure of angles of rotation is a graphical representation of the first derivative, and the epure of bending moments is a graphical representation of the second derivative of this function.

12.3 Control questions

1. What is the deflection of a beam?
2. What curve does the axis of the rod take the form of in pure bending?
3. What is the relationship between the radius of curvature ρ, the bending moment M and the stiffness of the beam $E\mathcal{J}_x$?
4. The rule of signs in a beam in bending.
5. What are the static initial parameters?
6. Write an equation of equilibrium for a rod in pure bending.
7. What is the neutral layer of a rod in bending?
8. What type of stress state is realised in a rod in pure bending?
9. Write the Navier's formula for determining the normal stresses in a rod in bending.
10. Where is the dangerous point of the rod in pure bending?
11. What are the forces in a rod cross-section in plane transverse bending?
12. Write Zhuravsky's formula for determining the tangential stresses in a rod in bending.
13. Formulate the basic assumptions about the nature of the distribution of tangential stresses in the cross-section of a rod in transverse bending, formulated by Zhuravsky. For which sections are these assumptions valid?
14. Formulate the sign rule for displacements in rods in bending.
15. Write the basic differential equation of the elastic line of a rod.
16. Condition of strength in bending.
17. Condition of stiffness in bending.

13. COMPLEX STRENGTH

Earlier, we studied the simplest **types of deformation**: longitudinal tension and compression, shear, torsion and plane transverse bending.

In the general case, all six components of internal forces $- N$, Q_y, Q_x, M_y, M_z, M_x, connected with four simple deformations can act on a beam in cross-sections.

In practice, the simplest types of deformation are almost never found in their pure form. Most parts operate in conditions of complex strength. Structural elements are subject to several simple types of deformation. Thus, in the case of **complex strength, a combination of several internal forces can occur in the cross-section of a rod.**

If the displacements of the axis points in the bar are small compared to the transverse dimensions, and the cross-sectional rotations are small compared to unity, then **the principle of independence of forces** is applied to determine the total stresses, according to which it is necessary to determine the stresses from each component of internal forces separately using known formulas, and then add them up.

13.1 Oblique bending

Complex (nonplanar) bending is a type of loading in which loads act in several planes passing through the axis of the beam (Fig. 13.1). In complex bending, four internal force factors occur in the cross-sections of the beam: Q_y, Q_x, M_y, M_z. When calculating the strength in complex bending, the influence of tangential stresses is usually neglected.

If the loads act in a single plane that does not coincide with any of the main planes of inertia, this type of bending is called oblique.

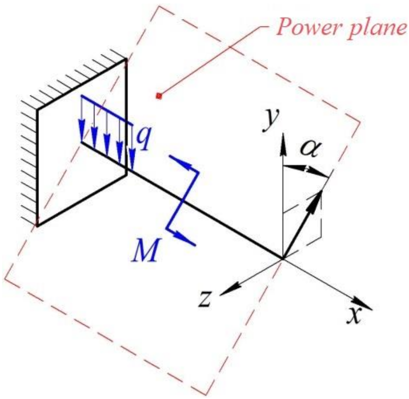


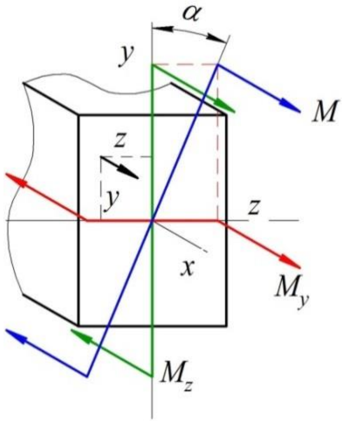
Figure 13.1 - Complex bending

In the case of a nonplanar bending, it is most convenient to reduce the bending to two planar bends. To do this, the load must be decomposed into components located in the principal planes xy and xz (here, the y and z axes are the principal axes of inertia of the section) (Fig. 1.2). Based on the principle of superposition, we find the stress at the point with coordinates, considering two plane bends from moments M_z and M_y. Then the normal stresses at the point will be:

$$\sigma' = \frac{M_z \cdot y}{\mathcal{J}_z} \quad and \quad \sigma'' = \frac{M_y \cdot z}{\mathcal{J}_y}.$$

Due to the fact that the stresses are of the same name, the normal stress is $\sigma = \sigma' + \sigma''$,

$$\sigma = \frac{M_z \cdot y}{\mathcal{J}_z} + \frac{M_y \cdot z}{\mathcal{J}_y}. \qquad (13.1)$$



a) b)

Figure 13.2 - Oblique bending

In the case of oblique bending, the moments M_z and M_y are related by dependencies:

$$M_z = M\cos\alpha; \quad M_y = M\sin\alpha\,. \tag{13.2}$$

Then, using formula (13.1), we have:

$$\sigma = \frac{M\cos\alpha \cdot y}{\mathcal{J}_z} + \frac{M\sin\alpha \cdot z}{\mathcal{J}_y} = M\left(\frac{\cos\alpha \cdot y}{\mathcal{J}_z} + \frac{\sin\alpha \cdot z}{\mathcal{J}_y}\right), \tag{13.3}$$

where M – is the bending moment in a given section in the $p - p$ force plane.

Formulas (13.1) and (13.3) allow us to determine the normal stresses at any point in the cross-section during complex or, as they say, **spatial bending.** The bending moment and the coordinates of the points at which the stresses are determined are substituted into the formulas with their signs.

The equation of the neutral line in the section is found by assuming $\sigma = 0$ and denoting the coordinates of the neutral line points (n.l.) by z_0 and y_0:

$$\frac{y_0}{\mathcal{J}_z}\cos\alpha + \frac{z_0}{\mathcal{J}_y}\sin\alpha = 0. \tag{13.4}$$

This equation is the equation of a line passing through the initial coordinate (the centre of gravity of the section), since it goes to zero at x_0= y_0 = 0.

The position of the neutral line is characterised by its angular coefficient

$$tg\varphi = \frac{y_0}{z_0} = -\frac{\mathcal{J}_z}{\mathcal{J}_y}\frac{M_y}{M_z}. \tag{13.5}$$

Analysis of the last formula shows that:

– in oblique bending, the neutral line is not perpendicular to the force plane. It passes through the centre of gravity of the section and through the II and IV quadrants at an angle of φ;

– the axis of the beam in oblique bending is distorted in the $n-n$ plane normal to the direction of the neutral line (see Fig. 13.2, b); this plane is called the **bending plane,**

– the direction of the bending plane ($tg\varphi$) can be perpendicular to the plane of external load ($tg\alpha$) only when the last coincides with one of the main planes of the beam, or when $\mathcal{J}_z = \mathcal{J}_y$ (circle, square, etc.); in general, the angle of inclination φ of the $n-n$ neutral line is not equal to the angle α of the force plane.

13.2 Calculations for strength and stiffness in oblique bending

Since the normal stress epure in the cross-section of the beam is linear, the maximum stresses occur at the point furthest from the neutral line. Let the coordinates of this point be (z_1, y_1). Then from equation (13.1) we obtain:

$$\sigma_{max} = \frac{M_z \cdot y_1}{\mathcal{J}_z} + \frac{M_y \cdot z_1}{\mathcal{J}_y}. \tag{13.6}$$

When the section is symmetrical about both axes, the determination of the maximum stresses is much simpler. For example, for a rectangular section, the maximum stresses will always be at the vertices of the rectangle, and they are easy to write down:

$$\sigma_{max} = \frac{M_z}{\mathcal{J}_z} y_{max} + \frac{M_y}{\mathcal{J}_y} z_{max}.$$

or

$$\sigma_{max} = \frac{M_z}{W_z} + \frac{M_y}{W_y}, \tag{13.7}$$

where

$W_z = \frac{\mathcal{J}_z}{y_{max}}$; $W_y = \frac{\mathcal{J}_y}{z_{max}}$ – are the strength moments of the section relative to the z and y.

When determining the dimensions of the section from **the condition of strength in oblique bending** by the formula

$$\sigma_{max} = \frac{M_z}{W_z} + \frac{M_y}{W_y} \le [\sigma] \tag{13.8}$$

the unknowns W_z and W_y cannot be found from the same equation. Therefore, it is necessary to determine the ratio $\frac{W_z}{W_y}$ by successive attempts that satisfy condition (1.8). In the case of a rectangular section

$$\frac{W_z}{W_y} = \frac{h}{b}.$$

Therefore, given the ratio, the value of W_z and the dimensions of the cross-section can be found from condition (13.8).

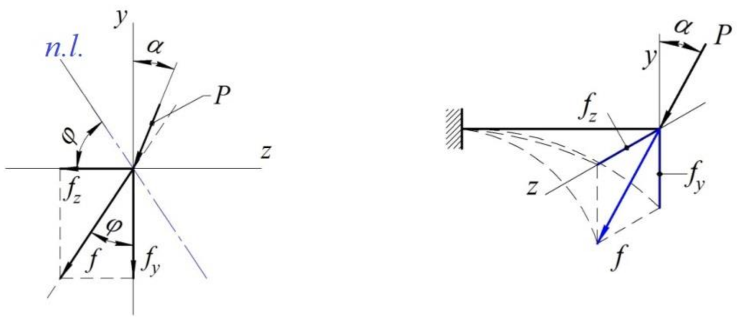
To determine the deflections in different sections of a beam under oblique bending, we use the method of superposition, i.e., the addition of acting forces, in the same way as when determining stresses. Denoting the deflection in the directions of the principal axes z and y by w and v, we write the differential equations of deflections in the x_z and z_y planes:

$$E\mathcal{J}_y \cdot \frac{d^2v}{dx^2} = M_y; \quad E\mathcal{J}_z \cdot \frac{d^2w}{dx^2} = M_z \,.$$

Total deflection f (Рис.13.3) is defined as the geometric sum of deflections w and v $(f_z = v;\ f_y = w)$

$$f = \sqrt{f_z^2 + f_y^2} = \sqrt{v^2 + w^2}\,. \qquad (13.9)$$

The deflection direction f in oblique bending in each section coincides with the bending plane perpendicular to the neutral line (n.l.) in that section.



a) **b)**

Figure 13.3 - Deflections in oblique bending

13.3 Compound bending and tension of a straight bar

Calculations for the combined action of bending and tension can be reduced to two types:

a) calculations for longitudinal-transverse loads;

b) calculations for off-centre tension (compression).

If a beam is subjected to longitudinal and transverse loads that bend the axis of the beam, bending moments M_y and M_z, transverse forces Q_y, Q_x, and a longitudinal force N occur in the cross sections (Fig. 13.4).

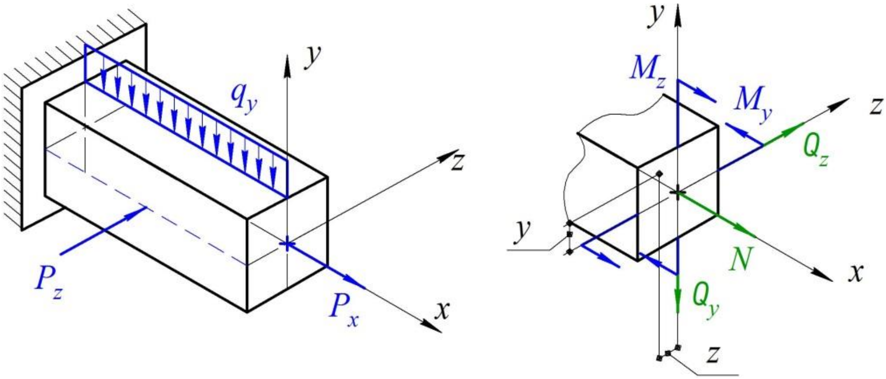


Figure 1.4 - Load diagram and internal force factors in tensile bending of a straight bar

Normal stresses at an arbitrary point of the section under such a load

$$\sigma = \frac{N}{F} + \frac{M_z}{J_z} y + \frac{M_y}{J_y} z. \tag{13.10}$$

The bending moments, longitudinal force and coordinates of the point at which the stress is determined are substituted into this formula with due regard to the signs.

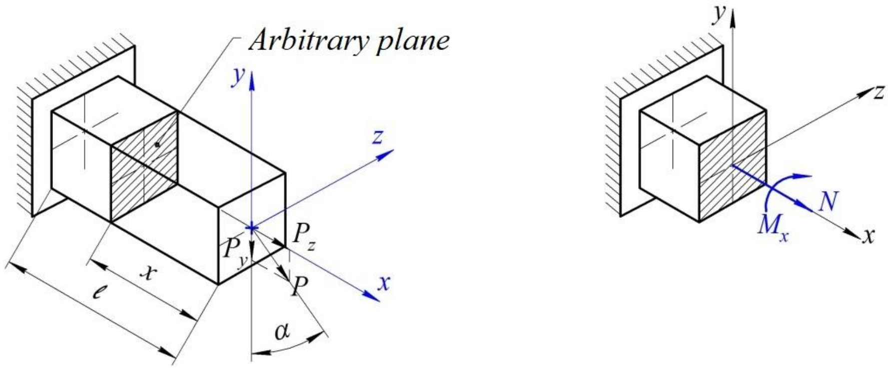
Neglecting the tangential stresses from the transverse forces, we can assume that the stress state at the danger point is linear.

Thus, **the strength condition** has a simple form:

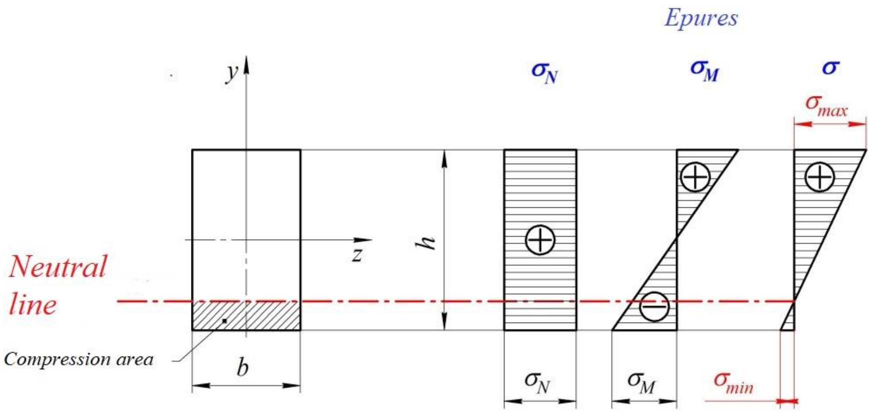
$$\sigma_{max} \leq [\sigma]. \tag{13.11}$$

If the section has two axes of symmetry, then one of the corner points will be dangerous and the stresses in it are determined by the formula:

$$\sigma = \frac{N}{F} + \frac{M_z}{W_z} + \frac{M_y}{W_y}. \tag{13.12}$$



a)



b)

Figure 13.5 - Bending and tension of a straight bar

Remarks. In compression bending, formulas (13.10) and (13.12) can be used only for short rods of high stiffness - for thin long rods, a loss of stability is possible.

In the case of plane bending with tension in the principal plane xOy formula (13.12) is simplified:

$$\sigma = \frac{N}{F} \pm \frac{M_z}{W_z}. \tag{13.13}$$

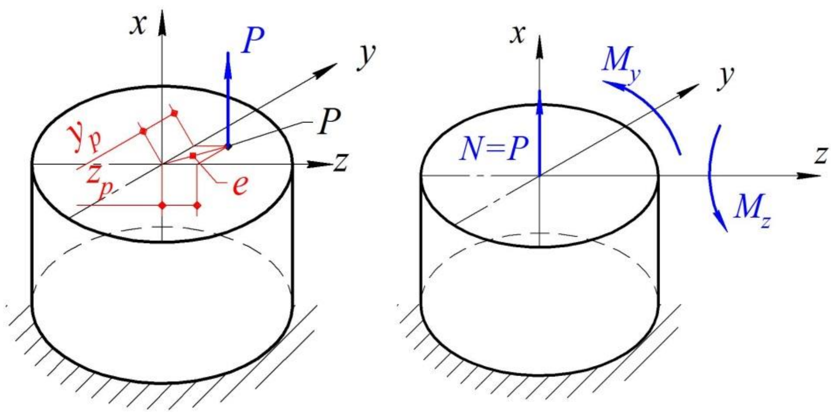
13.4 Off-centre tension and compression of a straight bar

The type of loading in which the equal external force does not coincide with the axis of the rod, but is displaced relative to its axis and remains parallel to it, is called off-centre tension or compression.

The point of application of the equal force P is called the pole of the force.

A bar of arbitrary cross-section is subject to a single force P, which is parallel to the axis of the bar and applied at point P of the cross-section. The point of application of external forces has coordinates z_p, y_p. Under such a load, a longitudinal force N = P and bending moments act in any cross-section of the bar (see Fig. 13.6, b):

$$M_z = Py_p; \quad M_y = Pz_p.$$



a) **b)**

Figure 13.6 - Off-centre tension (compression) axes

Thus, the off-centre tensile-compressive behaviour is identical to oblique bending. In contrast to the last one, not only bending moments, but also a longitudinal force $N = P$ occur in the cross-section of the beam.

At an arbitrary point with coordinates zy the normal stresses σ consist of axial tensile (compressive) stresses with force N and pure bending stresses of moments M_z,M_y see formula (13.10).

Obviously, the spatial epure forms a plane (since the z and y coordinates in the equation are included in the first degree) that does not pass through the centre of gravity of the section (since at $z = 0$ and $y = 0$ $\sigma \neq 0$).

Substituting their values into equation (13.10) instead of N, M_z, M_y we obtain:

$$\sigma = \frac{P}{F} + \frac{P \cdot y_P}{\mathcal{J}_z} y + \frac{P \cdot z_P}{\mathcal{J}_y} z. \qquad (13.14)$$

$\frac{P}{F}$ is put outside the brackets, and the moments of inertia are expressed in terms of radii of inertia

$$\mathcal{J}_z = i_z^2 F; \quad \mathcal{J}_y = i_y^2 F.$$

Let's get the formula for determining normal stresses at an arbitrary point in the section:

$$\sigma = \frac{P}{F}\left(1 + \frac{y_P}{i_z^2} y + \frac{z_P}{i_y^2} z\right). \qquad (13.15)$$

The variable components in formulas (13.14) and (13.15) are the last two terms, which reflect the effect of bending. Since the greatest stresses in bending will be at the points furthest from the neutral line, we need to find the position of the neutral line, just as in oblique bending.

Let's denote the coordinates of the points belonging to the neutral line by z_0 and y_0. On the neutral line $\sigma = 0$, i.e.

$$\sigma = \frac{P}{F}\left(1 + \frac{y_P . y_0}{i_z^2} + \frac{z_P \cdot z_0}{i_y^2}\right) = 0.$$

Since

$$\frac{P}{F} \neq 0, then \quad 1 + \frac{y_P . y_0}{i_z^2} + \frac{z_P \cdot z_0}{i_y^2} = 0. \qquad (13.16)$$

where z_P, y_P – are the coordinates of the points of application of the equal external force;

z_0, y_0 – coordinates of the neutral line points.

Equation (13.16) is **the equation of the neutral line,** which shows that the neutral line does not pass through the centre of gravity of the section. To construct this line, it is easier to determine the segments that are cut off by it on the coordinate axes. Let's denote these segments by z_n and y_n (Fig. 13.7).

To find the line segment z_n, which is cut off on the Ox axis, in equation (13.16), take $z_o = z_n, y_o = 0$.

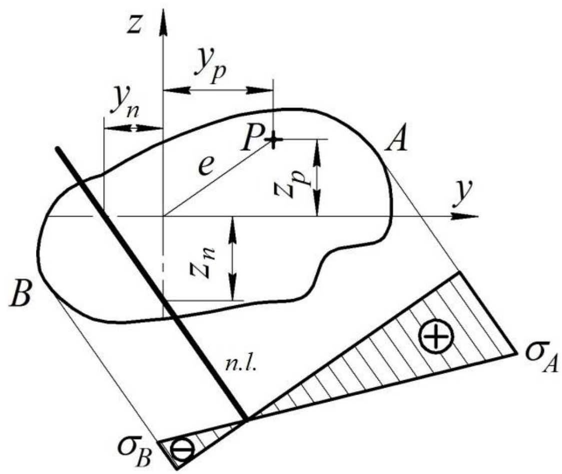


Figure 13.7 - Determining the location of the neutral line

Then we will get

$$1 + \frac{z_P \cdot z_n}{i_y^2} = 0.$$

Where

$$z_n = -\frac{i_y^2}{z_P}. \qquad (13.17)$$

Similarly, taking $z_0 = 0, y_0 = y_n$, we obtain

$$y_n = -\frac{i_z^2}{y_P}. \qquad (13.18)$$

It is clear from expressions (13.17) and (13.18) that if y_p and z_p are positive, then the segments z_n and y_n will be negative, i.e. **the neutral line is always located in the quadrant opposite to the one in which the point of application of the external force is located.**

If **the pole is located on one of the major axes, the neutral line is perpendicular to it.**

If **the pole approaches the centre of gravity of the section, the neutral line moves away from it.**

Now, by drawing tangents to the section contour parallel to the neutral line, we find the most stressed points A and B in the stretched and compressed zones of the section and construct a normal stress diagram σ (see Fig.13.7).

The stresses at these points and **the strength conditions** are as follows:

$$\sigma_A = \sigma_{\substack{max \\ strech}} = P\left(\frac{1}{F} + \frac{y_P . y_A}{i_z^2} + \frac{z_P \cdot z_A}{i_y^2}\right) \le [\sigma_+], \qquad (13.19)$$

$$\sigma_B = \sigma_{\substack{min \\ comp}} = P\left(\frac{1}{F} + \frac{y_P . y_B}{i_z^2} + \frac{z_P \cdot z_B}{i_y^2}\right) \le [\sigma_-]. \qquad (13.20)$$

For cross-sections in which both principal axes of inertia are axes of symmetry (rectangle, I-beam, etc.), the z, y coordinates of the vertex points simultaneously reach their maximum values. Therefore, formulas (13.19) and (13.20) can be written in the form:

$$\sigma max = \frac{P}{F} + \frac{M_z}{W_z} + \frac{M_y}{W_y} \le [\sigma]. \qquad (13.21)$$

Section core

In general, the neutral line can pass both through and beyond the cross section. Indeed, if the force P is applied at the centre of gravity ($z_p =$

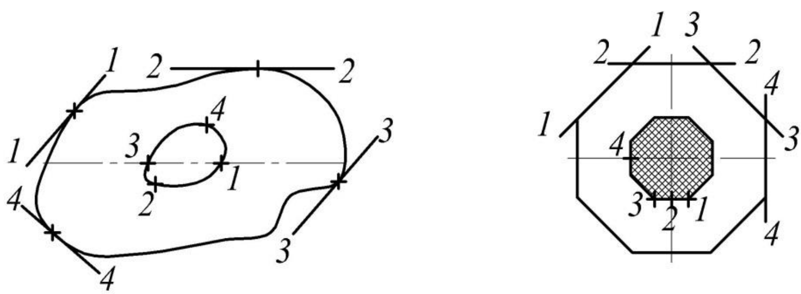
$y_p = 0$), then, according to formulas (13.17) and (13.18), the neutral line passes to infinity, and the stress in this case is distributed evenly across the cross-section As the **eccentricity** (distance from the centre of gravity of the cross-section to the point of application of **the longitudinal force**) increases, the neutral axis ***e*** approaches the centre of gravity of the cross-section. It is desirable for the designer to know in advance what eccentricity can be allowed for the selected type of section without risking causing stresses of different signs in the section of the rod. This is important to know when designing rods made of materials that work differently in tension and compression. It is necessary to determine the area of such distances of the force P from the axis at which the normal stress curve in the cross-section will remain with the same sign. It is desirable to achieve that the entire section works in compression.

The core of a section is the area around its centre of gravity, where the application of a force *P* inside which causes a stress of the same sign.

To construct the section core, it is necessary to set different positions of the neutral line and calculate the corresponding points of application of the force P using the formulas:

$$z_{\mathrm{P}} = -\frac{i_y^2}{z_n}; \;\; y_{\mathrm{P}} = -\frac{i_z^2}{y_n}. \tag{13.22}$$

The calculated coordinates z_p, y_p define the points that lie on the boundary of the section core (Figure 13.13).



a) **b)**

Figure 13.8 - Construction of the section core

When the neutral line is rotated around a fixed point on the section contour, the point of application of the force moves along a curve, since z_p, y_p and $z,$ y are linearly related (see Figure 13.8, a). To construct the core of a section of a polygon, it is necessary to draw tangents that coincide with its sides. The more tangents you draw, the more accurately the core area will be drawn. The core of the section will follow the shape of the cross section (see Fig. 13.8, b).

Let's construct a section core for a **rectangle** (Fig. 13.9) with sides b i h. First, draw a neutral line along one of the sides of the rectangle (position $I - I$). The coordinates of the neutral line are equal to

$$z_N = -\frac{b}{2};\ y_N = \infty,$$

and considering that

$$i_z^2 = \frac{\mathcal{J}_z}{F} = \frac{bh^3}{12bh} = \frac{h^2}{12}; \quad i_y^2 = \frac{\mathcal{J}_y}{F} = \frac{hb^3}{12bh} = \frac{b^2}{12},$$

from formulas (1.9) we obtain

$\mathrm{y_P} = 0;\ z_\mathrm{P} = -\frac{b^2 \cdot 2}{12 \cdot b} = \frac{b}{6}\ (point\ 1'\ ).$ Now draw a neutral line through the other side (position $II - II$). The coordinates of the neutral line at this position are

$$\mathrm{y_H} = \infty;\ z_\mathrm{H} = \frac{h}{2}.$$

Then the coordinates of the point 2' of the section core

$$\mathrm{y}_P = -\frac{h^2 \cdot 2}{12 \cdot h} = -\frac{h}{6};\ z_P = 0.$$

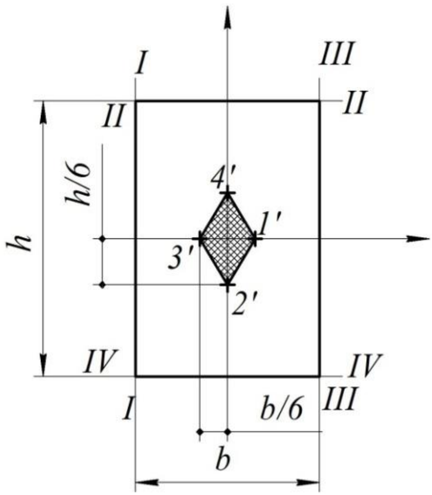


Figure 13.9 - Building a core of a rectangular section

Similarly, determine the coordinates of points 3' and 4'.

Since the neutral line rotates around the corner point of the section as it passes from one side to the other, the point of application of the force moves in a straight line, forming the contour of the core. Thus, **the core of a rectangular section will be a rhombus with diagonals equal to one third of the corresponding side of the section.**

13.5 Determination of torsional bending stresses

When torsion was considered, it was assumed that a torque was generated in the cross-sections of a circular rod (shaft). However, machine parts such as shafts rarely operate in pure torsion. Even a straight shaft is bent during operation by its own weight, the weight of pulleys, belt tension, etc. Thus, most torsional elements of machines operate under combined bending and torsional forces.

Under the action of **bending and torsion**, five internal force factors arise in the cross-sections of the shaft: torsional moment M_{tor}, bending moments M_z and M_y, and transverse forces Q_z and Q_y.

Thus, in any cross-section, normal bending stresses in two planes and torsional stresses occur simultaneously. To calculate the shaft, first of all, the epures of bending moments M_z, M_y and torsional moments $M_{tor} = M_x$ must be drawn.

The load acting on the shaft is decomposed into components along the coordinate axes, and then the epures are drawn:

- bending moments M_z from the vertical projections of forces P_{1y}, P_{2y}, P_{ny};
- bending moments M_y from horizontal projections of forces P_{1z}, P_{2z}, P_{nz};
– torsional moments, that $M_{tor} = M_x$.

The total bending moment M_{bend} is defined as the geometric sum of both components:

$$M_{bend} = \sqrt{M_z^2 + M_y^2}.$$

If the rod (shaft) is **not of circular cross-section**, then for each section we have its own bending moment plane, but in this example, since the shaft has a circular cross-section, in which the strength moments are the same relative to all central axes, we can combine the bending moment planes for the axes of the sections and build the total M_{bend} epure, placing it in the drawing plane, without affecting the calculation results. Since the total moment in different sections may have different directions, even in the absence of distributed loads, the M_{bend} epure, may be curved.

For the general case, this can be easily shown analytically. Then $M_z = a + by$, $M_y = c + dz$ (where a, b, c *and* d – are constant coefficients).

Then $$M_{bend} = \sqrt{M_z^2 + M_y^2} = \sqrt{(a + bz)^2 + (c + dz)^2}.$$

The expression below the radical is only in some cases a square (for example, when $c = d = 0$), and in most cases the epure is curved. Bending moment values M_{bend}, are calculated only for sections in which the M_z *and* M_y epures change sign to the opposite. These values are placed on one side and connected by a concave parabola. Next, we draw the torque epure M_{tor} and find the dangerous sections that combine the relative extremes of

M_{bend} and M_{tor}. The dangerous sections may be sections 1, 2 and 3. Now the dangerous points can be found in the dangerous section. Obviously, points A and B can be dangerous (Fig. 1.10) (points furthest from the neutral line, whose positions are easy to find, since $\varphi = \alpha$, and the neutral line is n-n perpendicular to the force line)

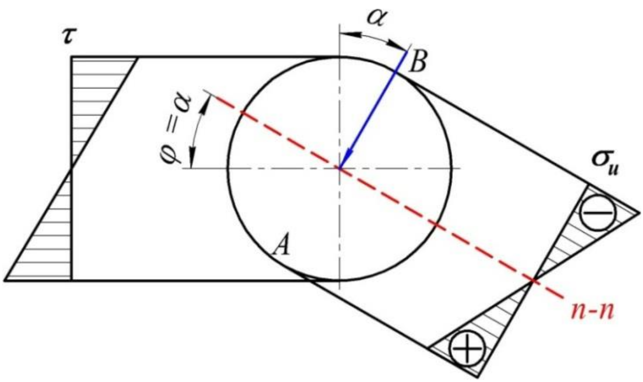


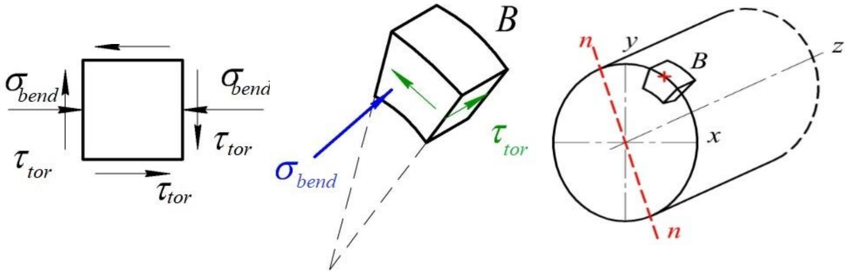
Figure 13.10 - Stress epure in a shaft cross-section under torsional bending

We draw epures of σ_i from the bending moment, which vary in proportion to the distance of the points from the neutral line. At points A *and* B normal bending stresses and torsional stresses have the highest values at the same time ($\tau$from bending at these points is zero and in general $\tau_{bend} << \tau_{tor}$).

$$\sigma_{bend} = \frac{M_{tor}}{W} = \frac{\sqrt{M_z^2 + M_y^2}}{W}; \qquad \tau_{bend} = \frac{M_{tor}}{W_\rho}. \quad (13.23)$$

Let's select an elementary particle of material (Figure 13.11) around the most dangerous point (for example, point B).

Four edges are subjected to tangential stresses, and two of them are subjected to normal stresses, while the other two edges are completely free of stress (Fig. 13.11, b). Thus, in torsional bending, the element is in a plane stress state at the danger point.



a) b) c)

Figure 13.11 - Stress at the hazardous point

Therefore, the principal stresses here must be determined using the same formulas as for bending:

$$\sigma_1 = \frac{1}{2}\left[\sigma + \sqrt{\sigma^2 + 4\tau^2}\right];\ \sigma_2 = 0;\ \ \sigma_3 = \frac{1}{2}\left[\sigma - \sqrt{\sigma^2 + 4\tau^2}\right].$$

The only difference between the formulas for transverse bending and torsional bending is that in the latter case the tangential stresses are caused by torque, while in bending they are caused by a transverse force. To check the strength of a shaft, we must determine the equivalent (reduced) stresses according to the relevant strength theory and compare them with the permissible stresses. Since shafts are usually made of ductile materials, the third and fourth theories of strength can be used.

$$\sigma_{equi}^{\mathrm{III}} = \sqrt{\sigma_{bend}^2 + 4\tau_{tor}^2} \le [\sigma];$$

$$\sigma_{equi}^{IV} = \sqrt{\sigma_{bend}^2 + 3\tau_{tor}^2} \le [\sigma]. \qquad (13.24)$$

Let's write the stresses σ and τ_{tor} in terms of bending and torsional moments:

$$\sigma_{bend} = \frac{M_{bend}}{W};\ \tau_{tor} = \frac{M_{tor}}{W_\rho} = \frac{M_{tor}}{2W}.$$

Substituting them into the strength theory, we obtain the third strength theory:

$$\sigma_{equi}^{\text{III}} = \sqrt{\frac{M_{bend}^2}{W^2} + 4\frac{M_{tor}^2}{(2W\quad)^2}} \le [\sigma].$$

From

$$\sigma_{equi}^{\text{III}} = \frac{\sqrt{M_{bend}^2 + M_{tor}^2}}{W} \le [\sigma]. \qquad (13.25)$$

According to the fourth theory of strength (energy theory)

$$\sigma_{equi}^{IV} = \sqrt{\frac{M_{bend}^2}{W^2} + 3\frac{M_{tor}^2}{(2W\quad)^2}} \le [\sigma];$$

or

$$\sigma_{equi}^{IV} = \frac{\sqrt{M_{bend}^2 + 0{,}75M_{tor}^2}}{W} \le [\sigma]. \qquad (13.26)$$

Formulas (13.25) and (13.26) are identical in structure to formula (13.23), so the strength test of a round shaft for the combined action of torsion and bending can be written in the form:

$$\sigma = \frac{M_{str}}{W} \le [\sigma], \qquad (13.27)$$

where the given moments equivalent to the action of three moments are equal to

$$M_{str}^{\text{III}} = \sqrt{M_{bend}^2 + M_{tor}^2} = \sqrt{M_z^2 + M_y^2 + M_{tor}^2} \qquad (13.28)$$

or

$$M_{str}^{IV} = \sqrt{M_{bend}^2 + 0{,}75M_{tor}^2} = \sqrt{M_z^2 + M_y^2 + 0{,}75M_{tor}^2}. \quad (13.29)$$

With the strength condition, **the design calculation or selection of the shaft cross-section can be carried out:**

$$W \quad \leq \frac{M_{str}}{[\sigma]}.$$

Since

$$W \quad = \frac{\pi d^3}{3r} = 0{,}1d^3$$

we get

$$d \geq \sqrt[3]{\frac{M_{str}}{0{,}1[\sigma]}}.$$

13.6 Calculations for strength and stiffness of shafts in torsional bending

According to the strength condition, the largest tangential stresses must not exceed the permissible ones, i.e.

$$\tau_{max} = \frac{M_{str}}{W_\rho} \leq [\tau].$$

From this, given a known torque and permissible stress, the required cross-sectional strength torque can be determined, and then the required radius or shaft diameter can be determined, i.e. the design calculation can be performed:

$$W_\rho \leq \frac{M_{str}}{[\tau]}, where\ [\tau] = (0{,}5 \ldots 0{,}6)[\sigma].$$

For solid sections $W_\rho = 0{,}2D^3$.

Source

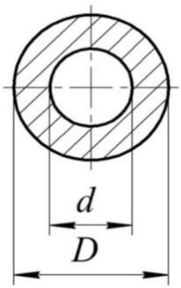
$$D \geq \sqrt[3]{\frac{M_{tor}}{0{,}2\ [\tau]}}.$$

For a hollow shaft, it should be remembered that, given the ratio

$$\frac{d}{D} = \alpha$$

(Fig. 1.12), we can obtain

$$W_\rho = \frac{\pi D^3}{16}(1 - \alpha^4) = 0{,}2D^3(1 - \alpha^4).$$



Substituting the expression W_ρ into the formula for $D,$ we obtain:

$$D \geq \sqrt[3]{\frac{M_{tor}}{0{,}2\,[\tau](1 - \alpha^4)}}.$$

Figure 13.12 - Section of a hollow shaft of the hollow shaft

According to the stiffness condition, **the maximum relative twist angle** must not exceed the permissible angle, i.e:

$$\theta_{max} = \frac{M_{str}}{G\mathcal{J}_\rho} \leq [\theta].$$

Where the polar moment of inertia comes from:

$$\mathcal{J}_\rho \geq \frac{M_{str}}{G[\theta]}.$$

Since for solid circular sections $\mathcal{J}_\rho = 0{,}1D^4$, then

$$D \geq \sqrt[4]{\frac{M_{tor}}{0{,}1G[\theta]}}.$$

13.7 Longitudinal and transverse bending

In practice, it is not uncommon for a bar to be subjected to combined bending and tensile or compressive forces. Such deformation may be caused by the combined action of longitudinal and transverse forces on the beam, or by longitudinal forces alone.

The first case is shown in Fig. 13.13. A beam AB is subjected to a uniformly distributed load with intensity q and longitudinal compressive forces P. Assume that the deflections of the beam compared to the dimensions of the cross section can be neglected; then, with a practical accuracy of $\pm 5\%$ it can be assumed that even after deformation, the forces P cause only axial compression of the beam.

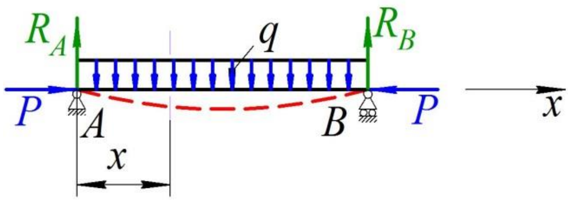


Figure 13.13 - A beam is simultaneously loaded with a distributed load q and longitudinal compressive forces P

Using the principle of superposition, we can find the normal stress at each point of any cross-section of the beam as the algebraic sum of the stresses caused by the forces P and the load q.

The compressive stresses σ_P due to the forces P distributed over the cross-sectional area F and are the same for all cross-sections:

$$\sigma_P = -\frac{P}{F};$$

are the normal bending stresses in the vertical plane in the section with the abscissa x, which is counted from the left end of the beam and expressed by the formula:

$$\sigma_q = \frac{M(x)_z}{\mathcal{J}_y}.$$

Thus, the total stress at the point with the x-coordinate (counting from the neutral axis) for this section is equal:

$$\sigma = \sigma_P + \sigma_q = -\frac{P}{F} + \frac{M(x)_z}{\mathcal{J}_y}.$$

Fig. 13.14 shows the epures of stress distribution in the considered section from the forces P, load q and the total epure.

The highest stress in this section will be in the upper fibres, where both types of deformation cause compression; the lower fibres may have compression or tension, depending on the numerical values of the stresses σ_P *and* σ_q. To write the strength condition, you need to find the largest normal stress.

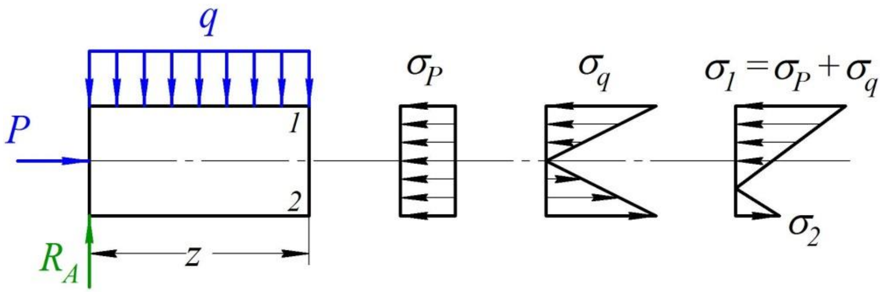


Figure 13.14 - Stress distribution epure in the considered section

Since the stresses due to the forces P are the same in all sections and are evenly distributed, the fibres that are most stressed by bending will be dangerous. These are the outermost fibres in the cross-section with the highest bending moment for them:

$$\sigma_{q\,max} = \pm\frac{M_{max}}{W}.$$

Thus, the stresses in the outermost fibres 1 and 2 (Fig. 13.14) for the middle section of the beam are expressed by the formula:

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = -\frac{P}{F} \pm \frac{M_{max}}{W}, \tag{13.30}$$

and **the calculated stress** will be equal to

$$|\sigma_{max}| = |\sigma_1| = \left|\frac{P}{F} \pm \frac{M_{max}}{W}\right|. \qquad (13.31)$$

If the forces P were tensile, the sign of the first term would be reversed, and the lower fibres of the beam would become dangerous.

Let us denote by the letter N the compressive or tensile force, and write down **the general formula for checking strength**

$$\sigma_{max} = \pm\left[\frac{N}{F} \pm \frac{M_{max}}{W}\right] \leq [\sigma]. \qquad (13.32)$$

When drawing up formula (13.32), we assume that the cross-section is symmetrical from the neutral axis and that the material has the same tension and compression strength (material strength hypothesis).

13.8 Control questions

1. What is called complex strength?
2. In what case is bending called oblique?
3. What is the stress tensor?
4. What type of loading is called off-centre tension or compression?
5. Does the neutral line in oblique bending pass through the centre of gravity of the cross section?
6. In what cases is it necessary to determine the core of the section?
7. What force factors occur in torsional bending?
8. The condition of strength in oblique bending.
9. Condition of strength in bending with tension.
10. Determination of stresses at dangerous angular points.
11. Condition of strength in off-centre tension-compression.
12. Strength condition in bending with torsion (III and IV theories of strength).
13. Strength condition for longitudinal-transverse bending.

14. GENERAL METHODS FOR DETERMINING DISPLACEMENTS

One of the most important tasks of material strength is to estimate the stiffness of a structure, i.e. the degree of its deformation under the influence of loads, stresses, and temperature changes. To solve this problem, it is necessary to be able to determine the displacements (linear and angular) of an elastic system (beam, frame, curved rod, truss, etc.) loaded in an arbitrary way.

The solution of this problem is necessary not only to determine the magnitude of the displacements themselves and to assess the stiffness of the structure. Based on the determination of displacements, general methods for determining internal force factors in statically indeterminate systems are created.

The determination of displacements is also necessary when studying the oscillations of elastic systems.

14.1 The concept of potential strain energy

Displacement determination methods are based on two basic principles of mechanics: **the principle of possible displacements and the law of conservation of energy.**

According to **the law of conservation of energy**, the work of external forces does not disappear, but is transformed into potential energy accumulated in an elastic body. Thus, the amount of accumulated potential strain energy is determined by the amount of work of external forces. This energy is expressed in the form of work performed during unloading by internal forces.

Let us consider the process of deformation of an elastic body from the energy point of view. External forces applied to an elastic body perform certain work. We will denote it by A. This work is partially converted into the potential energy U of the body's deformation, and partially used to impart velocity to the body's mass, i.e., it is converted into kinetic energy K.

The energy balance looks like this

$$A = U + K.$$

If the loading is carried out slowly and the rate of movement of the body masses is very small, then $K = 0$. Such a loading process can be considered static. The body is in a state of equilibrium at any given time. In this case, $A = U$ all the work of external forces is converted into potential strain energy.

When the body is unloaded, its internal forces perform work at the expense of the potential strain energy of the body. Thus, an elastic body is an "accumulator" of energy (e.g., a watch spring, a leaf spring, etc.).

Thus, it can be said that the complete conversion of one type of energy into another takes place if the deformation occurs without disturbing the equilibrium of the system. The measure of energy converted to another type is the amount of work done by the forces acting on the structure.

Let us denote the amount of accumulated potential energy by U, and the decrease in the potential energy of external loads by U_p . Then the value of U_p *i*s measured by the positive work of these loads A_p. On the other hand, the accumulation of potential energy of deformation corresponds to the negative work of internal forces ($-A$), since the movement of body points during deformation occurs in the opposite direction to the internal forces.

The law of conservation of energy during deformation of elastic systems takes the form

$$U_{\mathrm{p}} = U. \tag{14.1}$$

Replacing the values of U_p and U in this expression with the numerically equal values of A_p i $-A$, we get another formulation of this law

$$A_p = -A. \tag{14.2}$$

It follows from equation (14.1) that the potential energy of deformation U is numerically equal to the work of external forces A_p, performed by them during this deformation, i.e.

$$U = A_p. \tag{14.3}$$

Table 14.1 - Potential energy for simple types of load

Tension-compression	$U = \int_\ell \frac{N^2 d\ell}{2EF}.$ (2.4)
Pure displacement	$U = \int_\ell \frac{Q^2 dx}{2GF}.$ (2.5)
Torsion	$U_{tor} = \int_\ell \frac{M_{tor}^2 d\varphi}{2G\mathcal{J}_\rho}.$ (2.6)
Pure bending	$U_{bend} = \int_\ell \frac{M^2\, d\varphi}{2E\mathcal{J}_z}.$ (2.7)
Spring bending	$U_{spr.bend.} = \int_\ell \frac{2P^2R^3n}{Gr^4}.$ (2.8)

14.2 The theorem on the reciprocity of work and displacements (Castiliano's theorem)

K. Castiliano proposed a method for determining displacement based on the determination of the potential strain energy.

His results can be interpreted more widely: **the displacement of the point of application of a generalised force in the direction of its action is equal to the partial derivative of the potential strain energy under this force:**

$$\Delta_P = \frac{\partial U}{\partial P}, \tag{14.9}$$

where Δ_p – is the total displacement (linear displacement or angle of rotation) due to the generalised force;

P – the generalised force (force or moment of a pair of forces).

This conclusion is called **the Castiliano theorem**, which was published in 1875.

According to the method proposed by Castiliano, to determine the linear or angular displacement at a point where there is no force according to the condition of the problem, the corresponding dummy generalised force should be applied at this point. Then, having written an expression for the potential energy from a system of forces, including the specified dummy force, take the derivative of the dummy force and apply a **dummy load** of zero to the resulting expression for displacement. Despite the complexity of this method and the availability of other methods, Castiliano's theorem is used to determine displacements in non-core systems (plates, shells, and parts with all three dimensions of the same order).

The Castiliano method is also convenient to use in cases where it is necessary to determine the displacement at the point of application of a force.

14.3 Determination of displacements by Mohr's method

The determination of displacements using Castiliano's theorem has an obvious drawback: it allows you to determine the displacements of only the points where forces are applied and only in the direction of action of these forces. In practice, it is often necessary to determine the displacements of any points in the system in any direction.

The Maxwell-Mohr method for determining displacements is a universal method, valid, unlike the analytical method discussed above, for both beams and any rod systems. This method is used to calculate the displacements of arbitrarily loaded beams with any cross-sectional shape, both with a straight and curved axis.

If it is necessary to determine the displacement at a point where no external force is applied, we apply an external fictitious force $\bar{P}$ in the direction of interest at this point. Next, we make an expression of the potential energy of the system, taking into account the fictitious force. We find the displacement of this point in the direction of interest, i.e. in the direction of the applied force $\bar{P}$. Now it remains to "remember" that there is no real force. Thus, the desired displacement is determined.

Consider a beam loaded with given forces. The forces in any section are denoted by M_P, Q_P, N_P. Suppose you need to determine the displacement (in general) of any point in the direction 1. Let's introduce an auxiliary stage: we load the given system with only one unit (generalised) force $\bar{P} = 1$, applied at the point where we need to find the displacement Δ_{1P}. The forces in the auxiliary state caused by this force are denoted as force factors from the unit load $\overline{M}_{,}$, $\overline{Q}_{,}$, $\overline{N}_{,}$.

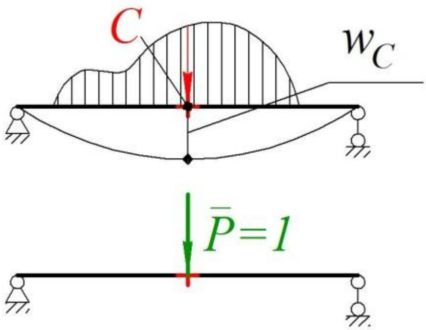


Figure 14.1 - Main and auxiliary load diagrams of a beam

According to the displacement reciprocity theorem, **the potential strain energy** is equal:

$$U_{1-2} = 1 \cdot w_C \ , \tag{14.10}$$

that is, it is equal to the work of the force to move the point from one state to another state.

The deflection at point C will be equal to

$$w_C = \int_{\ell} \frac{M_P \cdot \overline{M_1} \cdot dx}{E\mathcal{J}_z} \ . \tag{14.11}$$

In general, the load of a displacement system using the Mohr method can be determined by the formula:

$$\Delta = \sum \int_{\ell} \frac{N_x \cdot \overline{N_x} \cdot dx}{EF} + \int_{\ell} \frac{M_x \cdot \overline{M}_x \cdot dx}{EF} + \int_{\ell} \frac{M_y \cdot \overline{M}_y \cdot dx}{EF} +$$

$$+\int_{\ell} \frac{M_z \cdot \bar{M}_z \cdot dx}{EF}, \qquad (14.12)$$

where $N_{x,}\, M_x,\ M_y,\ M_z$ – are the values of the forces of a given beam in an arbitrary section under a given load.

$\bar{N}_{x,}\ \bar{M}_x, \overline{M_y}, \overline{M_z}$– are the values of internal forces in the auxiliary beam from the applied unit load.

Note that the unit load is applied only in the section where the displacement is to be determined.

If the problem requires determining a linear displacement, then the unit load will be the force $\bar{P} = 1$.

If it is necessary to determine the angular displacement (angle of rotation), then the unit load will be a concentrated moment $\bar{M} = 1$.

If, in addition to the mechanical load, a temperature load is applied to the system, the Mohr integral has the form:

$$\Delta = \sum \int_{\ell} \bar{N}_x \ell_x dx + \int_{\ell} \bar{M}_z \theta_x dx +$$

$$+\int_{\ell} \bar{M}_y \theta_y dx + \int_{\ell} EdF, \qquad (14.13)$$

where ℓ_x – the relative linear deformation of the system;

θ_x – the relative angular deformation in the xz area;

θ_y – is the relative angular deformation or change in the curvature of the rod in the yx area;

$$\theta_x = \frac{M_z + \int\limits_F \delta E_y dF}{\int\limits_F E y^2 dF},$$

where $\delta = \beta\Delta t\ell \pm \Delta i$ – linear temperature and installation displacements;

β – coefficient of linear expansion of the material;

Δt – temperature difference;

ℓ – the length of the rod;

Δi – he difference in rod manufacturing inaccuracy.

If $\delta = 0$, i.e. there are no temperature loads, then

$$\ell_z = \frac{N_z}{EF};$$

$$\theta_z = \frac{1}{\rho_z} = \frac{M_z}{E\mathcal{J}_z};$$

$$\theta_y = \frac{1}{\rho_y} = \frac{M_y}{E\mathcal{J}_y}.$$

The procedure for determining displacements using Mohr's method.

1. Let's consider a loaded system and determine the support reactions.

2. Consider each section of the system and write down the analytical expressions of bending moments from a given load (M_z) for each section.

3. Apply to the unloaded system at the point where it is necessary to determine the displacement, a unit force (when determining the linear displacement) or a unit moment (when determining the angle of rotation) in the direction of the desired displacement.

4. For each section of the system, write an analytical expression for the bending moment from the unit force factor.

5. Calculate the integrals for the corresponding moment expressions for each section. Summarise the results for the entire structure.

If the result is positive, it means that the direction of displacement coincides with the direction of the unit force. A negative sign indicates that the actual direction of the desired displacement is opposite to the direction of the unit force.

14.4 Determination of displacements by the Vereshchagin method

In many cases, Mohr integration can be avoided and other methods of determining displacement can be used. One of these methods is the Simpson method, but you can also determine the displacement using the **Vereshchagin** method. This method was proposed by **A.K. Vereshchagin** in 1924, when he was a student.

Let's consider the sequence of actions according to the Vereshchagin rule.The initial stage is the same as according to the Mohr method, i.e., first, a load diagram is built from the acting loads (actual state), then we consider the beam in the auxiliary state.

The auxiliary state is obtained in the same way as by Mohr's method: first, all the specified load must be removed, then a "unit force factor" must be applied at the point where the displacement is to be determined and in the direction of this desired displacement. A unit moment diagram or a unit load diagram is constructed.

The displacement is then calculated using the formula:

$$\Delta = \frac{1}{E\mathcal{J}_z}\sum \omega_i \cdot \bar{y}_{C_i}, \qquad (14.14)$$

where ω_i – area of the load epure;

$\bar{y}_{C_i}$ – the ordinate of a unit diagram (necessarily straight) taken under the centre of gravity of the load diagram;

$E\mathcal{J}_z$ – the stiffness of the section.

It should be noted that this method can be used only if **two conditions** are met: the stiffness of the beam in this section must be constant ($E\mathcal{J}_z$ =const), and one of the two moment epures in this section (load or unit) must be linear. At the same time, both epures must not have a fracture within this section.

Let's prove this formula.

Let the load epure M_p be arbitrary and the unit epure be linear (since a unit load is a concentrated force or a pair of forces, the unit epure $\bar{M}$ is bounded by straight lines) (Fig. 14.2).

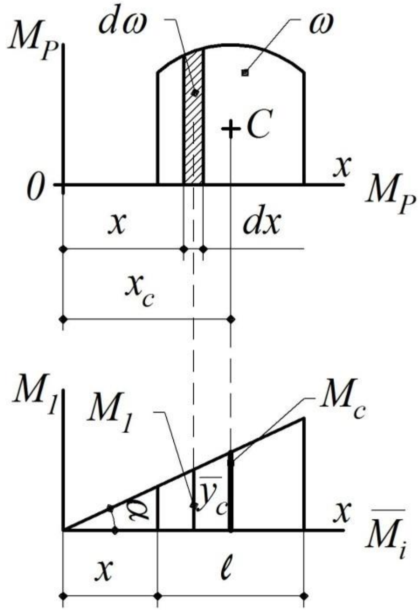


Figure 14.2 - Load and unit load epures

Let us denote by ω he area of the epure M_p, and by $\bar{y}_C$ – the ordinate of the epure from a unit load under the centre of gravity C of the load epure.

Obviously, $M_p dx = d\omega$ −is the differential of the area of the M_p epure.

$$\overline{M} = x \cdot tg\alpha.$$

The Mohr's integral will have the form

$$\int_\ell M_P \cdot \overline{M} \cdot dx = tg\alpha \int_\ell x d\omega.$$

The integral in the right-hand side is the static moment of the area of the epure M_p, relative to the y-axis, it is equal to

$$\int_{\ell} x d\omega = x \cdot X_C,$$

where X_C – is the abscissa of the centre of gravity of the epure M_p. Since

$$tg\alpha \cdot X_C \cdot \omega = \omega \cdot \bar{M}_c \,, \qquad \text{integral} \int_{\ell} M_P \cdot \bar{M} \cdot dx = \omega \cdot \bar{y}_C \;.$$

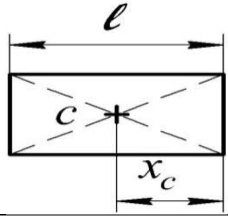
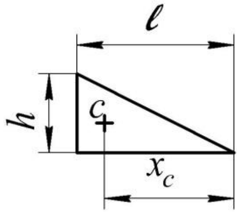
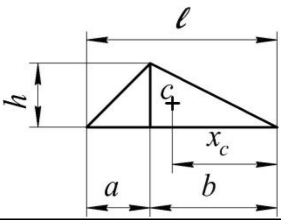
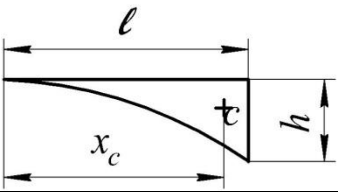
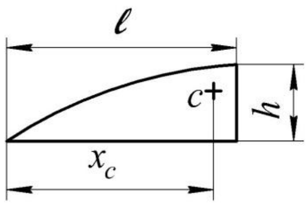
Thus, we obtain formula 14.13.

In other words, **the displacement is equal to the product of the area of the epure from the external load and the ordinate of the rectilinear epure from the unit load located under the centre of gravity of the epure figure from the specified external load.** Calculations according to formula (14.12) are performed by sections. If the epure M_P consists of epures from several loads, then it should be divided into simple shapes for which it is easy to determine the area and position of the centre of gravity. In this case, each of the areas should be multiplied by the ordinate of the single epure under the centre of gravity of the corresponding area.

If the epures from a given load and a single load are opposite in sign, their product has a minus sign, which confirms that the displacement occurred in the direction opposite to the single load. If a given beam is loaded by several forces, it is more advantageous to use the principle of independence of forces, i.e., to plot the diagrams for each load separately.

To facilitate the calculation of areas and ordinates, Table 14.2 is provided.

Table 14.2 - Area values and coordinates of centres of gravity for the most commonly used types of unit epures

View of a single epure	ω	X_C
	$h\ell$	$\dfrac{\ell}{2}$
	$\dfrac{1}{2}h\ell$	$\dfrac{2}{3}\ell$
	$\dfrac{1}{2}h\ell$	$\dfrac{1}{3}(\ell + a)$
	$\dfrac{1}{3}h\ell$	$\dfrac{3}{4}\ell$
	$\dfrac{2}{3}h\ell$	$\dfrac{5}{8}\ell$

If all the epures in the beam sections are straight (there is no distributed load), it is more advantageous to use the **M.V. Karnaukhov formula.** Consider an example when the load epure has the shape of a trapezoid (Fig. 14.3).

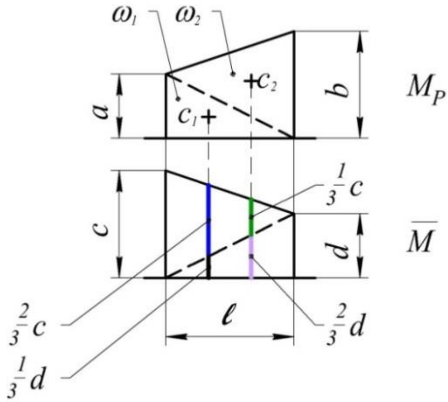


Figure 14.3 - Determination of displacement using the Karnaukhov formula

$$\omega_1 = \frac{1}{2}a\ell; \omega_{\mathbf{2}} = \frac{1}{2}b\ell.$$

Then the sum is

$$\omega\bar{y}_C = \frac{a\ell}{2}\left(\frac{2}{3}c + \frac{d}{3}\right) + \frac{b\ell}{2}\left(\frac{2}{3}d + \frac{c}{3}\right) = \frac{2a\ell c}{6} + \frac{a\ell d}{6} + \frac{b\ell c}{6} =$$
$$= \frac{\ell}{6}(2ac + 2bd + ad + bc).$$

If there are several sections on the beam that satisfy these conditions, the formula for determining displacements takes the form:

$$\Delta = \sum_{i=1}^{n} \frac{\ell_i}{6E\mathcal{J}_z}\left(a \cdot \bar{a} + 4 \cdot c \cdot \bar{c} + b \cdot \bar{b}\right). \qquad (14.15)$$

The obtained dependence is called **the rule of trapezoids** or **the Simpson-Karnaukhov formula.**

14.5 Control questions

1. When is Castiliano's theorem used?
2. The formula for displacement by Mohr's method.
3. Vereshchagin's formula for determining displacements.
4. Two conditions for using the Vereshchagin method.
5. Conditions for using the Karnaukhov formula.
6. What is displacement? Types of displacements.

15. STRESS-STRAIN STATE OF FLAT CURVED BARS OF LARGE CURVATURE

Bars with a curved axis are often used in various structures. These include hooks, chain rings, pulley and wheel rims, arches, etc. The axes of these bars are flat curves.

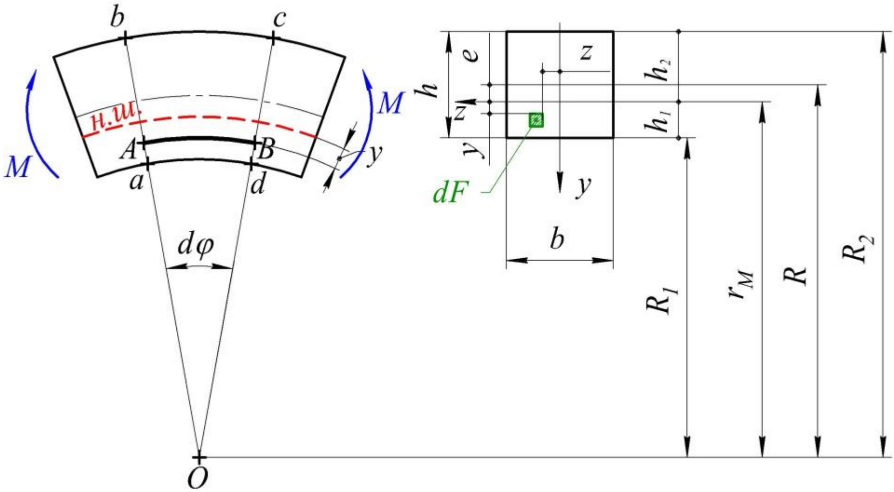
Studies show that in bending, the distribution of normal stresses in the cross-section of a curved beam differs significantly from that of a beam with a straight axis. Other things being equal, this difference is greater, the greater the ratio of the height $\boldsymbol{h}$ of the cross-section to the radius R of curvature of its axis (Fig. 15.1).

In this regard, there is a distinction between beams of different curvature:

if $h/R < 1/5$ – these are **beams of low curvature;**

if $h/R \geq 1/5$ – these are **bars of high curvature.**

When bending beams of small curvature, normal stresses can be determined using the derivation formulas for beams with a straight axis (the error is $\sim 2\%$).



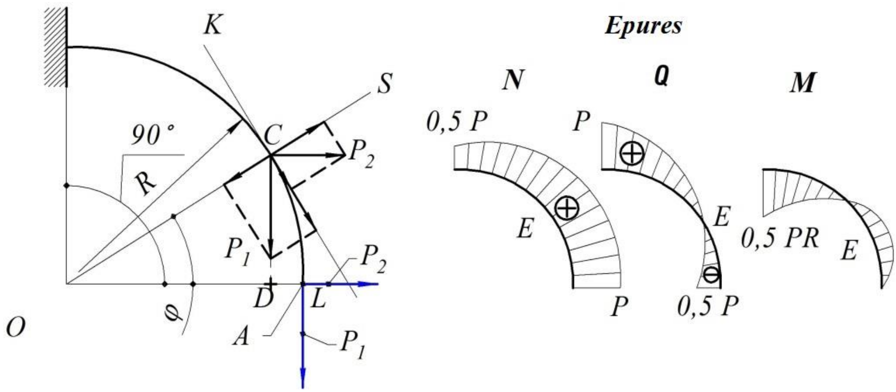
a) b)

Figure 15.1 - Flat beam with large curvature

15.1 Internal force epures in curved bars

In the cross-sections of a flat curved beam, there are generally three internal force factors – N, Q and M. In practice, we deal with rods whose axis is defined along the arc of a circle. In this case, it makes sense to switch to the polar coordinate system, so that the longitudinal, transverse force and bending moment will be functions of the angle φ: $N(\varphi)$, $Q(\varphi)$, $M(\varphi)$.

As an example, let us consider a flat curved beam (Fig. 15.2, a). Let's write the expressions $N(\varphi)$, $Q(\varphi)$, $M(\varphi)$ for an arbitrary section C.



a) **b)**

Figure 15.2 - Internal force epures

To get the expression for finding $N(\varphi)$ we need to determine the projections of the forces P_1 and P_2 on the tangent KL. For convenience, let's move the projections to the point C (Fig. 15.2, a). Then

$$N(\varphi) = P_1 cos\varphi + P_2 sin\varphi.$$

To obtain the expression for finding $Q(\varphi)$, we need to find the projections of the forces applied on the arc section AC onto the section plane, i.e., the SO direction.

$$Q(\varphi) = P_1 sin\varphi - P_2 cos\varphi.$$

When formulating the expression for the bending moment, we agree to **consider the bending moment to be positive when it increases the curvature of the rod.** We have

$$M_C(\varphi) = P_1 \cdot AD - P_2 \cdot CD =$$
$$= P_1 R(1 - cos\varphi) - P_2 R sin\varphi.$$

The obtained formulas allow us to construct the epures of N, Q and M. We assume $P_1 = P$ and $P_2 = 0{,}5P$. Then:

$$N(\varphi) = (cos\varphi + 0{,}5 sin\varphi)P;$$

$$Q(\varphi) = (sin\varphi - 0{,}5 cos\varphi)P; \qquad (15.1)$$

$$M_C(\varphi) = PR(1 - cos\varphi - 0{,}5 sin\varphi).$$

Using formulas (15.1), we determine the values of N, Q and M in the cross-section of the rod through 10° and construct the N, Q and M plots to scale (Fig. 15.2, b).

15.2 Determination of stresses in flat curved bars

Let us consider the case of pure bending of a curved beam (Fig. 15.1). Assume that the neutral layer has an unknown **radius of curvature** r_u, which is generally different from the radius R of the bar axis. Let's derive the formulas for the stresses σ in bars of large curvature.

When deriving these formulas, it is assumed that:

1) the curved beam is flat (i.e., its axis is a curve located in the same plane);

2) the beam is symmetrical about the plane in which its axis is located, and external forces act in this plane;

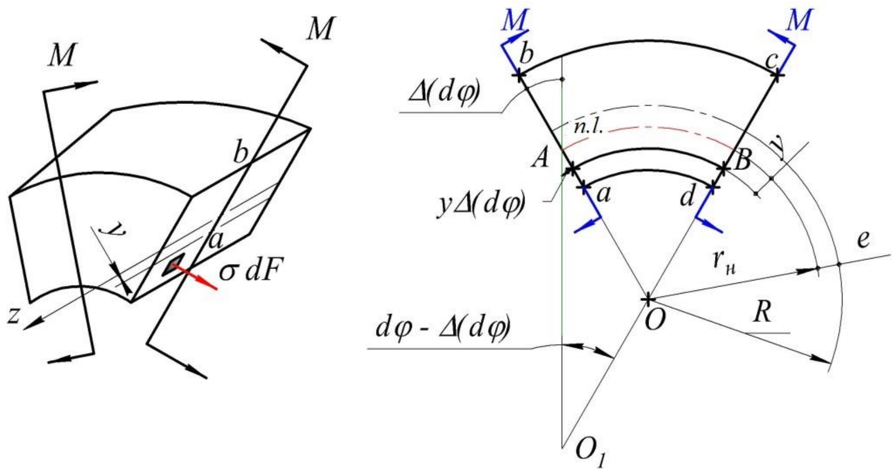
3) the cross-sections of the beam, which are flat before deformation, remain flat after deformation (flat section hypothesis);

4) the pressure of longitudinal fibres of the beam on each other does not significantly affect the distribution of stresses in the beam, and therefore it can be ignored.

To derive the equations of the static aspect of the problem, we divide the beam into two parts by any cross section *ab* (Fig. 15.1, a), and select an area element *dF* in the cross section at a distance *y* from the neutral line (Figs. 15.1, b and 15.3, a). The element is subjected to a force σdF. From the conditions of symmetry at *N=0*, and

$M_z = M$, we have:

$$\int_F \sigma \cdot dF = 0; \int_F \sigma \cdot y \cdot dF = M. \qquad (15.2)$$



a) b)

Figure 15.3 - Static and geometric aspects of the problem

The condition of equality of the moment M_y to zero

$$M_y = \int_F \sigma \cdot z \cdot dF = 0 \qquad (15.3)$$

is performed automatically due to the symmetry of the cross-section with respect to the *y*-axis.

Let us consider **the geometrical aspect of the problem.**

The relative elongation of an arbitrarily selected fibre *AB*, located at a distance *y* from the neutral layer, which received an elongation $y\Delta d\varphi$ as a result of deformation, is equal to

$$\varepsilon = \frac{y\Delta(d\varphi)}{(r_u - y)d\varphi}, \tag{15.4}$$

where $(r_u - y)d\varphi$ – the length of the element before deformation.

The physical aspect, as for a beam, can be expressed by Hooke's formula:

$$\sigma = \varepsilon \mathrm{E} = \frac{E\Delta(d\varphi)}{d\varphi} \cdot \frac{y}{r_u - y}. \tag{15.5}$$

Condition (3.3) can be rewritten as

$$\int_F \sigma \cdot dF = \frac{E\Delta(d\varphi)}{d\varphi} \cdot \int_F \frac{ydF}{r_u - y} = 0.$$

Since here

$$\frac{E\Delta(d\varphi)}{d\varphi} \neq 0,$$

then

$$\int_F \frac{ydF}{r_u - y} = 0. \tag{15.6}$$

From (15.2) we find

$$\int_F \sigma \cdot y \cdot dF = \frac{E\Delta(d\varphi)}{d\varphi} \int_F \frac{y^2 dF}{r_u - y} = M. \tag{15.7}$$

The integral in (3.7) can be written as follows:

$$\int_F \frac{y^2 dF}{r_u - y} = \int_F \frac{y^2 + r_u y - r_u y}{r_u - y} dF =$$

$$= -\int_F (y - \frac{r_u y}{r_u - y}) dF = -\int_F y dF + r_u \int_F \frac{y}{r_u - y} dF. \quad (15.8)$$

The first integral in (15.8) is the static moment S_z of the cross-sectional area relative to the neutral z-axis, and the second integral, according to (15.6), is zero. Then we write expression (15.8) as follows:

$$\int_F \frac{y^2 dF}{r_u - y} = -S_z = -(-e) dF = e \cdot dF, \quad (15.9)$$

where e – the distance from the centre of gravity of the curved beam section to the neutral axis;

F – the cross-sectional area of the beam.

Substitute (15.9) into (15.7) and obtain

$$\frac{E\Delta(d\varphi)}{d\varphi} \cdot eF = M,$$

whence

$$\frac{E\Delta(d\varphi)}{d\varphi} = \frac{M}{eF}\,. \quad (15.10)$$

Taking into account expression (15.10), formula (15.5) for determining the stresses can now be written as follows:

$$\sigma = \frac{M \cdot y}{eE(r_u - y)} \quad (15.11)$$

or

$$\sigma = \frac{M \cdot y}{S_z(r_u - y)},$$

where M – the bending moment in the section;

S_z – the static moment of the cross-sectional area of the curved beam relative to the neutral line.

The analysis of formula (15.11) shows that the normal stress along the width of the section is the same (independent of z) and changes only with the distance of the point from the neutral line:

The highest modulus stresses will be at the extreme points of the section located near the concave surface of the beam.

The maximum stress values will be at the extreme points of the curved bar and are calculated by the following formulas:

$$\sigma_1 = \frac{Mh_1}{F \cdot e \cdot R_1}; \ \ \sigma_2 = \frac{Mh_2}{F \cdot e \cdot R_2}, \tag{15.12}$$

where R_1 and R_2 – are the radii of curvature of the inner and outer layers of the fibres of the curved beam;

h_1 and h_2 – distances from the neutral line to these fibres (Fig. 15.1).

The signs of stresses are easily determined by the direction of the bending moment in the cross-section.

If a curved bar is subjected to an axial force N, then in addition to the stress caused by the bending moment, normal stresses will act in the bar

$$\sigma_{st} = \frac{N}{F}.$$

To determine the stresses in a curved beam under bending using formulas (15.11) and (15.12), it is necessary to first calculate the value of e (distance from the neutral layer to the centre of gravity) or the radius r_n – of the neutral layer, since

$$e = R - r_n, \tag{15.13}$$

where $R -$ the radius of the layer containing the centres of gravity of the cross-sections of the curved beam (Fig. 15.1).

The radius r_n will be determined from equation (15.6):

$$\int_F \frac{y \cdot dF}{r_n - y} = 0.$$

Let's make the following substitution here:

$$r = r_n - y, \text{ or } y = r_n - r.$$

Then equation (3.6) can be rewritten as follows:

$$\int_F \frac{r_n - r}{r} dF = 0, \text{or } r_n \cdot \int_F \frac{dF}{r} - F = 0,$$

whence

$$r_n = \frac{F}{\int_F \frac{dF}{r}}. \qquad (15.14)$$

Considering that $F = b \cdot h,\ dF = b \cdot dr,$
We have

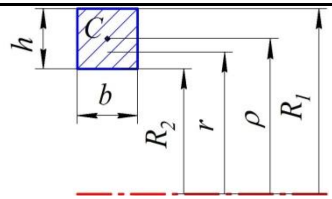
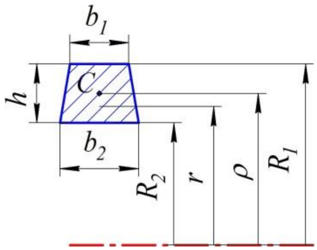
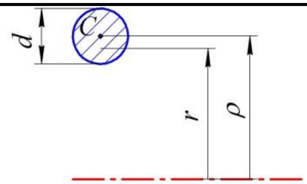
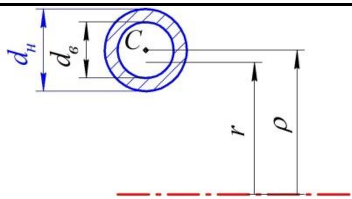
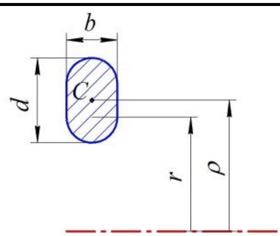
$$r_n = \frac{b \cdot h}{\int_{R_1}^{R_2} \frac{b \cdot dr}{r}} = \frac{h}{\ell n \frac{R_2}{R_1}} = \frac{h}{2{,}303 \ell g \frac{R_2}{R_1}}. \qquad (15.15)$$

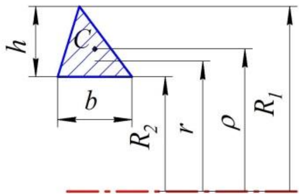
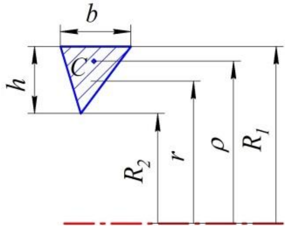
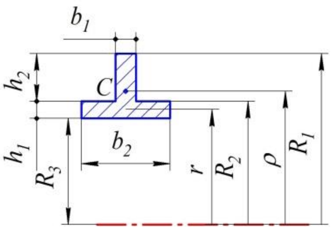
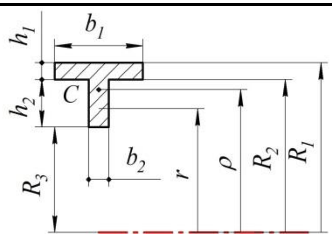
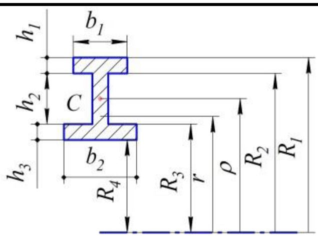
(Here, 2.303 is the modulus of conversion to decimal logarithms).

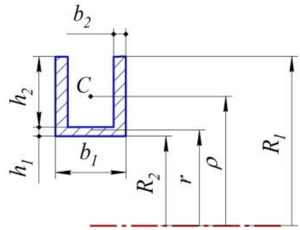
Using formula (15.14), an expression for e can be found for any cross-section of the beam.

The values of r_n or commonly encountered cross-sections of various shapes are given in Table 15.1.

Table 15.1 - Values of the radius of curvature of the neutral layer

Sectional shape	**Radius of curvature of the neutral layer**
	$r_n = \dfrac{h}{\ln\dfrac{R_1}{R_2}};$ $\ln\dfrac{R_1}{R_2} = \dfrac{h}{\rho}\left[1 + \dfrac{1}{3}\left(\dfrac{h}{2\rho}\right)^2 + \dfrac{1}{5}\left(\dfrac{h}{2\rho}\right)^4 + \cdots\right]$
	$r_n = \dfrac{h(b_1 + b_2)}{2\dfrac{b_2R_1 - b_1R_2}{h}\ln\dfrac{R_1}{R_2} - (b_2 - b_1)}$
	$r_n = \dfrac{d^2}{4(2\rho - \sqrt{4\rho^2 - d^2})}$
	$r_n = \dfrac{{d_n}^2 - {d_e}^2}{4\left(\sqrt{4\rho^2 - {d_e}^2} - \sqrt{4\rho^2 - {d_n}^2}\right)}$
	$r_n = \dfrac{d^2}{4(2\rho - \sqrt{4\rho^2 - d^2})}$

Sectional shape	**Radius of curvature of the neutral layer**
	$r_n = \dfrac{h}{2\left(\frac{R_1}{h}\ln\frac{R_1}{R_2} - 1\right)}$
	$r_n = \dfrac{h}{2\left(1 - \frac{R_2}{R_1}\ln\frac{R_1}{R_2}\right)}$
	$r_n = \dfrac{b_1h_1 + b_2h_2}{b_1\ln\frac{R_1}{R_2} + b_2\ln\frac{R_2}{R_3}}$
	$r_n = \dfrac{b_1h_1 + b_2h_2}{b_1\ln\frac{R_1}{R_2} + b_2\ln\frac{R_2}{R_3}}$
	$r_n = \dfrac{b_1h_1 + b_2h_2 + b_3h_3}{b_1\ln\frac{R_1}{R_2} + b_2\ln\frac{R_2}{R_3} + b_3\ln\frac{R_3}{R_4}}$

Sectional shape	Radius of curvature of the neutral layer
	$r_n = \dfrac{b_1 h_1 + b_2 h_2}{b_1 \ln \dfrac{R_1 - h_2}{R_2} + 2 b_2 \ln \dfrac{R_1}{R_1 - h_2}}$

15.3 Strength calculations for beams with large curvature

If a longitudinal force *N* acts in addition to the bending moment in the cross-section during bending of a curved bar, the strength calculation is performed taking into account both force factors. Tangential stresses do not have a significant effect on the strength and are usually not determined.

For rods of large curvature, the strength condition is written as follows based on formula (15.11):

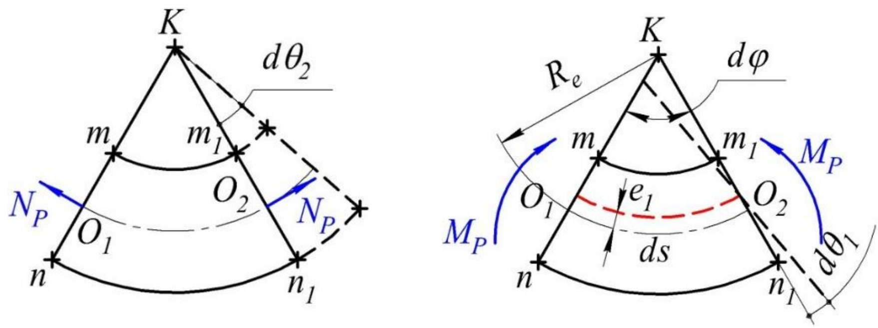
$$\sigma_{max} = \frac{M \cdot y}{S_z \cdot r} + \frac{N}{F} \leq [\sigma]. \tag{15.16}$$

In this case, it is necessary to consider the sections in which the total stresses are the highest. In these sections, one of the extreme points will be dangerous. For these points, $y = h_1$ or $y = h_2$ and $r = R_1$ або $r = R_2$ should be substituted into formula (15.16), respectively.

15.4 Determining displacements in curved bars

To determine displacements in rods of any curvature, it is convenient to use Mohr's method. In rods of small curvature, the longitudinal and shear strains can be neglected. In this case, in the case of plane bending, you can use Mohr's formula in the form (2.10).

In the case of plane bending of a bar of large curvature, the deformation of the element under the action of forces also consists of the elongation of the axis segment and the relative rotation of the sections that bound the element (Fig. 3.4, a, b).



a) b)

Figure 15.4 - Planar bending of a beam element of large curvature

The mutual angle of rotation of the sections $\Delta d\varphi = d\theta_1$, due to the bending moment can be determined from (3.10).

$$d\theta_1 = \frac{M_p d\varphi}{ES} = \frac{M_p dz}{ESR_0},$$

where $S = |S_y| = \ell F$.

The angle of rotation of the cross-sections, from axial forces due to different fibre lengths of the element (Fig. 3.4, b), is equal to

$$d\theta_2 = \frac{N_p dS}{EFR_0}.$$

Total angle of rotation:

$$d\theta = d\theta_1 + d\theta_2 = \frac{M_p dS}{EFR_0} + \frac{N_p dS}{EFR_0}. \qquad (15.17)$$

Elongation of an element due to axial forces:

$$\Delta(dS)_1 = \frac{N_p dS}{EF}.$$

Elongation of the element due to rotation of the section by an angle $d\theta$:

$$\Delta(dS)_2 = \ell d\theta_1 = \frac{M_p dS}{ESR_0}\ell = \frac{M_p dS}{EFR_0}. \qquad (15.18)$$

Total elongation of the axial fibre:

$$\Delta(dS) = \Delta(dS)_1 + \Delta(dS)_2 = \frac{N_p dS}{EF} + \frac{M_p dS}{EFR_0}. \qquad (15.19)$$

Substituting (3.18) and (3.19) into the formula for possible displacements, we find a general formula for determining the displacements of a large curvature beam:

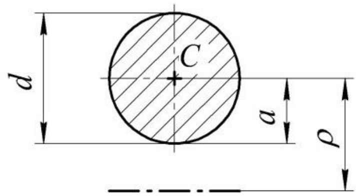
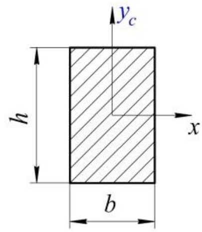
$$\Delta_{\mathrm{ip}} = \int_S \left[\frac{\bar{M}_i M_p}{ESR_0} + \frac{\bar{N}_i M_p + \bar{M}_i N_p}{EFR_0} + \frac{\bar{N}_i N_p}{EF} + k\frac{\bar{Q}_i Q_p}{GF}\right] dS, \quad (15.20)$$

where k – the coefficient selected according to Table 15.2, depending on the ratio ρ/a, a – the distance from the centre of gravity of the section to the internal fibres.

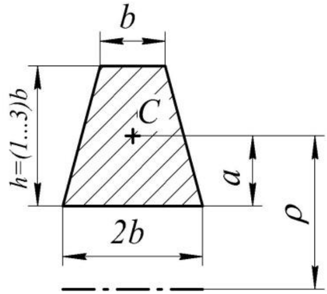
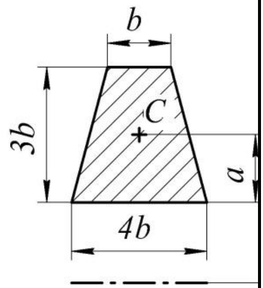
15.5 Control questions

1. What is the difference between small and large curvature bars?
2. What method is used to determine displacements in curved bars?
3. Strength condition for bars of large curvature.

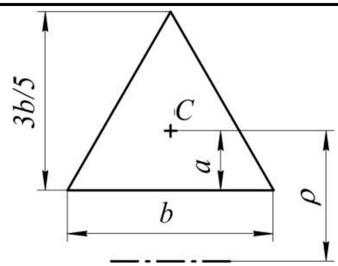
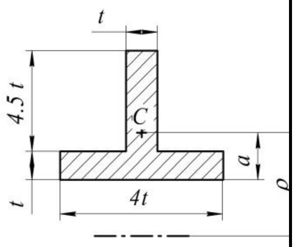
Table 15.2 - Values of the coefficient k

$\frac{\rho}{a}$		
1,2	0,224	0,305
1,4	0,151	0,204
1,6	0,108	0,149
1,8	0,084	0,112
2,0	0,069	0,090
2,2	0,058	0,077
2,4	0,049	0,065
2,6	0,042	0,055
2,8	0,036	0,047
3,0	0,030	0,041
3,5	0,022	0,028
4,0	0,016	0,021
6,0	0,0070	0,0093
8,0	0,0039	0,0052
10,0	0,025	0,0033

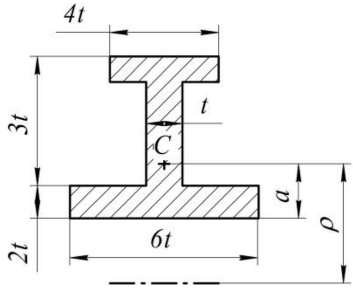
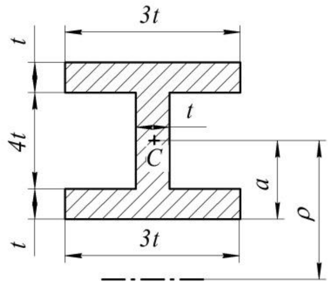
Continuation of Table 15.2

$\frac{\rho}{a}$		
1,2	0,336	0,352
1,4	0,229	0,243
1,6	0,168	0,179
1,8	0,128	0,138
2,0	0,102	0,110
2,2	0,084	0,092
2,4	0,071	0,078
2,6	0,061	0,067
2,8	0,063	0,058
3,0	0,046	0,050
4,0	0,024	0,028
6,0	0,011	0,012
8,0	0,0060	0,0060
10,0	0,0039	0,0039

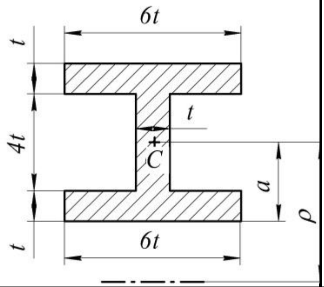
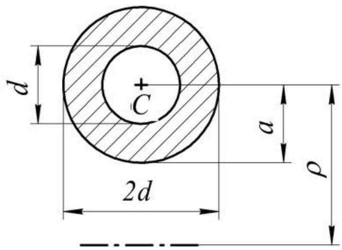
Continuation of Table 15.2

$\dfrac{\rho}{a}$		
1,2	0,361	0,418
1,4	0,251	0,299
1,6	0,186	0,229
1,8	0,144	0,183
2,0	0,116	0,149
2,2	0,096	0,125
2,4	0,082	0,106
2,6	0,070	0,091
2,8	0,060	0,089
3,0	0,052	0,079
3,5	0,038	0,052
4,0	0,029	0,040
6,0	0,013	0,018
8,0	0,0060	0,010
10,0	0,0039	0,0065

Continuation of Table 15.2

$\frac{\rho}{a}$		
1,2	0,409	0,408
1,4	0,292	0,285
1,6	0,224	0,208
1,8	0,178	0,160
2,0	0,144	0,127
2,2	0,120	0,104
2,4	0,103	0,088
2,6	0,089	0,077
2,8	0,077	0,067
3,0	0,067	0,058
3,5	0,049	0,041
4,0	0,038	0,030
6,0	0,018	0,018
8,0	0,010	0,0076
10,0	0,0065	0,0048

Continuation of Table 15.2

$\dfrac{\rho}{a}$		
1,2	0,453	0,269
1,4	0,319	0,182
1,6	0,236	0,134
1,8	0,183	0,104
2,0	0,147	0,083
2,2	0,122	0,068
2,4	0,104	0,057
2,6	0,090	0,049
2,8	0,078	0,043
3,0	0,067	0,038
3,5	0,048	0,028
4,0	0,036	0,020
6,0	0,016	0,0087
8,0	0,0089	0,0049
10,0	0,0057	0,0031

16 STRENGTH OF COMPRESSED RODS

16.1 Differential equation of the elastic line of a deformed rod

Bars used in engineering structures must not only have sufficient strength but also sufficient stiffness. The permissible deflections are set by standards. **The shape of the bent axis of a beam can be determined using the expression for curvature** (see mathematics course):

$$\frac{1}{\rho} = \pm \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}.$$

Substituting here the value of $1/\rho$, which is equal to

$$\frac{1}{\rho} = \frac{M_z}{E\mathcal{J}_z},$$

we get

$$\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \pm \frac{M_z}{E\mathcal{J}_z} \ . \tag{16.1}$$

Since for a beam of constant cross-section $\mathcal{J}_z = const$, the right-hand side of equation (16.1) depends only on M_z. If the function M_z is known, then the differential equation (16.1) can be used to determine the elastic line of the beam. This equation is nonlinear and nonhomogeneous of the second order. Its integration is associated with great difficulties. However, this equation can be simplified if we take into account that for most structures the maximum deflection is usually a very small part of the span ℓ: $y_{max} < (0{,}003 \ldots 0{,}002)\ell$.

Therefore, the angle of rotation of the section $\varphi(x)$ will be small compared to unity $\left[\varphi(x) = \frac{dy}{dx} \leq 1\right]$ and the value $\left(\frac{dy}{dx}\right)^2$ will be even smaller. Then we can write $\frac{1}{\rho} \approx \frac{d^2y}{dx^2}$. The problem of finding the equation of the elastic line will be reduced to finding the function $y(x)$ from the differential equation $\frac{d^2y}{dx^2} = \pm\frac{M_z}{E\mathcal{J}_z}$. If $M_z > 0$ the curvature $\frac{1}{\rho} > 0$; if $M_z = 0$ the curvature $\frac{1}{\rho} = 0$. Thus, for the adopted coordinate system, a plus sign must be put on the right-hand side of the differential equation:

$$\frac{d^2y}{dx^2} = \frac{M_z}{E\mathcal{J}_z} \tag{16.2}$$

where $E\mathcal{J}_z$ – is the bending stiffness of the beam; $\mathcal{J}_z = \int_F y^2 dF$ – the moment of inertia of the entire section relative to the neutral z -axis

16.2 Euler's problem for a centrally compressed rod

The problem of loss of stability of a compressed rod was first solved in 1744 by the great mathematician and mechanic, member of the Russian Academy of Sciences Leonard Euler. That is why, when talking about the stability of a compressed rod, the expression "Euler's problem" is used, and the critical force found by solving this problem is sometimes called the "Eulerian force". Euler himself called it the "column force", since at that time columns were the most common compressed elements of technical structures.

16.2.1 Critical force. Let's consider a rod with cross-sectional area F (Fig. 16.1), which is compressed by a longitudinal force F, the line of action of which coincides with the geometric axis of the rod. If the force P is small, then the rod will be subjected to uniform pressure with a stress

$$\sigma = \frac{P}{F}.$$

By applying a small transverse force, we can induce bending of the rod, and it will be in equilibrium, remaining bent. When the transverse force is removed, the rod will become straight again. The state in which the rod is under the action of a small force P is a **stable equilibrium.**

If you increase the force P, then after it reaches a certain value, the equilibrium of the rod becomes indifferent. **The smallest value of the axial compressive force at which two forms of equilibrium of the rod** (straight and curved, infinitely close to each other) **become equivalent is called the critical force.** In this case, when the force reaches the critical value, the rod is in a state of unstable equilibrium. In this case, a small transverse force will throw the rod out of balance, causing sudden transverse bending and fracture.

A rod is hinged.

Suppose a hinged rod is in equilibrium, maintaining its curved shape under the action of a critical force $\overline{P}_{cr}$ (Fig. 16.1, b).

a)

b)

c)

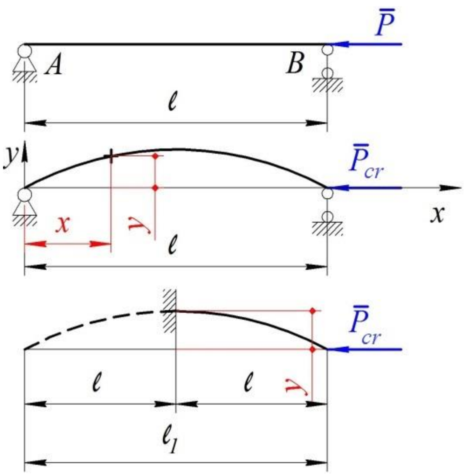


Figure 16.1 - Stability of compressed rods

A bending moment acts in the section at a distance x from the origin:

$$M_z = -P_{cr} \cdot y. \tag{16.3}$$

The differential equation of an elastic line based on expression (16.2):

$$\frac{d^2y}{dx^2} = -\frac{P_{cr} \cdot y}{E\mathcal{J}_z}, \tag{16.4}$$

whence

$$\frac{d^2y}{dx^2} \pm k^2 y = 0, \tag{16.5}$$

where

$$k^2 = \frac{P_{cr}}{E\mathcal{J}_z}.$$

The differential equation (16.5) is a second-order linear homogeneous equation with constant coefficients.

The solution of equation (16.5) is given in the form:

$$y = C_1 \cos kx + C_2 \sin kx. \tag{16.6}$$

The arbitrary constants C_1 and C_2 are found from the condition of fixing the ends of the rod:

at $x = 0$, $y = 0$ and, respectively, $C_1 = 0$

at $x = \ell$, $y = 0$, $C_2 \sin kx = 0$.

If $C_2 = 0$, then we obtain a trivial solution for equation (16.6): $y = 0$. This solution corresponds to the equilibrium of a non-curved rod. If $C_2 \neq 0$, then $\sin k\ell = 0$. owever, this condition is possible when $k\ell = 0;\ \pi,\ 2\pi,\ 3\pi$ ... .

Thus, the equilibrium occurs when $\ell\sqrt{\frac{P_{cr}}{E\mathcal{J}_z}} = 0;\ \pi; 2\pi; 3\pi;$ ...

The first condition $\left(\ell \cdot \sqrt{\frac{P_{cr}}{E\mathcal{J}_z}} = 0\right)$ gives a trivial solution:

$F_{cr} = 0$. The second condition $\left(\ell \cdot \sqrt{\frac{P_{cr}}{E\mathcal{J}_z}} = \pi\right)$ leads to **the Euler's**

formula for determining the critical force:

$$P_{cr} = \frac{\pi^2 E\mathcal{J}_z}{\ell^2}. \qquad (16.7)$$

16.2.2 Effect of anchoring conditions.

The value of the critical force changes when the anchoring conditions change. From the diagram (Fig.16.2, b), which shows a rod of length ℓ, fixed at one end, and its mirror image, it is clear that the critical force for this case can be determined by formula (16.7) if ℓ is substituted for $\ell_1 = 2\ell$.

$$P_{cr} = \frac{\pi^2 E\mathcal{J}_z}{4\ell^2} = \frac{\pi^2 E\mathcal{J}_z}{(2\ell)^2}\,. \qquad (16.8)$$

As we can see, when the anchoring conditions change, the numerical factor near ℓ^2 n the denominator of the right-hand side of (16.7) changes.

Since, when a rectilinear equilibrium form loses stability, bending always occurs in the plane of least stiffness $E\mathcal{J}_{min}$, the neutral line is the one of the main central axes of inertia for which the moment of inertia of the cross-section is minimal ($\mathcal{J}_{min}$). Then **Euler's formula in general form:**

$$P_{cr} = \frac{\pi^2 E\mathcal{J}_{min}}{(\mu\ell)^2}\,, \qquad (16.9)$$

where μ – characterises **the condition of fixing the rod ends** (Fig. 16.2).

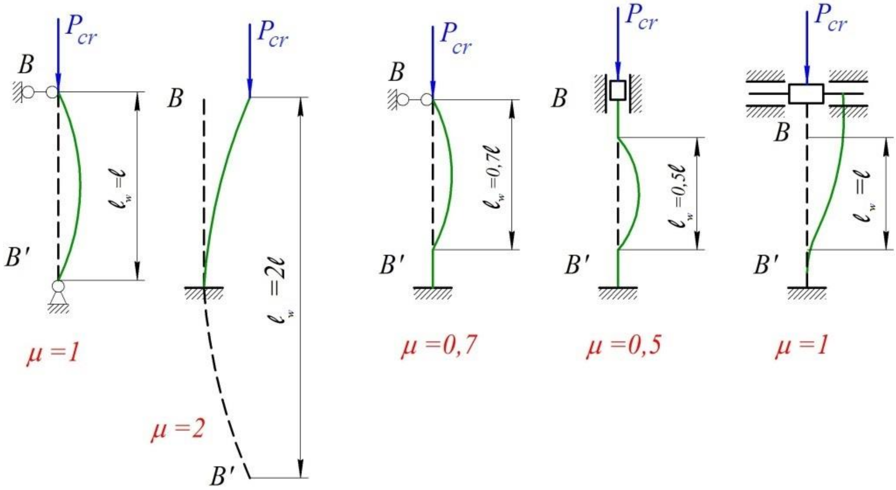


Figure 16.2 - Dependence of μ on the fixation conditions

The sum $\mu\ell$ is called **the reduced length of the bar. The critical stress** is found by dividing the critical force by the cross-sectional area F:

$$\sigma_{cr} = \frac{P_{cr}}{F} = \frac{\pi^2 E\mathcal{J}_z}{(\boldsymbol{\mu\ell})^2 F} = \frac{\pi^2 E i_{min}}{(\boldsymbol{\mu\ell})^2}, \quad (16.10)$$

where

$$i_{min} = \sqrt{\frac{\mathcal{J}_{min}}{F}}$$

– minimum radius of inertia.

16.3 Conditions for applying Euler's formula

In deriving Euler's formula, the differential equation of the bent axis of the rod was used, which is valid only within the limits of Hooke's law.

It follows that Euler's formula is also valid only if the loss of stability occurs at a stress σ_{cr}, that is less than the proportionality limit σ_{pr}. The condition for the validity of Euler's formula can be represented as $\sigma_{cr} \leq \sigma_{pr}$. Substituting the value of σ_{cr} from (16.10) into this inequality, we obtain:

$$\left(\frac{i_{min}}{\mu\ell}\right)^2 \le \frac{\sigma_{pr}}{\pi^2 \mathrm{E}}.$$

The ratio ${}^{\mu\ell}/_{i_{min}} = \lambda$ is called **the flexibility of the rod**. This is **the limit value of flexibility:**

$$\lambda_{lim} = \sqrt{\frac{\pi^2 \mathrm{E}}{\sigma_{pr}}}\,. \qquad (16.11)$$

Condition (16.11) defines the limits of the Euler's formula. If the rod flexibility is less than the limit value λ_{lim}, the Euler's formula cannot be used.

Formula (16.10) can be used provided that the flexibility of the rod, which is determined by equation:

$$\lambda = \mu\ell\sqrt{\frac{F}{\mathcal{J}_{min}}} \qquad (16.12)$$

satisfies the inequality

$$\lambda_{cr} \le \pi\sqrt{\frac{\mathrm{E}}{\sigma_{pr}}},$$

That is, at $\lambda > \lambda_{cr}$ the loss of stability occurs within the limits of material proportionality.

16.4 Stability calculations using reduction factors for the principal permissible stress

The above relations refer to the case of loss of stability in the elastic region, i.e. they are valid under the **condition** that the maximum stress does not exceed the proportionality limit ($\sigma_{cr} \le \sigma_{pr}$) until the critical state is reached.

If $\sigma_{cr} > \sigma_{pr}$, then the loss of stability occurs beyond the proportionality limits.

The sequence of calculations is not stable:

1. 1. Determine the flexibility by formula (16.12) and the maximum critical flexibility by formula (16.11).

2. If $\lambda > \lambda_{cr}$, then the calculation is performed using formulas (16.9) and (4.10).

3. At a flexibility that satisfies the condition $\lambda > \lambda_{cr}$, the critical stress can be found approximately by the formula:

$$\sigma_{\text{y}} = \varphi[\sigma] \, , \tag{16.13}$$

where φ – **the coefficient of reduction of the permissible stress** $[\sigma]$, depends on the material of the rod and flexibility λ (Table. 16.1).

4. Check the condition of stability of the compressed rod by the formula:

$$\sigma_{\text{y}} = \frac{P}{F} \leq \varphi \cdot [\sigma] \; . \tag{16.14}$$

According to the given safety factor n_{saf} we determine the value of **the permissible external load:**

$$P \leq [P] = \frac{P_{cr}}{n_{saf}} = \frac{\pi^2 E \mathcal{I}_{min}}{(\mu\ell)^2 n_{saf}} \; . \tag{16.15}$$

From where the smallest axial moment of inertia is determined

$$\mathcal{I}_{min} = \frac{P \cdot n_{saf}(\mu\ell)^2}{\pi^2 E} \; . \tag{16.16}$$

Table 16.1 - Values of the coefficients φ for some materials

Flexibility	Steel			Cast iron	
	St. 0 St. 2 St. 3 St. 4	St. 5	15KHSND	G4 15-32 G4 12-28 G4 18-36 G4 21-40	G4 24-44 G4 28-48
0	1,00	1,00	1,00	1,00	1,00
10	0,99	0,98	0,98	0,97	0,95
20	0,97	0,95	0,95	0,91	0,87
30	0,95	0,93	0,93	0,81	0,75
40	0,92	0,90	0,90	0,69	0,60
50	0,89	0,84	0,83	0,57	0,43
60	0,86	0,80	0,78	0,44	0,32
70	0,81	0,74	0,71	0,34	0,23
80	0,75	0,66	0,63	0,26	0,18
90	0,69	0,59	0,54	0,20	0,14
100	0,60	0,50	0,45	0,16	0,12
110	0,52	0,43	0,39		
120	0,45	0,38	0,33		
130	0,40	0,32	0,29		
140	0,36	0,28	0,25		
150	0,32	0,27	0,23		
160	0,29	0,24	0,21		
170	0,26	0,21	0,19		
180	0,23	0,19	0,17		
190	0,21	0,17	0,15		
200	0,19	0,15	0,13		

In order to find the cross-sectional area F from (16.14), it is necessary to know the value of the coefficient φ depending on the flexibilityλ. But to determine the flexibility, we need to know the dimensions of the cross-section. In this regard, the problem is solved by the method of successive approximation. First, using approximate values of the stress reduction factor φ_1 we determine the cross-sectional area. Taking the shape of the section, we obtain the value i_{min}. Using the found values of i_{min} and λ determine φ_1^*. If φ_1^*, is close to the value of φ_1 then the calculation ends. Otherwise, the calculation is repeated until the input and output values of the coefficient φ are sufficiently close (method of successive approximations).

16.5 Calculations for stability beyond the limit of flexibility of materials

If it is impossible to apply the Euler's formula ($\lambda < \lambda_{cr}$) the critical stress is determined by empirical formulas compiled by **F.S. Yasinsky** on the basis of research:

$$\sigma_{cr} = a - b\lambda. \tag{16.17}$$

In this formula, the dependence of the critical stress on flexibility is linear; a and b – coefficients determined by experience, constant for a given material (see Table 16.2).

At a certain value of flexibility σ_{cr}, according to formula (16.17), becomes equal to the yield strength σ_y for ductile material) or the ultimate strength σ_{B} (for brittle materials).

Then the flexibility is calculated by the formula:

$$\lambda_0 = \pi\sqrt{\frac{\mathrm{E}}{\sigma_y}} \text{ or } \lambda = \pi\sqrt{\frac{\mathrm{E}}{\sigma_{\mathrm{B}}}}.$$

Thus, depending on the flexibility, compressed rods are conditionally divided into three categories (see Fig. 16.4):

I. Cores of high flexibility ($\lambda \geq \lambda_{cr}$), for which the calculation of stability is carried out according to **the Euler formula** and the dependence of σ_{cr} on λ – hyperbolic (AB):

$$\sigma_{cr} = \frac{\pi^2\mathrm{E}}{\lambda^2} - \text{ (Euler's hyperbole).}$$

II. Cores of medium flexibility ($\lambda_0 \leq \lambda < \lambda_{cr}$), which are calculated using the Yasinsky formula (16.17). For them, the dependence of σ_{cr} on the flexibility of λ is linear (BC):

$$\sigma_{cr} = a - b\lambda\,.$$

III. Bars of low flexibility ($\lambda < \lambda_0$), which are calculated not for stability but for strength. For them, the critical stress is considered to be constant (CD) $\sigma_{cr} = \sigma_y$ or $\sigma_{cr} = \sigma_{\text{B}}$.

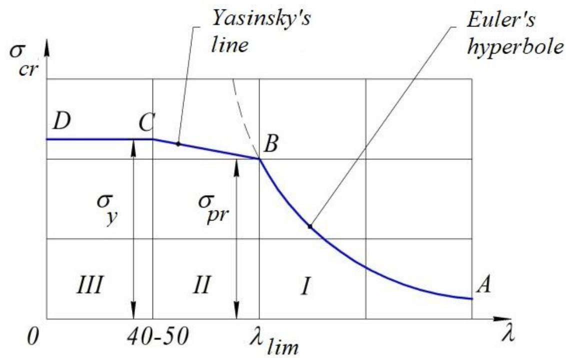


Figure 16.3 - Use of sustainability calculation methods

Table 16.2 - Values of the coefficients *a, b* for some materials

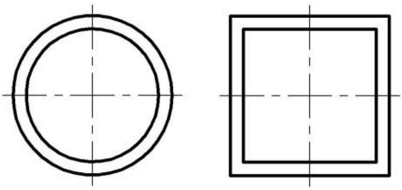
Material	a, MPa	b, Mpa	λ_0	λ_{cr}
Steel:				
St2	264	0,7	60	105
St3	310	1,14	60	100
20, St4	328	1,15	40	96
45	449	1,67	52	85
Duralumin D16 T	406	1,83	30	83
Pine, spruce	29,3	0,194		70

16.6 Selection of material and rational cross-sectional shape for compressed bars

When calculating for stability, the modulus of elasticity E is the only mechanical characteristic of materials that determines the material's strength to loss of stability. Since E has little dependence on strength, it is not advisable to use high-strength materials for rods whose load-bearing capacity is determined by loss of stability ($\lambda \geq \lambda_{lim}$– high flexibility).

For rods of low flexibility ($\lambda \leq \lambda_0$), whose limit state is determined by strength rather than stability, it is advisable to use strong steels to increase σ_y.

The better the cross-sectional profile, the greater the moment of inertia $\mathcal{J}$ for the same area. It is desirable that the flexibility of the rod in its principal planes is the same. These requirements are best met by hollow rods of round and square cross-section with a thin wall. It should be noted that the lower limit of the wall thickness is determined by the risk of local loss of stability (bulging, distortion). To prevent this, diaphragms are installed.



To evaluate the optimality of the section, the parameter is used $\xi = \dfrac{i_{\min}}{\sqrt{F}}$ – **the specific radius of inertia** (see Table 16.3).

Table 16.3 - Values of the specific radius of inertia for different cross-sectional shapes

Section	ξ
Tubular	$(\alpha = {}^{d_2}/_{d_3} = 0{,}95 - 0{,}8)$, 2,25–1,64 $(\alpha = 0{,}7 - 0{,}8)$, 1,2–1,0
Corner	0,5 – 0,3
I-beam	0,41 – 0,27
Channel	0,41 – 0,29
Square	0,289
Circle	0,283

The phenomenon of loss of stability of structural elements is very dangerous. Often, the cause of the destruction of structures and buildings is not a violation of strength, but a loss of stability.

In the history of engineering, there are many cases of major accidents and disasters when the destruction of bridges, buildings, ships and other structures was caused by the phenomenon of loss of stability. For example, we can mention the destruction of a large gas holder in Hamburg, which collapsed during a test filling on 7 December 1909 due to a loss of stability in one of the elements of the support structure.

16.7 Control questions

1. The concept of the Eulerian force.
2. Euler's formula for determining the critical force.
3. What coefficient characterises the conditions for fixing the ends of the rod?
4. What is called the reduced length of the rod?
5. What characterises the flexibility of the rod?
6. How is the stress reduction factor used?
7. In what case is the Yasinsky formula used?
8. What are the three categories of rods depending on their flexibility?
9. How is the rational selection of material and cross-sectional shape for compressed rods evaluated?

17. CALCULATIONS OF STATICALLY INDETERMINATE SYSTEMS BY THE METHOD OF FORCES

17.1 Basic concepts and definitions. Stages of calculation of statically indeterminate systems

Statically indeterminate systems are those in which internal forces cannot be determined using the equations of statics alone. In such systems, there are more links than are necessary for equilibrium. Consequently, some of the links are redundant in this sense, and the forces in them are redundantly unknown. The number of redundant links or redundant unknown forces is used to determine **the degree of static uncertainty of the system as the difference between the number of redundant links and the number of equations.**

To calculate such beams, in addition to the statics equations, it is necessary to develop additional equations, which are called the **displacement equation (or deformation equation).**

In Chapter 6, we considered the simplest examples of statically indeterminate systems, with elements subjected only to axial tension or compression. In this chapter, we will consider more general cases, focusing on statically indeterminate beams and frames.

The following sequence should be followed when solving these systems.

Step 1: Determine the number of "extra" unknown connections. The term **"extra unknowns"** refers to the connections without which the system remains "geometrically" unchanged, i.e. in equilibrium.

Step 2. Eliminating the **"extra connections"**, we replace the original system with a statically determinate one, which is called the **basic system.**

For a single statically indeterminate initial system, different variants of the basic systems are possible. The main thing is that each of them was geometrically unchanged.

Thus, **the basic system is a statically determinate variant of the system under consideration, obtained by getting rid of "extra connections".**

Stage 3. We load the main system with a given load and extra unknown forces that replace the effect of the eliminated links. Such a system is called **an equivalent system.**

Step 4. In order for the main system to be equivalent to the original system, the unknown forces must be selected in such a way that the deformation of the main system does not differ from the deformation of the original statically indeterminate system. In other words, the work of the equivalent system should be identical to that of the specified system. To do this, the equation of compatibility of deformations of the points of application of extra links is drawn up (i.e., the deformation of the points of application of extra links in the direction of their action is equated to zero) and the magnitude and direction of the link reactions, i.e., unknown loads, are determined from the solution of these equations. The displacements of the corresponding points of the system can be determined in any way, but the best way is to use the Mohr method or the Vereshchagin method. Having found the extra unknown forces, the reactions of the supports are determined, internal force diagrams are constructed, and the dimensions and shape of the section are selected and the element is tested for strength using the already known methods.

This system of calculation of statically indeterminate systems is called **the method of forces**, since the forces of extra connections are chosen as the main unknowns.

Examples of the definition of "extra connections". Fig. 17.1(a) shows a beam supported on hinged supports, statically determinate and geometrically indeterminate. All the reactions (R_A, H_A, R_B) are determined from the equilibrium conditions of the plane force system, and then it is easy to find the force factors Q and M in any cross-sections of the beam.

Let us add one more connection - the hinged-moving strength in section C (Fig. 17.1, b). From the point of view of geometric invariability, this connection is unnecessary. It is impossible to determine four reactions (R_A, H_A, R_B, R_c) from the three equations of equilibrium. Thus, the beam shown in Fig. 5.1 **is statically indeterminate once.**

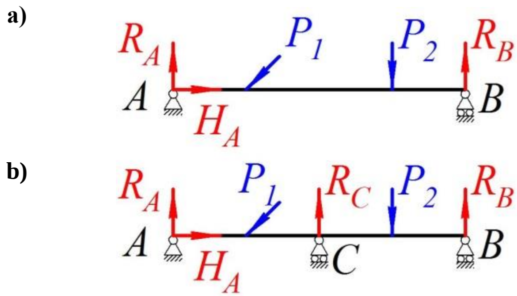


Figure 17.1 - Hinged and supported beam

Figure 17.2(a) shows a twice statically indeterminate beam. There are only three equations to determine the five reactions (R_A, H_A, M_A, R_B, R_C) The system has two "extra bonds".

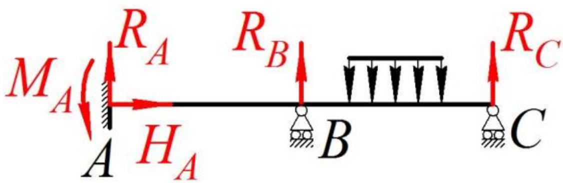
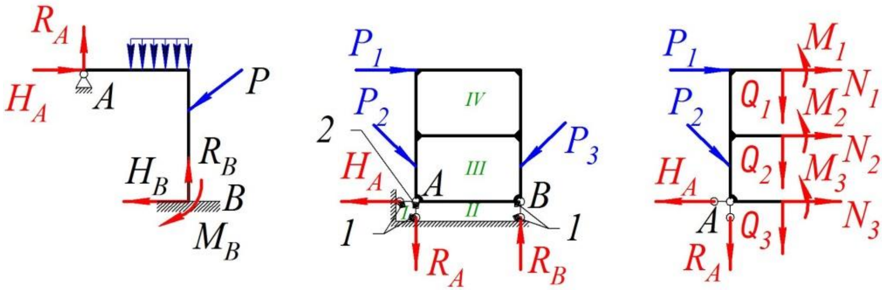


Figure 17.2 - Double statically indeterminate beam

In structures, **frames** are often used in which, unlike trusses, where the rods are connected by hinges and loaded with forces, **the rods are rigidly connected, have no relative movements and rotations.**

Fig. 17.3(a) shows a twice statically indeterminate plane truss. Here, we have only three equations of equilibrium to determine the five reactions of the external connections.



a) b) c)

Figure 17.3 - Statically undetectable frames

The static uncertainty can be caused not only by unnecessary connections, but also by the conditions of the system formation. For the frame (Fig. 17.3, b), the reactions R_A, H_A, R_B the external connections can be easily determined from the equilibrium equations. However, the equilibrium equations do not allow determining all the force factors in the frame elements.

Let's cut the frame into two parts and consider the equilibrium of one of the parts (Fig. 17.3, c). The action of the discarded part in each section is replaced by three force factors: axial force N, transverse force Q and bending moment P. Thus, nine unknown forces must be determined from the three equations of equilibrium. The system is statically undecidable six times. It consists of two closed, hingeless contours, each of which is statically indeterminate three times. Installing a hinge on the axis of the rod turns the bending moment to zero and reduces the degree of static indeterminacy by one. Such a joint is called a **single** joint.

A hinge located at a node where n rods coincide reduces the degree of static uncertainty by $n - 1$. Such a hinge is called a **common** hinge.

The degree of static uncertainty S of plane systems can be determined by the formula

$$S = 3k - h, \qquad (17.1),$$

where k – the number of closed circuits (complete absence of hinges);

h – number of hinges in terms of single hinges

The base (ground) is treated as a rod. The frame (Fig. 17.3, b) has four closed loops; the corresponding reactions of single joints are indicated near each frame support with the corresponding indices. In this case, the rods that are rigidly connected to each other are considered to be one rod.

17.2 Calculation of a simple statically indeterminate beam

Here is a calculation of a beam to which loads are applied. One end of the beam is fixed, and the other end is supported by an articulated support (Fig. 17.4, a). The fixed support A and the articulated support B produce four reactions ($R_A, H_A, M_A,\ R_B$). Thus, the beam is statically indeterminate once. To construct the basic system, one link, the hinged-moving support, must be eliminated. The basic system is a console (Fig. 17.4, b).

We apply a given distributed load q, to the main system, and instead of the rejected support, we apply an unknown reaction $R_B = X_1$ (Fig. 17.4, b). In the following, we will denote extra connections with the letter X regardless of whether it is a force or a moment.

The total displacement of point B of the main system (from the given q and the extra force) in the direction X_1 must be zero (point B is stationary). The additional equation of displacements is written as follows:

$$\Delta_1 = 0. \qquad (17.2)$$

The total deflection Δ_1 can be defined as the sum of the deflections due to the external load

$$\Delta_{1\mathrm{p}} = -\frac{q\ell^4}{8EJ}$$

(Fig. 17.4, c), and the unknown response (Fig. 17.4, d).

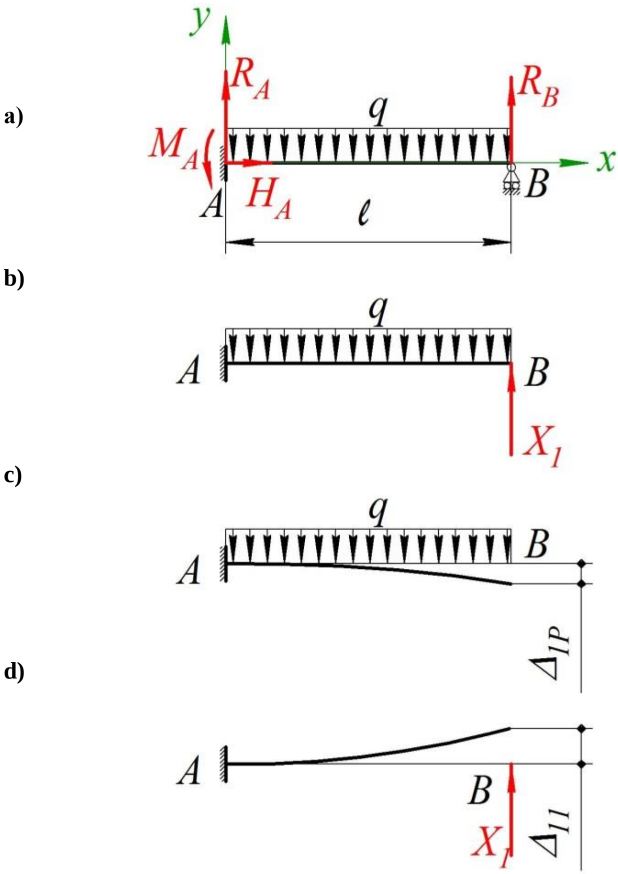


Figure 17.4 - Statically indeterminate beam

Then equation (17.2) is written in the form

$$\Delta_1 = \Delta_{1p} + \Delta_{11} = 0$$

or

$$-\frac{q\ell^4}{8E\mathcal{J}} + \frac{X_1\ell^3}{3E\mathcal{J}} = 0.$$

Hence the desired reaction

$$X_1 = \frac{3}{8}q\ell, \qquad (X_1 = R_B).$$

From the statics equations, we determine the last reactions of the supports:

$$\sum P_{kx} = 0;\;\; H_A = 0;$$

$$\sum P_{ky} = 0;\;\; R_A - q\ell + R_B = 0;$$

$$\sum M_{kA} = 0;\;\; M_A - \frac{q\ell^2}{2} + R_B\ell = 0;$$

$$H_A = 0;\; M_A = \frac{q\ell^2}{8}\;;\; R_A = \frac{5}{8}q\ell;\;\; R_B = \frac{3}{8}q\ell.$$

Fig. 17.5 shows the Q and M diagrams and the values of the reactions of the supports.

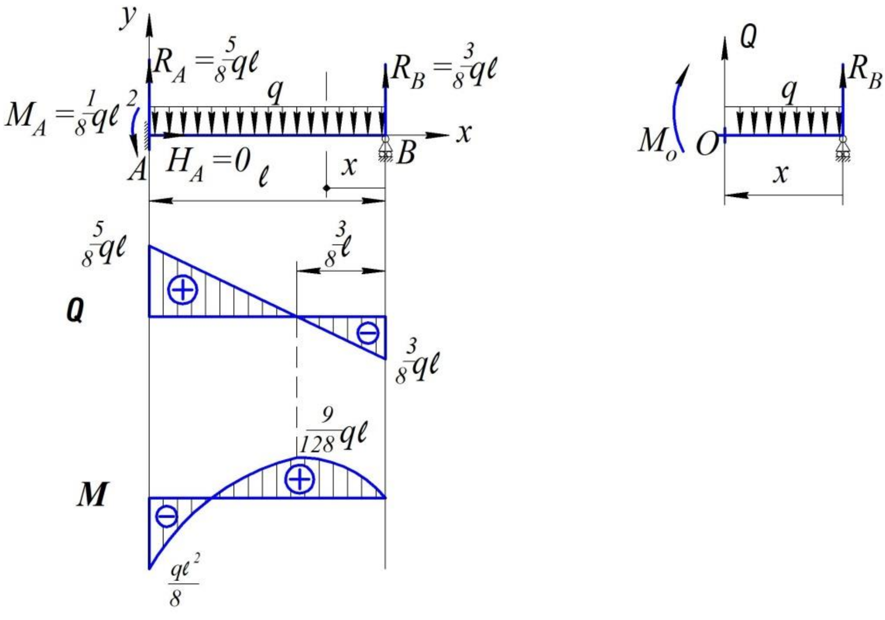


Figure 17.5 - *Q* and *M* epures

Let's define the internal forces Q and M (Fig. 17.5, b).

$$\sum y_k = Q + R_B - qx = 0 \quad Q = -R_B + qx.$$

When $x = 0$

$$Q = -R_B = -\frac{3}{8}q\ell;$$

when $x = \ell$

$$Q = -R_B + q\ell = -\frac{3}{8}q\ell + q\ell = \frac{5}{8}q\ell.$$

$$\sum M_{Ok} = R_B x - q\frac{x^2}{2} - M_O = 0 \quad M_O = R_B x - q\frac{x^2}{2};$$

when $x = 0$

$$M_O = 0 \;\; ;$$

when $x = \ell$

$$M_O = \frac{3}{8}q\ell^2 - \frac{1}{2}q\ell^2 = -\frac{q\ell^2}{8}.$$

Defining the extremum.

$$x_0 = \frac{R_B}{q}; \; x_0 = \frac{\frac{3}{8}q\ell}{q} = \frac{3}{8}\ell.$$

When $x = x_0$

$$M_O = \frac{9q\ell^2}{128}.$$

17.3 Canonical equations of the force method

External loads encountered in problems of material strength are a group of constant forces. The work of a group of constant forces can be represented as the product of two quantities $A = P \cdot \Delta_P$, where the multiplier P depends only on the forces of the group and is called the **generalised force**, and Δ_P depends on the displacements and is called the **generalised displacement.**

Thus, **the generalised force is understood as any load (concentrated forces, concentrated moments, distributed loads, etc.) that is capable of performing work at the corresponding generalised displacement.**

It is common to denote generalised displacements (both linear and angular) by the letters Δ_{ik} or δ_{ik} with the corresponding double indices. **The first index indicates the point and direction of displacement, and**

the second index indicates the force factor that caused this displacement. For example, Δ_{PP} indicates the displacement of the point of application of a force P in the direction of its action caused by the same force P (Fig. 17.6, a).

To denote the total displacement caused by several force factors, for example, force P, moment M and "extra" unknown force X_1 (see Fig. 17.6, b), only the first index is retained in Δ:

$$\Delta_P = \Delta_{PP} + \Delta_{PM} + \Delta_{PX_1}, \tag{17.3}$$

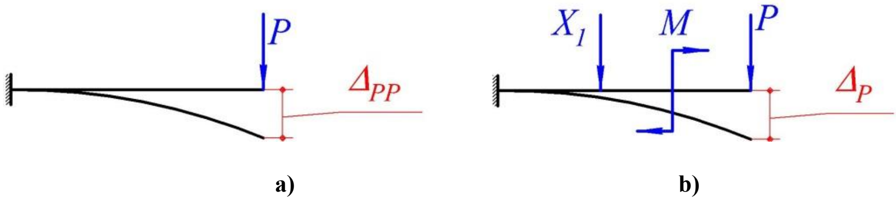


Figure 17.6 - Movement of the point of application of the force P

The displacement caused by **a unit force** ($\bar{X}_1 = 1$), is usually denoted by the letter δ and is called **the specific displacement**. If a unit force $\bar{X}_1 = 1$ caused a displacement δ_{11} , then the total displacement Δ_{11} can be given as:

$$\Delta_{11} = \bar{X}_1 \cdot \delta_{11}. \tag{17.4}$$

Additional equations of displacements, expressing the equality of zero displacements in the directions of "extra" unknowns, are conveniently written in the so-called **canonical form**, i.e., according to a certain regularity.

Let us show this by the **example** of solving a simple statically indeterminate system (Fig. 17.7, a).

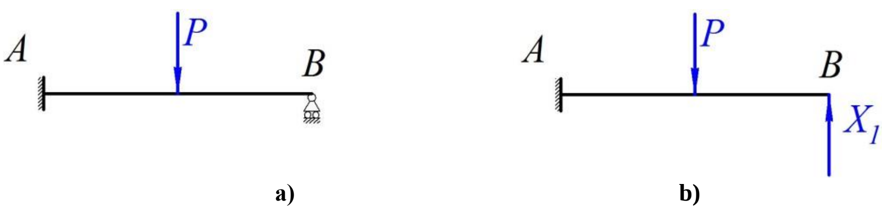


Figure 17.7 - An example of a simple statically undefined system

Let's choose a console as the main system, which we load with an external force P and an "extra" unknown $\bar{X}_1$ to obtain an equivalent system (Fig. 17.7, b). The additional equation of displacements, which expresses the equality of the displacement of point B due to the forces P and $\bar{X}_1$, to zero, is

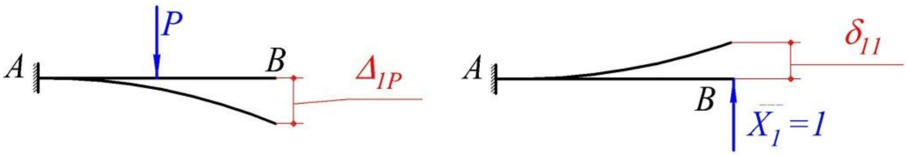
$$\Delta_1 = \Delta(P, \bar{X}_1) = 0. \tag{17.5}$$

Based on the principle of independence of forces, we write

$$\Delta_1 = \Delta_{1P} + \delta_{11}, \tag{17.6}$$

where Δ_{1P} –the displacement of point B under a given load P (Fig. 17.8, a);

Δ_{11} – displacement of point B from the force $\bar{X}_1$.



a) b)

Figure 17.8 - Displacement of point *B* from a given load *P* and a unit force $\bar{X}_1 = 1$

Denoting the displacement of point B from $\bar{X}_1 = 1$ in its direction by δ_{11} (Fig. 17.8, b), we obtain $\Delta_{11} = \bar{X}_1 \cdot \delta_{11}$.

$$\Delta_1 = \bar{X}_1 \cdot \delta_{11} + \Delta_{1P}\,. \tag{17.7}$$

By analogy, the displacement equations for any n times statically indeterminate system can be written in the canonical form:

Main

$$\begin{cases} \delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \delta_{13} \cdot X_3 + \cdots + \delta_{1n} \cdot X_n + \Delta_{1P} = 0, \\ \delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \delta_{23} \cdot X_3 + \cdots + \delta_{2n} \cdot X_n + \Delta_{2P} = 0, \\ \delta_{31} \cdot X_{12} + \delta_{32} \cdot X_2 + \delta_{33} \cdot X_3 + \cdots + \delta_{3n} \cdot X_n + \Delta_{3P} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \delta_{n1} \cdot X_1 + \delta_{n2} \cdot X_2 + \delta_{n3} \cdot X_3 + \cdots + \delta_{nn} \cdot X_n + \Delta_{nP} = 0, \end{cases} \tag{17.8}$$

Side

where δ_{ik} – **the unit coefficients of the canonical equations;**

Δ_{iP} – **the load coefficients of the canonical equations.**

Single coefficients with the same indices are called main coefficients, and those with different indices are called side coefficients.

The number of equations of the system (17.8) is equal to the degree of static uncertainty of the rod system.

The physical meaning of the canonical equations (17.8) **is that the displacements in the direction of the discarded couplings are zero.**

To determine **the coefficients of the canonical equations**, perform the following steps:

1. Apply the forces $\bar{X}_1 = 1$ to the fundamental system (FS) one by one and construct unit epures.

2. Find the unit coefficients δ_{ik} by "multiplying" the corresponding unit epures.

3. Apply an external load to the main system and draw a load diagram.

4. By "multiplying" the load epure with each unit epure in turn, obtain the load coefficients Δ_{iP}.

5. After determining the coefficients of the canonical equations and solving system (5.8), find the unknown forces X_1,. X_2,… X_n.

It can be shown that the determination of unit displacements can be simplified by establishing a relationship between displacements δ_{ik} and δ_{ki}.

17.4 Reciprocity theorems for work and displacements

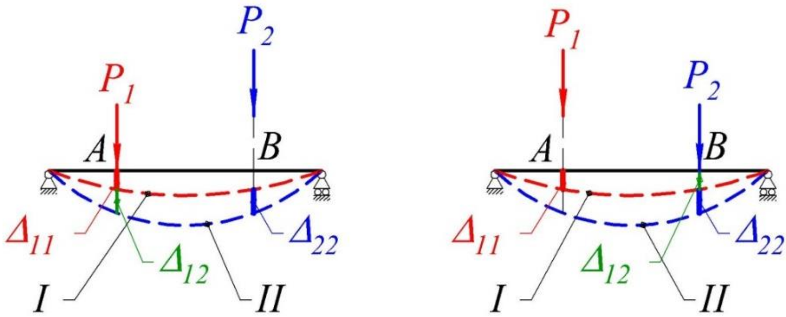
The reciprocity of work theorem is one of the general theories of material strength. It follows from the principle of independence of forces and is applicable to all systems.

Consider a beam to which forces P_1 and P_2 are applied (Fig. 17.9).

Let's apply a direct load, i.e. load the beam at point A with force P_1 (see Fig. 17.9, a). It will do the work **(Clapeyron's formula)**

$$A = \frac{1}{2} P_1 \cdot \Delta_{11},$$

where Δ_{11} – is the displacement of p. A in the direction of force P_1, caused by force P_1.



a) **b)**

Figure 17.9 - On the reciprocity theorem

Next, at point B, we apply a force P_2 . It will do work that has the same expression: $\frac{1}{2}P_2 \cdot \Delta_{22}$.

At the same time, the force P_1, will also do work, because when the force P_2 is applied, point A will also move. The force P_1 remains constant during this displacement, and its work will be equal to $P_1\Delta_{12}$.

Thus, the total work under direct loading of the beam (see Fig. 17.9, a) will be equal to

$$A_1 = \frac{1}{2}P_1 \cdot \Delta_{11} + \frac{1}{2}P_2 \cdot \Delta_{22} + P_2 \cdot \Delta_{21}. \qquad (17.9)$$

Let's perform a reverse load (see Fig. 17.9, b), i.e. first apply a force P_2 at point B, and then a force P_1 at point A. Then, accordingly, the total work will take the form

$$A_2 = \frac{1}{2}P_2 \cdot \Delta_{22} + \frac{1}{2}P_1 \cdot \Delta_{11} + P_2 \cdot \Delta_{21}. \qquad (17.10)$$

The total (total) work in both cases will be the same, so equating the work $A_1 = A_2$, we find

$$P_1 \cdot \Delta_{12} = P_2 \cdot \Delta_{21}. \qquad (17.11)$$

This is **the reciprocal work theorem: the work of the first force in moving the point of its application under the action of the second force is equal to the work of the second force in moving the point of its application under the action of the first force**.

If $P_1 = P_2 = P$, then

$$\Delta_{12} = \Delta_{21}. \tag{17.12}$$

The reciprocity theorem: the displacement of the first unit force in its direction under the action of the second unit force is equal to the displacement of the second unit force in its direction under the action of the first unit force.

This theorem can be illustrated (Figure 17.10).

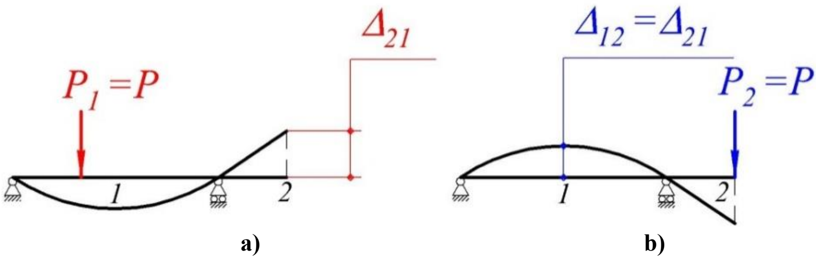


Figure 17.10 - On the reciprocity theorem of displacements

17.5 Control questions

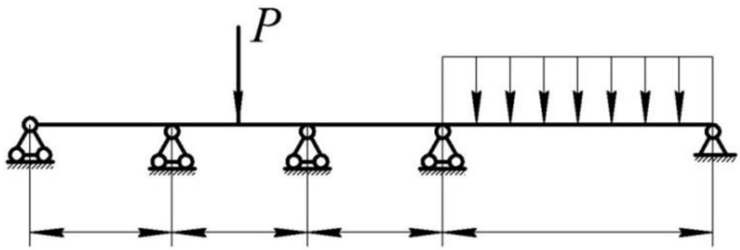
1. What systems are called statically undecidable?
2. What is meant by the term "extra unknowns"?
3. The sequence of solving statically undecidable problems.
4. Give an example of a statically indeterminate beam.
5. What is the difference between a frame and a truss?
6. Give an example of a statically indeterminate frame.
7. The formula for determining the degrees of static indeterminacy S of plane systems.
8. What is called a generalised force?
9. Theorem on the reciprocity of work and displacements.
10. Give examples of structures on an elastic base.
11. Differential equations of the bent axis of a beam on an elastic base.

18. MULTI-SPAN CONTINUOUS BEAMS

18.1 Calculation of continuous beams. Basic concepts

Continuous beam is a statically indefinite beam that spans several spans (at least 2) and passes over all intermediate supports with which it is hinged. The end supports can be either hinged or rigidly fixed (Figure 18.1). In the first case (Fig. 18.1, a), when a vertical load is applied, the horizontal reaction in the fixed support is zero, and therefore the number of extra vertical reactions (and the degree of static uncertainty) **will be equal to the number of intermediate supports.** In the second case (Fig. 18.1, b), **the degree of static uncertainty is equal to the number of intermediate hinged supports.**

a)



b)

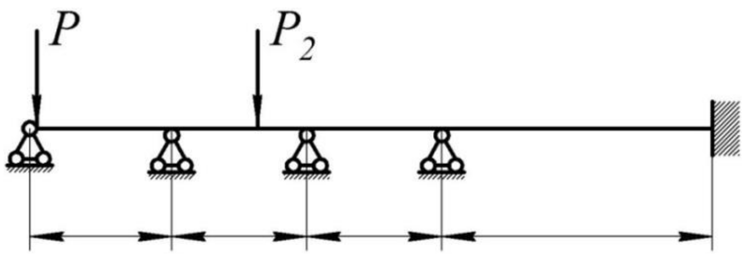


Figure 18.1 - Types of continuous beams with end hinged supports (a) and rigid clamping (b)

Continuous beams are designed in such a way that there is no possibility of the beam detaching from the supports.

First, we will consider **the case of hinged supports**. One of the supports of a continuous beam is usually made articulated and fixed, while the others are articulated and movable. We will number the supports and spans from left to right, designating the leftmost support as number 0 and the leftmost span as number 1. Span lengths will be denoted by the letter ℓ with a number corresponding to the span number. The cross-sections of the beam in all spans are assumed to be the same and, therefore, the stiffness of the beam $E\mathcal{I}$ – constant. Fig. 18.2 shows an unbraced beam, indicates the conventional notation and shows the possible reactions of the supports. As can be easily seen, the number of extra support reactions is equal to the number of intermediate supports.

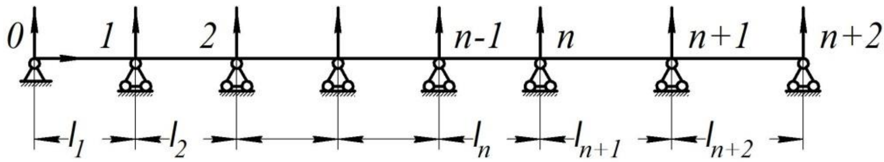


Figure 18.2 - Statically indeterminate continuous beam on hinged supports

Applying the above method of solving the problem, it would be necessary to take the reactions of the intermediate supports as additional unknowns, and the beam on the hinged supports at points 0 and $n + 2$. as the main system. Additional equations would be the conditions of zero deflections at the points of the main system corresponding to the intermediate supports; in this case, all unknowns would be included in all equations. However, another method is simpler and more common, associated with a different type of basic system and additional unknowns; in this method, no more than three unknowns are included in each of the equations.

The operations of selecting the redundant unknown and the main system are inextricably linked; the main statically determinate system is obtained from the statically indeterminate one by discarding the support fixtures that cause the support reactions considered "redundant". You can do it differently: turn an indeterminate structure into a statically determinate one in some way, and then see what forces or reactions would have to be discarded. These quantities will be the "extra" unknowns in our statically indeterminate system.

Thus, in an undivided beam (Fig. 18.3, a), the reaction of the middle support B can be assumed to be unnecessarily unknown; then the main system will be a beam on two supports A and C; but it is possible to turn this beam into a statically determinate one by adding an additional hinge at point D (Fig. 18.3, b). Then we will have a basic system consisting of a cantilever beam CBD and a suspended beam AD. The placement of the hinge in section D requires that the bending moment M_D, i.e., the normal stress caused by it, in this section be zero.

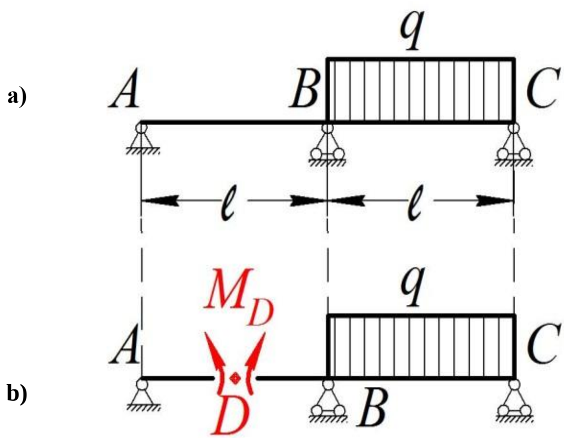


Figure 18.3 - Double span continuous beam

Thus, in the transition to the basic system, we discarded the normal stress in section *D*, which is transmitted from the left to the right, and vice versa; summarized in pairs of forces equal to the bending moment in section *D*; these pairs, again applied to the basic system, are shown in Fig. 18.3. By transforming our beam into a statically determined one by introducing the hinge *D*, we choose the value of the bending moment in this section as the "extra" unknown, rather than the external force - one of the supporting reactions.

The position of the section *D* can be taken arbitrarily; the calculations are the simplest if we combine the point *D* with the reference section over the intermediate support - point *B*,. take the "extra" unknown

bearing moment in the section *B* as the "extra" unknown. Then the main system will be two simple beams, on hinged supports at points *A*, *B* and *C*, which have a common support at point *B*. This is exactly how the basic system is chosen when calculating continuous beams. The values of the bending moments M_1, M_3,…., M_{n-1}, M_n, M_{n+1} over all intermediate supports are chosen as "extra" unknowns. With such a choice of "extra" unknowns, the equations for their determination are simplified and can be compiled in a general form using the three-moment theorem.

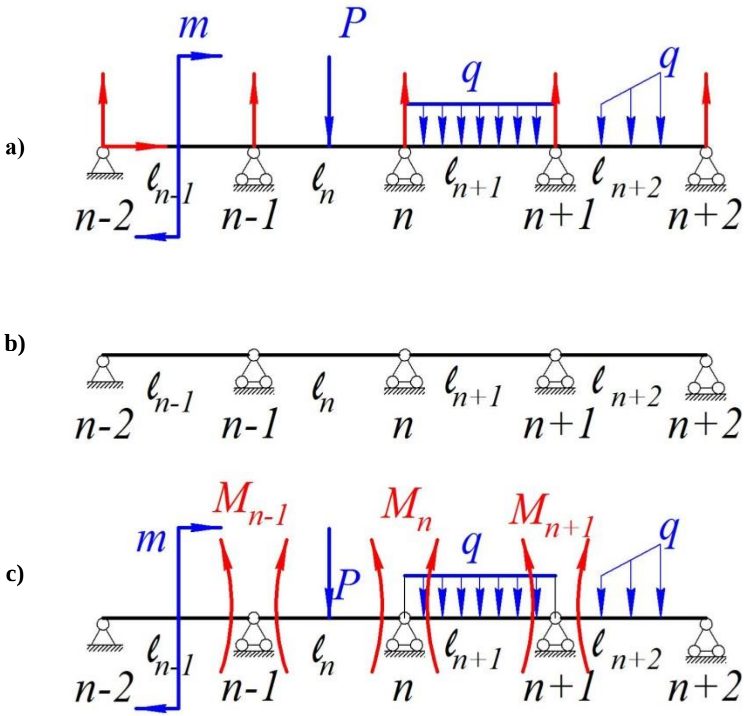
18.2. Equation of three moments

To derive **the three-moment theorem,** consider an inextensible beam with a number of spans of different lengths ℓ_1, ℓ_2 etc. with different loads (Fig. 18.4, a). **The supports are numbered from left to right, denoting the leftmost one with the number 0, and the span number is determined by the number of the rightmost support.** First, let's show all the reactions that occur in this case; from the statics point of view, the horizontal reaction $H = 0$.

In order to obtain the basic system (Fig. 18.4, b), we add hinges in the support sections. Additional unknowns will be the support moments on the intermediate supports M_{n-1}, M_n, M_{n+1}. **The moments on the extreme supports are zero.**

Let us load the main system with external loads and bearing moments (equivalent system) (see Fig. 18.4, c).

The inserted hinges in the main system allow the beam to deform from the external load within one span. In order for the main system to work as specified, in the equivalent system, **we apply fictitious bending moments at the points of hinge insertion** and select their **values and directions so that the cut beam works as a whole. The moment has the same name as the support against which it is applied.**



a) a given system; b) the basic system; c) equivalent system.
Figure 18.4 - Multi-span beam

In the basic system, both sides of the n −th support section can rotate under load independently of each other (Fig. 18.5).

The angle of rotation of the section at the support n of the left span from the section support is denoted by $\varphi_{n\,left}$, and the angle of rotation for the right span is denoted by $\varphi_{n\,right}$.

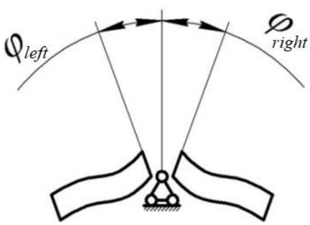


Figure 6.5 - Angles of rotation of the beam sections on the support

In a continuous beam, both span sections coincide, they are only different sides of the same support section, so the condition for compatibility of deformations will be

$$\varphi_{left} + \varphi_{right} = 0. \qquad (18.1)$$

We can make this condition for each of the intermediate supports, and write as many additional equations as we have unknown moments.

Let's explain how the conditions (6.1) are formed using the example (Fig. 18.6). Let's select two adjacent spans from the beam (Fig. 18.4, c) and write down condition (18.1):

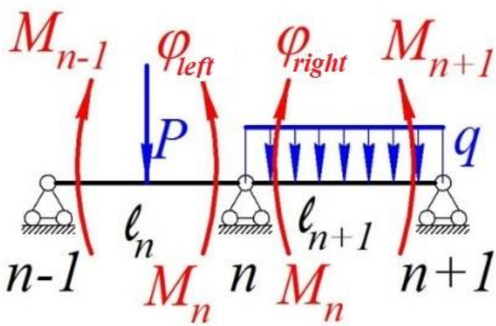


Figure 18.6 - Two adjacent spans

$$\varphi_{left} = \varphi_{M_{n-1}} + \varphi_{M_n} + \varphi_P^{left}; \qquad (18.2)$$

$$\varphi_{right} = \varphi_{M_{n+1}} + \varphi_{M_n} + \varphi_q^{right}, \qquad (18.3)$$

where $\varphi_{M_{n-1}}$ – is the angle of rotation of section n from the bending moment M_{n-1} on span ℓ_n.

φ_{Mn} – is the angle of rotation of section n from the bending moment M_n on span ℓ_n.

φ_P^{left} – is the angle of rotation of section n from force P on span ℓ_{n+1}.

Using condition 18.1, write down:

$$\varphi_{M_{n-1}} + 2\varphi_{M_n} + \varphi_{M_{n+1}} = -\left(\varphi_P^{left} + \varphi_q^{right}\right). \qquad (6.4)$$

To determine the angles of rotation included in equation 6.4, consider two independent beams at spans ℓ_n i ℓ_{n+1}.

The angles of rotation of the sections in the main system at support n depend on the deformations of only two adjacent spans ℓ_n and ℓ_{n+1}. Let us consider these two spans. Span ℓ_n s subjected to an external force P and the supporting moments M_n nd M_{n-1}; span ℓ_{n+1} is subjected to an external distributed load of intensity q and the supporting moments M_n and M_{n+1}.

For clarity, both adjacent spans are shown separately.

To calculate the angles of rotation, we use **the graph-analytical method.** The dummy beams are also hinged-supported beams (Figure 6.7, a, and Figure 18.8, a).

The load epure is the epure of moments from the total load.

Dummy loads of the left span ℓ_n:

a) load bending moment epure of the external load P with area ω_n and distance from the center of gravity of this area to the left support a_n;

b) triangular bending moment epure from the additional supporting moment $\overline{M}_{n-1} = 1$;

c) triangular epure of the bending moment from the positive support moment $\overline{M}_n = 1$.

The right dummy beam of span ℓ_{n+1} is subjected to a distributed load. We construct the epures (Fig. 18.8, b):

a) the bending moment epure from external forces *(q)* with the loaded area ω_{n+1} and the distance from the center of gravity of this area to the right support b_{n+1} .

b) triangular epure of the bending moment from the positive supporting moment $\overline{M}_{n+1} = 1$;

c) triangular epure of the bending moment from the positive supporting moment $\overline{M}_n = 1$.

Let us consider the left span of length $\boldsymbol{\ell_n}$.

Using Vereshchagin's method, we find the angles of rotation of section n from the bending moments $\overline{M}_{n-1}, \overline{M}_n$ and load P.

$$\varphi_{Mn-1} = \frac{\omega_1 \cdot \overline{y}_1}{E\mathcal{J}_x} = \frac{1}{E\mathcal{J}_x}\frac{P\ell_n^2}{4}\frac{\overline{M}_{n-1}}{2} = \frac{P\ell_n^2\overline{M}_{n-1}}{8E\mathcal{J}_x}, \qquad (18.5)$$

where ω_1 – area of the load epure of moments from the force P:

$$\omega_1 = h_1 \ell_n = \frac{P\ell_n}{4}\ell_n = \frac{P\ell_n^2}{4};$$

$$h_1 = \frac{P}{2} \cdot \frac{\ell_n}{2} = \frac{P\ell_n}{4}.$$

$\overline{y}_1$– the ordinate under the center of gravity of the load epure on the unit moment epure $\overline{M}_{n-1}$

$$\overline{y}_1 = \frac{\overline{M}_{n-1}}{2}.$$

$$\varphi_{Mn} = \frac{\omega_1 \cdot \overline{y}_2}{E\mathcal{J}_x} = \frac{1}{E\mathcal{J}_x}\frac{P\ell_n^2}{4}\frac{\overline{M}_n}{2} = \frac{P\ell_n^2 \overline{M}_n}{8E\mathcal{J}_x}, \tag{18.6}$$

where ω_1 – the area of the load epure of moments from the force$P$:

$$\omega_1 = h_1 \ell_n = \frac{P\ell_n}{4}\ell_n = \frac{P\ell_n^2}{4};$$

$$h_1 = \frac{P}{2} \cdot \frac{\ell_n}{2} = \frac{P\ell_n}{4}.$$

$\overline{y}_2$– the ordinate under the center of gravity of the load epure on the unit moment epure $\overline{M}_n$

$$\overline{y}_2 = \frac{\overline{M}_n}{2}.$$

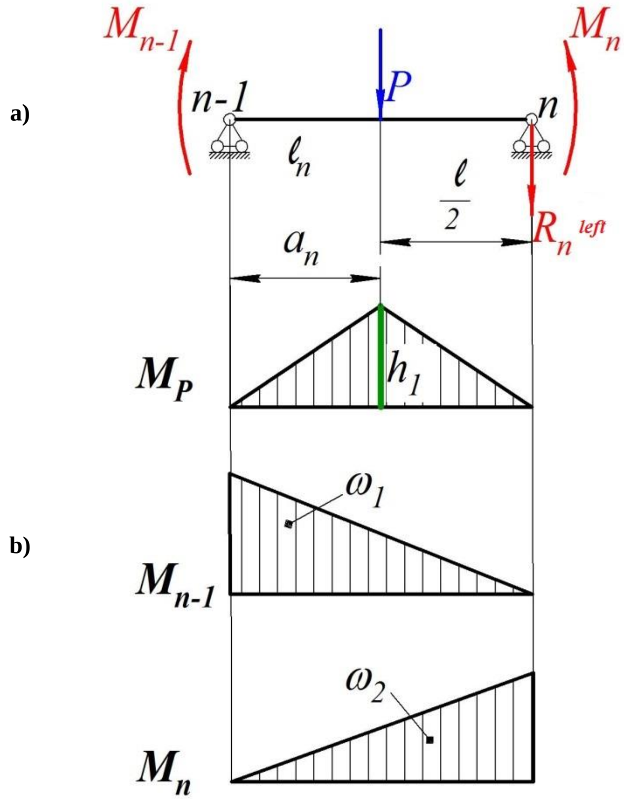


Figure 18.7 - Left span of a beam with epures

Let's consider the right span of length ℓ_{n+1}.

Using Vereshchagin's method, we find the angles of rotation of section n from the bending moments $\overline{M}_{n+1}$, $\overline{M}_n$ and the distributed load q.

$$\varphi_{Mn+1} = \frac{\omega_2 \cdot \overline{y}_1}{E\mathcal{J}_x} = \frac{1}{E\mathcal{J}_x}\frac{q\ell_{n+1}^3}{6}\frac{\overline{M}_{n+1}}{2} = \frac{q\ell_{n+1}^3\overline{M}_{n+1}}{12E\mathcal{J}_x}, \qquad (18.7)$$

where ω_2 – the area of the load moment epure from the distributed load q;

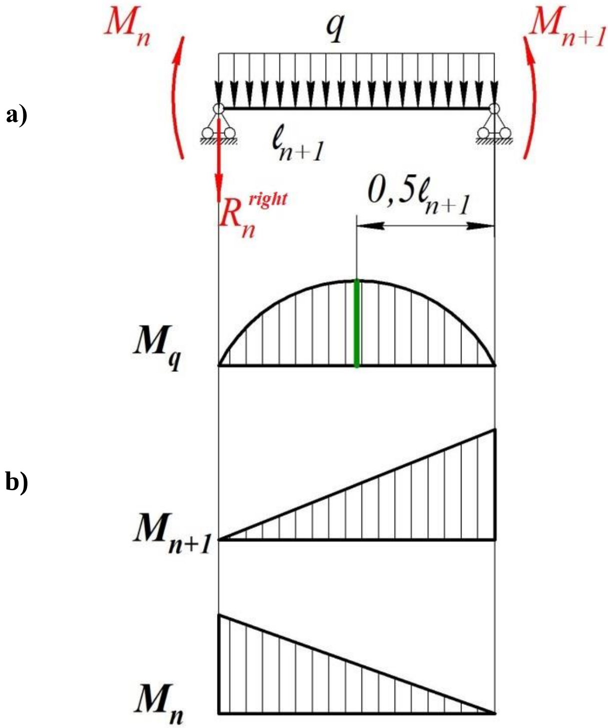


Figure 18.8 - Right beam span with epures

$$\omega_2 = \frac{2}{3} h_2 \ell_{n+1} = \frac{2}{3}\,\frac{q\ell_{n+1}^2\,(\ell_{n+1})}{4} \quad = \frac{P\ell_{n+1}^3}{6};$$

$$h_2 = \frac{q\ell_{n+1}}{2} \cdot \frac{\ell_{n+1}}{2} = \frac{q\ell_{n+1}^2}{4}.$$

$\overline{y}_1$– the ordinate under the center of gravity of the load epure on the unit moment epure $\overline{M}_{n+1}$

$$\overline{y}_1 = \frac{\overline{M}_{n+1}}{2}.$$

$$\varphi_{Mn} = \frac{\omega_2 \cdot \overline{y}_2}{E\mathcal{J}_x} = \frac{1}{E\mathcal{J}_x}\frac{q\ell_{n+1}^3}{6}\frac{\overline{M}_n}{2} = \frac{q\ell_{n+1}^3\overline{M}_n}{12E\mathcal{J}_x}, \tag{18.8}$$

ω_2 – the area of the load moment epure from the distributed load q:

$$\omega_2 = \frac{2}{3}h_2\ell_{n+1} = \frac{2}{3}\,\frac{q\ell_{n+1}^2\,(\ell_{n+1})}{4} \quad = \frac{q\ell_{n+1}^3}{6};$$

$$h_2 = \frac{q\ell_{n+1}}{2}\cdot\frac{\ell_{n+1}}{2} = \frac{q\ell_{n+1}^2}{4}.$$

$\overline{y}_2$– the ordinate under the center of gravity of the load epure on the unit moment epure $\overline{M}_n$

$$\overline{y}_2 = \frac{\overline{M}_n}{2}.$$

Substituting the values of the angles of rotation of the sections (18.5-18.8) into equation (18.4), we have

$$\frac{P\ell_n^2\overline{M}_{n-1}}{8E\mathcal{J}_x} + \frac{P\ell_n^2\overline{M}_n}{8E\mathcal{J}_x} + \frac{q\ell_{n+1}^3\overline{M}_{n+1}}{12E\mathcal{J}_x} + \frac{q\ell_{n+1}^3\overline{M}_n}{12E\mathcal{J}_x} = -\left(\varphi_P^{left} + \varphi_q^{right}\right).$$

$$\frac{P\ell_n^2\overline{M}_{n-1}}{8E\mathcal{J}_x} + \frac{q\ell_{n+1}^3\overline{M}_{n+1}}{12E\mathcal{J}_x} + \frac{\overline{M}_n(3P\ell_n^2 + 2q\ell_{n+1}^3)}{24E\mathcal{J}_x} = -\left(\varphi_P^{left} + \varphi_q^{right}\right)$$

$$3\overline{M}_{n-1}P\ell_n^2 + 2\overline{M}_{n+1}q\ell_{n+1}^3 + \overline{M}_n(3P\ell_n^2 + 2q\ell_{n+1}^3) =$$

$$= -24E\mathcal{J}_x\left(\varphi_P^{left} + \varphi_q^{right}\right). \tag{18.9}$$

Thus, we obtained **the equation of three moments.**

To solve the problem using three moments, you need to create as many equations as there are intermediate supports for a given beam. In this case, the same span should be included in the equation twice. The system of equations is solved by the method of successive elimination of unknowns.

The angle of rotation of the support section n of the left span is equal to the transverse force divided by the stiffness for the corresponding dummy beam:

$$\varphi_n^{left} = \frac{Q_n^{left}}{E\mathcal{J}_x} = \frac{R_n^{left}}{E\mathcal{J}_x}, \tag{18.10}$$

since the transverse force Q_n^{left} in the reference section is equal to the reference reaction of the dummy beam R_n^{left}.

For the right span, the angle of rotation:

$$\varphi_n^{right} = \frac{Q_n^{right}}{E\mathcal{J}_x} = \frac{R_n^{right}}{E\mathcal{J}_x}. \tag{18.11}$$

Let us substitute the expressions from (18.10) and (18.11) into equation (6.1) instead of the angles φ and obtain

$$\varphi_n^{left} + \varphi_n^{right} = \frac{Q_n^{left}}{E\mathcal{J}_x} + \frac{Q_n^{right}}{E\mathcal{J}_x} = \frac{R_n^{left}}{E\mathcal{J}_x} + \frac{R_n^{right}}{E\mathcal{J}_x} = 0, \tag{18.12}$$

or

$$R_n^{left} + R_n^{right} = 0. \tag{18.12'}$$

The values R_n^{left} and R_n^{right} represent the reactions at the support n of the left and right spans of the dummy beam (Figures 18.7 and 18.8). thus, the three-moment theorem can be formulated as follows:

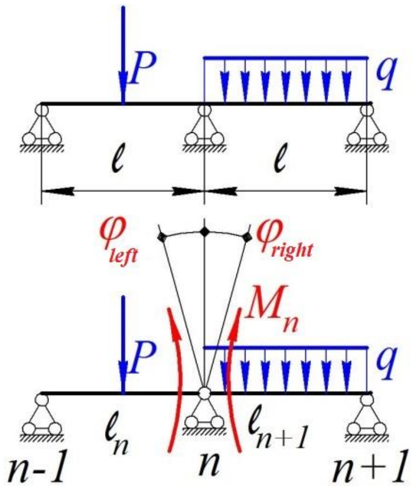
Dummy Reactions Theorem: The sum of dummy reactions at each intermediate support must be zero.

18.3 Special cases of application of the equation of three moments

1. If there are no concentrated moments at the extreme supports of the beam, then in the equation of three moments, the moments at these supports will be zero.

a)

b)



a) a given system;
b) equivalent system
Figure 18.9 - There are no centered moments at the extreme supports of the beam

For such a system, the equation of three moments is as follows:

$$\bar{0}\ell_n + 2M_n(\ell_n + \ell_{n+1}) + 0\ell_{n+1} = -6\, E\mathcal{J}_x(\varphi_{left} + \varphi_{right}), \quad (18.13)$$

where φ_{left} – the angle of inclination of the right end of the beam at span ℓ_n due to the force P;

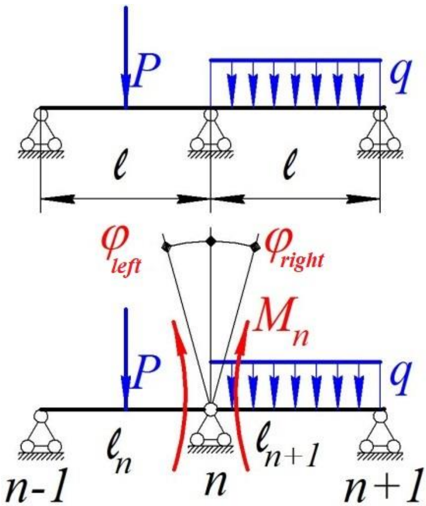
φ_{right} – the angle of inclination of the left end of the beam at span ℓ_{n+1} due to the distributed load.

The span angle is determined by any known method (Mohr or Vereshchagin).

1. If the beam has a **cantilever with a load**, then in the equivalent system, the bending moment on this support is equal to the moment from the load applied to the cantilever relative to this support.

a)

b)



a) a given system;
b) equivalent system
Figure 18.10 - A beam has a cantilever with a load

The equation of three moments for such a system is

$$\bar{0}\ell_n + 2M_n(\ell_n + \ell_{n+1}) - \frac{P\ell}{2}\ell_{n+1} = -6\,E\mathcal{I}_x(\varphi_{left} + \varphi_{right}). \quad (18.14)$$

2. If **the end of the beam is clamped**, then the clamping must be replaced by an additional span of infinitesimal length or infinitely large stiffness (Figure 18.11).

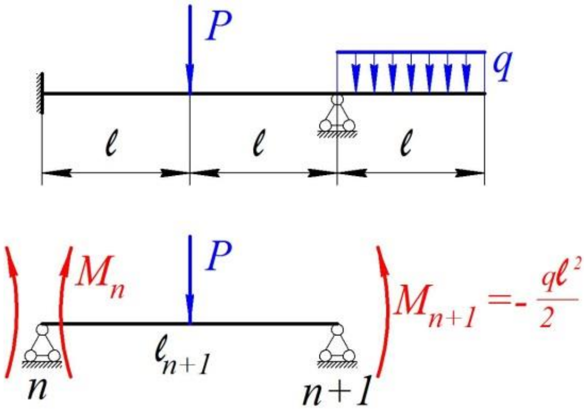
The equation of three moments for such beams is as follows

$$\bar{0}\ell_n + 2M_n(\ell_n + \ell_{n+1}) - \frac{P\ell}{2}\ell_{n+1} = -6\,E\mathcal{I}_x(\varphi_{left} + \varphi_{right}), \quad (18.15)$$

where ℓ_n – a dummy span on which $M_{n-1} = 0,\ M_n = 0.$

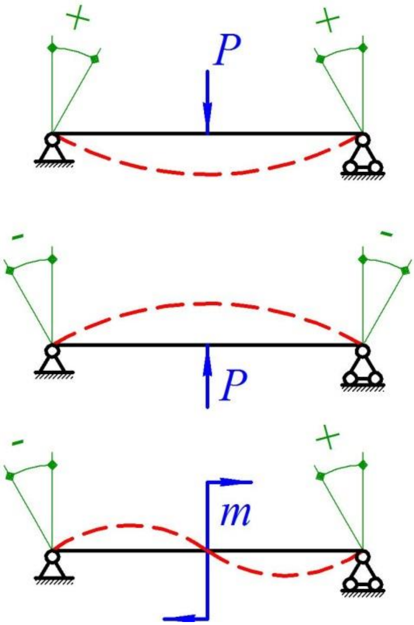
a)

b)



a) a given system;
b) equivalent system
Figure 18.11 - Beam with a clamped end

The rule of signs in the equation of three moments for the angle of inclination (Figure 18.12).



a)

b)

c)

Figure 18.12 - Sign rule for the angle of inclination

If **the angle of inclination is in the middle of the span, then it is positive in the equation of three moments** (Figure 18.12, a). If the angle of inclination goes beyond the span, it is negative in the equation of three moments (Fig. 18.12, b).

If the calculated **bending moment compresses the upper fibers of the span beams, it is positive.** If the calculated **moment is negative, then it must be directed so that it compresses the lower fibers of the beam in the span.**

Thus, we can recommend the following **procedure for calculating a continuous beam:**

1. Having numbered the supports and spans, an equivalent system is depicted under the given beam, taking into account the comments in individual cases.
2. Make equations of three moments for every two adjacent spans.
3. Determine the angles of inclination of the sections from the external load in each span and substitute them into the equation of three moments.
4. Solving the equation, determine the unknown bending moments on the intermediate supports where the hinges are installed, and their direction.
5. Taking into account the calculated bending moments, determine the reactions of the supports and build the epures Q_y and M_x on the span, and then for the entire beam as the sum of the span epures.

18.4 Control questions

1. What is called a continuous multi-span beam?
2. What is the static indeterminacy of an unbraced beam?
   a) when the extreme support is a fixed joint;
   b) when the extreme support is a rigid anchorage?
3. Numbering of supports and spans in an unbreakable beam.
4. The theorem of fictitious reactions.
5. Write the equation of three moments in general form.
6. The rule of signs in the equation of three moments for the angle of rotation φ.
7. Sequence of calculation of an unbroken beam.
8. Formula for determining the static indeterminacy of a beam.

19. STRENGTH CALCULATIONS UNDER DYNAMIC LOADS

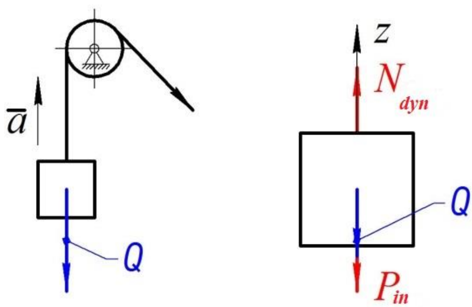
Strength calculations taking into account inertial forces have to be performed when structural elements undergo large accelerations during operation, which cause significant inertial forces.

The general calculation method for dynamic loads is based on the **D'Alembert principle.** According to this principle, any moving body can be considered in instantaneous equilibrium. To do this, it is necessary to apply inertial forces to the external load, which are equal to the product of the body mass and its acceleration and are directed in the opposite direction to this acceleration.

If the forces of inertia are unknown and their determination is difficult, then the dynamics problems in this case are solved on the basis of the law of energy conservation.

19.1 Strength calculation with inertial forces

Let any body of weight Q be lifted with acceleration $\bar{a}$, (Fig. 19.1 a). Determine the dynamic stresses σ_{dyn} in the rope without taking into account the load of the rope itself. The body, in addition to the force of weight Q is also subject to the force of inertia P_{in}, which is equal to the product of body mass and acceleration, i.e. $P_{in} = ma = \frac{Q}{g}a$ and is directed in the opposite direction to the acceleration $\bar{a}$.



a) b)

Figure 19.1 - Calculation scheme of dynamic load

Let's fix the body at some point and draw up **the equilibrium condition** (Figure 19.1, b):

$$\sum P_{\text{iz}} = 0, \qquad N_{dyn} - Q - \frac{Q}{\mathrm{g}}a = 0, \tag{19.1}$$

where g – acceleration of free fall; g=9,81 м/с^2; N_{dyn} – dynamic reaction in the rope

$$N_{dyn} = Q\left(1 + \frac{a}{\mathrm{g}}\right). \tag{19.2}$$

But when stretched:

$$\sigma_{dyn} = \frac{N_{dyn}}{F}, \qquad N_{dyn} = \sigma_{dyn}F. \tag{19.3}$$

Equating the right-hand sides of 7.2 and 7.3, we get

$$\sigma_{dyn}F = Q\left(1 + \frac{a}{\mathrm{g}}\right); \tag{19.4}$$

$$\sigma_{dyn} = \frac{Q\left(1 + \frac{a}{\mathrm{g}}\right)}{F} = \frac{Q}{F}\left(1 + \frac{a}{\mathrm{g}}\right). \tag{19.5}$$

When stretched

$$\frac{Q}{F} = \sigma_{static} = \sigma_{st.}$$

We accept

$$1 + \frac{a}{\mathrm{g}} = K_d.$$

K_d **- dynamic coefficient.**

Then 7.5 will be rewritten:

$$\sigma_{dyn} = \sigma_{st} \cdot K_d. \tag{19.6}$$

Similarly, we can write down that

$$\Delta\ell_{dyn} = \Delta\ell_{st} \cdot K_d \quad . \tag{19.7}$$

From formulas 19.6 and 19.7, we can say that any desired **dynamic value is equal to the static value multiplied by the dynamism coefficient.**

The static value is the value obtained by applying a static load.

The strength condition can be written from formula (19.6),

$$\sigma_{dyn} = \sigma_{st} \cdot K_d \leq [\sigma], \tag{19.8}$$

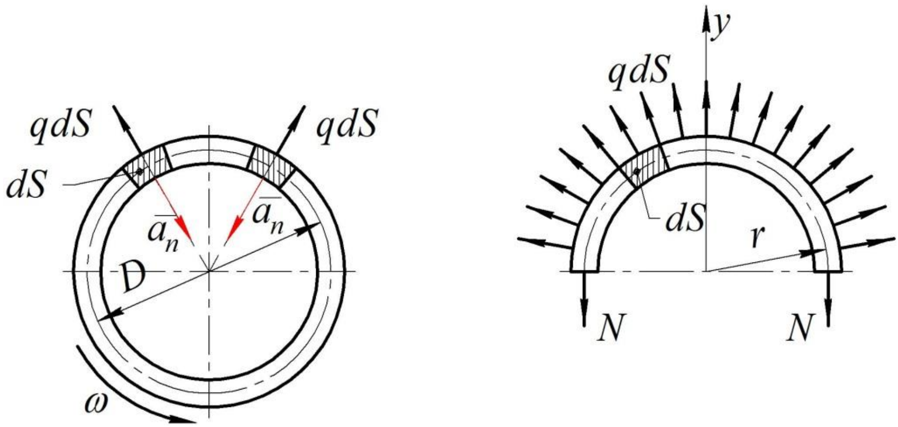
and the **stiffness condition** from formula (19.7), respectively:

$$\Delta\ell_{dyn} = \Delta\ell_{st} \cdot K_d \leq [\Delta\ell]. \tag{19.9}$$

19.2 Determination of stresses in rotating rings

Consider the calculation of stresses in a rapidly rotating ring of constant cross-section (Fig. 19.2,a). A bicycle wheel or flywheel rim is in similar conditions, if we ignore the influence of the spokes.

The ring is considered thin if its thickness is small compared to the radius r . In this case, we can assume that all points of the ring are at the same distance from the axis of rotation, which is equal to $D/_2$.



a) **b)**

Figure 19.2 - Calculation of the ring with uniform rotation

Let F denote the cross-sectional area of the ring,

γ – the volume weight of the material, n – the number of revolutions per unit time

$$\omega = \frac{\pi \cdot n}{30}, \left(\frac{1}{sec}\right) - \text{ the angular velocity of rotation.}$$

Let's select a ring element of length ds. When rotating, this element moves in a circle with a constant angular velocity ω. The angular acceleration ε is zero. Therefore, the tangential acceleration of the element is zero:

$$a_\tau = \varepsilon \cdot \frac{D}{2} = 0,$$

normal (centripetal) acceleration of the element

$$a_n = \frac{\omega^2 \cdot D}{2} = \omega^2 r$$

and directed to the center of the ring (Fig. 19.2, a).

Each element of the ring of unit length is subject to a force of inertia in the form of a centrifugal force, the magnitude of which (intensity) is equal to

$$q = \frac{\gamma \cdot F}{\mathrm{g}} \cdot \omega^2 \cdot r = ma_n. \qquad (19.10)$$

Centrifugal forces are directed along the radius. Their effect on the ring is similar to that of a uniform internal pressure of intensity q. Due to the axial symmetry of the system and the load in the transverse sections, the bending moments and transverse forces are zero. To determine the normal forces acting in the transverse (radial) sections of the ring, consider the equilibrium of its half (see Fig. 19.2, b).

The equivalent of the distributed load of intensity q is equal to the product of q times the diameter, perpendicular to the diameter and acting along the axis passing through its middle, i.e., along $\boldsymbol{y}$. The equilibrium condition of the half-ring will take the form:

$$\sum P_{iy} = 0, \qquad 2N - \mathrm{g}2r = 0\,.$$

Whence

$$N = \mathrm{g} \cdot r. \tag{19.11}$$

Normal stresses in the ring cross-section

$$\sigma = \frac{N}{F} = \frac{\mathrm{g} \cdot r}{F}. \tag{19.12}$$

Substituting the value of q from formula (19.10), we obtain:

$$\sigma = \frac{\gamma}{\mathrm{g}}\omega^2 r^2, \tag{19.13}$$

or

$$\sigma = \frac{\gamma}{\mathrm{g}}\left(\frac{\pi \cdot n}{30}\right)^2 r^2. \tag{19.14}$$

The stress can be expressed in terms of its linear velocity v. Taking into account equation (19.13), we have $v = \omega \cdot r$:

$$\sigma = \frac{\gamma}{\mathrm{g}} v^2. \tag{19.15}$$

As can be seen from formula (19.15), **the stress does not depend on the cross-sectional area of the ring.**

Strength condition of a rotating ring

$$\sigma = \frac{\gamma}{\mathrm{g}} v^2 \leq [\sigma]. \tag{19.16}$$

Determine the value of the permissible linear speed (critical)

$$v_{cr} = \sqrt{\frac{[\sigma]\mathrm{g}}{\gamma}}. \tag{19.17}$$

As can be seen from formula (19.17), the critical speed at which the ring can fracture depends only on the material and is independent of the ring dimensions.

The relative elongation of the ring in the circular direction is

equal to

$$\varepsilon_0 = \frac{\sigma}{\mathrm{E}} = \frac{\gamma}{\mathrm{g} \cdot E}\omega^2 r^2$$

or

$$\varepsilon_0 = \frac{2\pi r_1 - 2\pi r}{2\pi r} = \frac{r_1 - r}{r} = \frac{\Delta_r}{r}, \qquad (19.18)$$

where $\Delta_r = r_1 - r$ – **change in the size of the ring in the radial direction.**

The same is true for the relative elongation of the ring in the radial direction:

$$\varepsilon_r = \frac{r_1 - r}{r} = \frac{\Delta_r}{r}. \qquad (19.19)$$

So $\varepsilon_\theta = \varepsilon_r$.

19.3 Strength calculations for a straight rotating bar

A straight rod of rectangular cross-section rotates about an axis perpendicular to the axis of the rod with constant angular velocity ω. It is necessary to construct an epure of longitudinal forces due to the action of inertia.

The inertial forces during the rotation of the rod cause tensile deformation. Consider the equilibrium of the element dz, which is located at a distance z from the axis of rotation.

$$dN_z = q_{in} dz \qquad (19.20)$$

Taking into account (19.10 and 19.11), replacing R by z we have:

$$N_z = \int_z^{\ell/2} \frac{\gamma F \omega^2}{\mathrm{g}} z dz = \frac{\gamma F \omega^2 z^2}{2\mathrm{g}}\bigg|_z^{\ell/2} = \frac{\gamma F \omega^2}{2\mathrm{g}}\left[\left(\frac{\ell}{2}\right)^2 - z^2\right].$$

When $z = 0$

$$N_z = \frac{\gamma F \omega^2 \ell^2}{8\mathrm{g}}$$

(on the axis of rotation).

When $z = \ell/2$ $\quad N_z = 0$ (at the end of the rod).

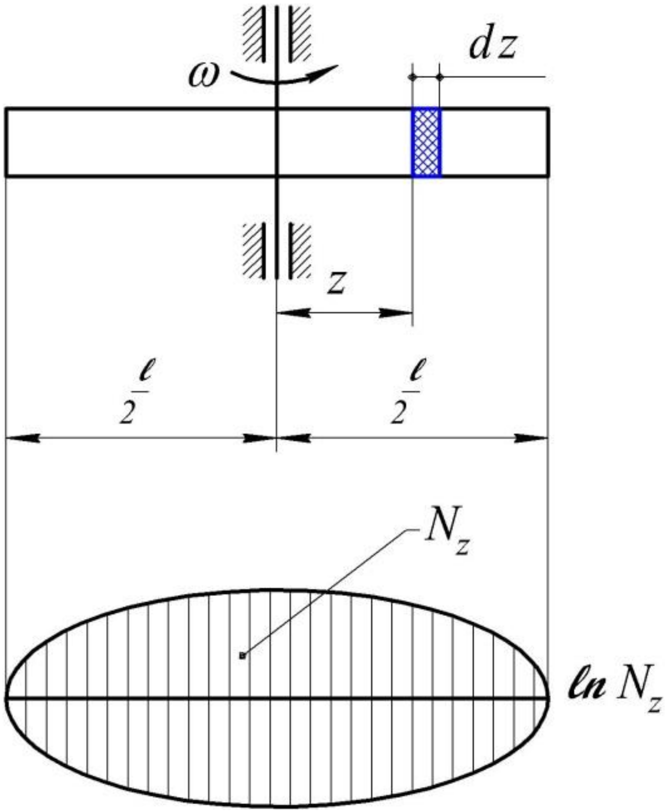


Figure 19.3 - Strength calculation of a straight rotating bar

As can be seen from the N_z epure, the longitudinal forces are distributed according to the parabolic law (z^2). The greatest stresses will be on the axis of rotation and are equal:

$$\sigma_{max} = \frac{N_{z_{max}}}{F} = \frac{\gamma \omega^2 \ell^2}{8g} \leq [\sigma]. \qquad (19.21)$$

19.4 Control questions

1. What is the principle of the dynamic load calculation method?
2. What is the coefficient of dynamism?
3. How are dynamic stresses calculated?
4. How are dynamic deformations calculated?
5. Strength condition under dynamic loading?
6. Stiffness condition under dynamic loading?
7. Strength condition of a rotating ring.
8. Relative elongation of the ring in the circumferential and radial directions.
9. Change in the size of the ring in the radial direction.

20. STRESS DETERMINATION AND CALCULATIONS FOR STRENGTH UNDER IMPACT LOADS

20.1 Impact tolerances

The phenomenon of impact occurs when the velocity of the structural element in question or its parts in contact with it changes by a finite amount within a very short period of time. The large accelerations that occur in this case lead to the appearance of significant inertial forces that act in the direction opposite to the direction of acceleration, i.e., in the direction of body movement.

The following figures (Fig. 20.1) show different types of impact loading in terms of deformation: a) – vertical or horizontal (axial), impact loading causes compression deformation; b) – tensile deformation from impact; c) – bending impact; d) – torsional impact for a section of a rod with a circular cross-section and bending impact for a rod with a square cross-section.

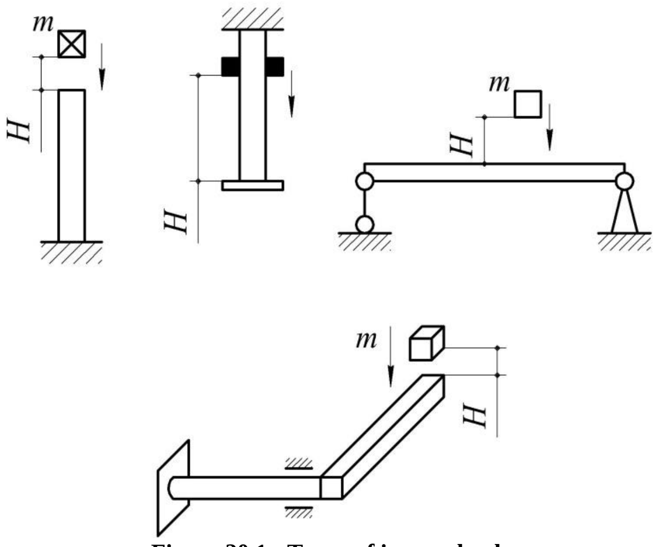


Figure 20.1 - Types of impact load

The theory of impact is very complex, as the speed of structural elements changes from maximum to zero within a short period of time. This results in significant inertial forces caused by large accelerations (decelerations). The time of impact is unknown, and plastic deformations occur at the point of impact, local temperature increases, and other factors that make the calculation difficult.

Thus, due to the complexity of the processes that occur during an impact, the so-called **technical** or **engineering theory of impact** (proposed by T. Jung) is used to calculate structures based on the following **assumptions**.

1. During an impact, the bodies **do not separate from each other** until the greatest deformations develop. In this case, there are no elastic waves and associated rebounds of the impacting body. **The impact is considered inelastic.**

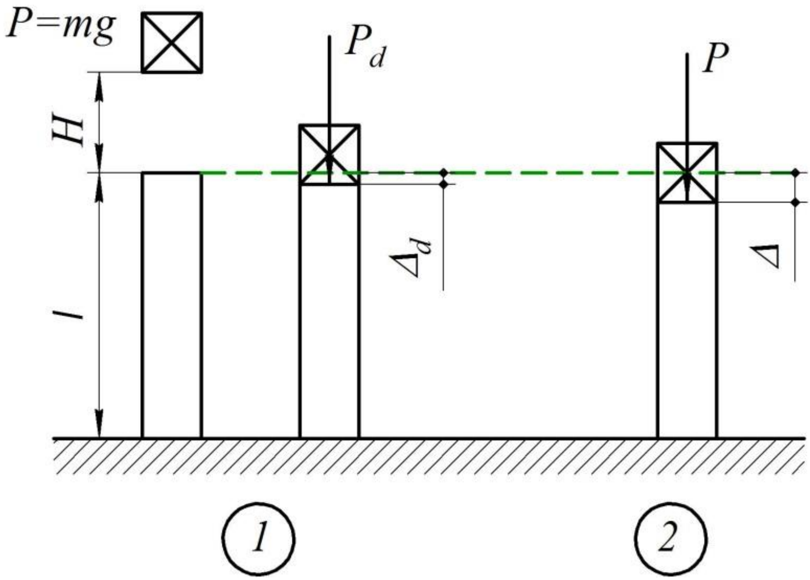
2. During the period of mutual impact, deformations propagate throughout the entire volume of the impacted body, and **the dependence between the forces and deformations** that occur **corresponds to Hooke's law.**

3. In a mutual impact of moving bodies, **the decrease in the kinetic energy of the system is equal to the increase in the potential energy of deformation of the bodies.** At the same time, the losses of thermal energy and energy for local plastic deformations, as well as the inertia of the mass of the body undergoing impact are neglected.

4. It is assumed that a system of bodies under impact has one degree of freedom, i.e., **the position of the system is determined by one coordinate.**

20.2 Impact stress. Strength condition

The formulas for determining the dynamic stresses and strains under axial impact will be derived using the example of a system (Fig. 20.2) consisting of a vertically placed elastic prismatic rod with tensile (compressive) stiffness $c = EF/\ell$, on the end of which a load P falls freely from a height H.



1 – a load of mass *m* falls on a structure from a height *H*:
***P* – the weight of the load; P_d – is the dynamic force; Δ_d – the dynamic displacement.**
2 – static load with force *P*:
***P* – the weight of the load; Δ – the static displacement.**
Figure 20.2- Dynamic and static loading of a vertically located rod

At the moment of mutual impact (or rather, in a very short time), the force P_d, which is caused by the fall of the load $P = mg$ from a height H, causes a dynamic contraction of the rod by the value Δ_d.

The displacement under longitudinal dynamic loading is determined by the formula:

$$\Delta_d = \frac{P_d \ell}{EF} = \frac{P_d}{c}, \tag{20.1}$$

where $\frac{EF}{\ell}$ – the stiffness of the rod in tension-compression;

ℓ – the initial length of the rod;

E – Young's modulus; F – cross-sectional area.

If the load is simply placed on the structure, i.e., under static loading, the displacement is determined as follows:

$$\Delta = \frac{P\ell}{E} = \frac{P}{c}, \tag{20.2}$$

So, $P = \Delta \cdot c$, $P_d = \Delta_d \cdot c$.

The relation

$$\frac{\Delta_d}{\Delta} = \frac{P_d}{P} = K_d$$

is called **the dynamic coefficient.**

Hence, the displacement on impact is determined by the formula

$$\Delta_d = \Delta \cdot K_d. \tag{20.3}$$

The dynamic coefficient indicates how many times the maximum stress and strain under impact is greater than the corresponding stress and strain under static load.

Since within Hooke's law, stress is directly proportional to force, then

$$\frac{\Delta_d}{\sigma} = K_d \rightarrow \sigma_d = K_d \cdot \sigma. \tag{20.4}$$

The potential strain energy of an impact is equal to the work of the dynamic force P_d on the dynamic displacement Δ_d (according to the Clapeyron formula):

$$U = \frac{1}{2} P_d \cdot \Delta_d = \frac{1}{2} C\Delta_d^2. \tag{20.5}$$

The lost kinetic energy of the falling load and the deformation of the rod are equal:

$$T = mg(H + \Delta_d) == P(H + \Delta_d). \tag{20.6}$$

Let us write the law of energy conservation $U = T$, equating (20.5) and (20.6)

$$\frac{1}{2} P_d \Delta_d = P(H + \Delta_d),$$

$$\frac{1}{2} \mathrm{C}\Delta_d \Delta_d - \mathrm{C}\Delta H - C\Delta\Delta_d = 0,$$

$$\Delta_d^2 - 2\Delta\Delta_d - 2\Delta H = 0.$$

We obtained a quadratic equation with respect to Δ_d, and its solution:

$$\Delta_d = \Delta \pm \sqrt{\Delta^2 + 2\Delta H} \text{ or } \Delta_d = \Delta\left(1 \pm \sqrt{1 + \frac{2H}{\Delta}}\right).$$

Since from (20.3) $\Delta_d = \Delta \cdot K_d$, then

$$K_d = 1 \pm \sqrt{1 + \frac{2H}{\Delta}}. \qquad (20.7)$$

The condition for impact strength is as follows:

$$\sigma_{dmax} = \sigma_{st} \cdot K_d \leq [\sigma_d] = \frac{\sigma_T}{n_T}. \qquad (20.8)$$

The coefficient of safety n_T can be chosen the same as for static loading (1.4...1.6), since the dynamism is already reflected in the calculation formula for the coefficient K_d. However, taking into account the approximation of the considered calculation method, this coefficient is taken to be somewhat larger than $n_T = 2$.

20.3 Mechanical properties of the material under impact

To test the ability of a material to resist impact loads, a special type of impact bending test is used to determine the impact strength of notched specimens. This test is carried out on pendulum mills (Fig. 20.3) on standard-shaped material samples with a cut on one side.

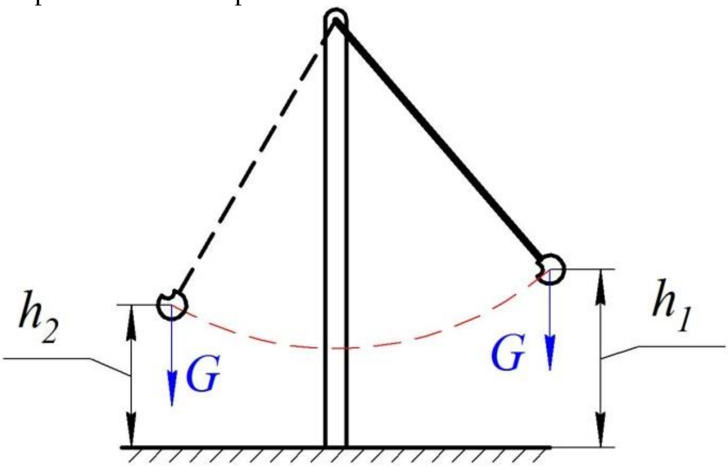


Figure 20.3 - Scheme of the copra

Thus, **the impact strength of the KC material is the work required to fracture the sample, referred to the area of its cross-section at the cut point.**

$$KC = \frac{A}{F} = \frac{G(h_1 - h_2)}{F}, \qquad J \cdot m^2.$$

Low impact strength is grounds for rejecting the material or not using it in critical elements and structures. The impact strength of a steel depends on its structure, and this dependence cannot be detected in static tests. At low temperatures, most ferrous metals become brittle, and their impact strength also decreases. For such metals, a so-called **critical temperature is established at which the impact strength suddenly decreases.**

Unlike other types of mechanical property testing, the laws of similarity do not apply to impact strength testing. Therefore, the test should be performed only on standard specimens (Figu20e 8.4).

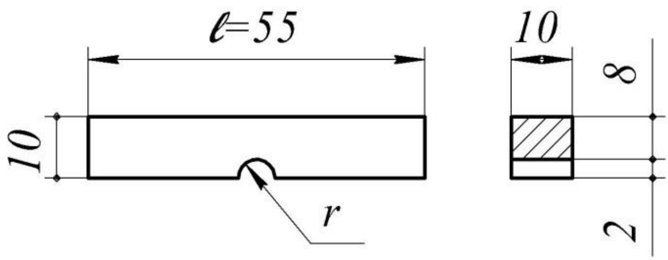


Figure 20.4 - Standard specimen for impact testing with a U-center

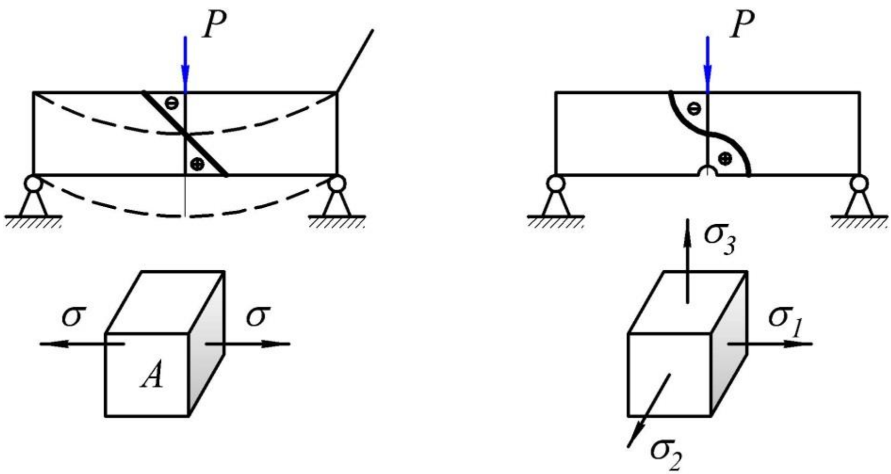
Depending on the type of cut, the impact strength of the material is distinguished:

KCU – with a U-shaped cut with a radius of 1 mm;

KCV – with a V-shaped cut;

KCT – with a developed crack.

The cuts are made to ensure that the material has a volumetric stress state at the beginning of crack development.



Linear stress state **Volumetric stress state**

Figure 20.5 - Linear and volumetric stress states

20.4 Control questions

1. What is the phenomenon of impact?
2. Types of impact load
3. What assumptions are taken into account when calculating structures under impact?
4. The concept of the coefficient of dynamism?
5. How to choose the safety factor at impact?
6. What are the types of impact strength depending on the type of cut?
7. What is the impact strength of a material?
8. For which metals is the critical temperature of brittleness determined in impact tests?

21. STRESS DETERMINATION AND VIBRATION STRENGTH CALCULATIONS

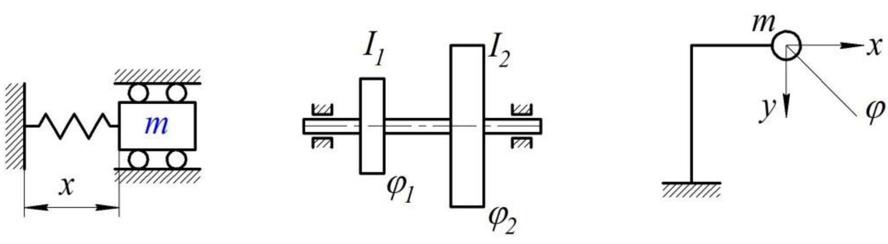
21.1 Oscillations: basic concepts

Vibration theory is a vast branch of modern mechanics that covers a very wide range of issues in mechanics, electrostatics, radio engineering, optics, etc. The theory of vibrations is of particular importance for applied problems encountered in engineering practice, in particular, in the issues of the strength of machines and structures. There have been cases when a building structure designed with a large margin of safety for static loading collapsed under the influence of relatively small periodic forces, while an equally lightweight and seemingly less durable structure perceives these forces quite calmly. Therefore, the issues of oscillations and, in general, the behavior of elastic systems under variable loads require special attention from the designer.

When studying vibrations, elastic systems are usually distinguished, first of all, by the number of **degrees of freedom.**

The number of degrees of freedom means the number of independent coordinates that determine the position of the system.

Consider several systems (Fig. 21.1) that can perform oscillatory movements.



a) b) c)

Figure 21.1 - Degrees of freedom of structures

These systems have one, two (see Fig. 21.1, b) or three degrees of freedom. The movement of the load *m* (see Fig. 21.1, a) can be described by a single coordinate x , so the system has one degree of freedom - horizontal movement.

The rotation of the shaft (see Fig. 21.1, b) and the rotation of one pulley relative to the other can be described by two coordinates - the angles $\varphi_1 and\ \varphi_2$ i.e., the system has two degrees of freedom.

The position of the oscillating load m (see Fig. 21.1, c) is given by three coordinates: the displacements of the center of mass x and y and the angle of rotation φ of the mass relative to its center of gravity. In the case of a small moment of inertia, the mass m can be considered as a point mass, in which case the system has only two degrees of freedom.

The number of degrees of freedom is determined by the choice of the calculation scheme, i.e., it depends on the degree of approximation with which we consider it necessary to study a real object.

When studying elastic **oscillatory systems**, we distinguish between **natural and forced oscillations, parametric and self-oscillations.**

Natural (free) oscillations are those that occur in an isolated system as a result of an external excitation ("push") that causes initial deviations from the equilibrium position or initial velocities at the system's points, and that continue thereafter due to the presence of internal elastic forces.

Forced oscillations are understood as the movement of an elastic system that occurs under the influence of time-varying external forces.

An example of forced oscillations is the motion that an elastic system performs when it is subjected to a motor that is not fully balanced.

The force acting on the elastic system by the motor is the **exciting force.** In this case, **the frequency of forced oscillations** is determined by the formula:

$$\Omega = \frac{\pi n}{30} = \frac{2\pi n}{60}\ \left(\frac{\text{рад}}{s}\, or\ \text{c}^{-1}\right),$$

where n – the number of engine revolutions.

The time interval between two subsequent maximum deviations of an elastic system from the equilibrium position is called the period of (natural or forced) oscillations (Figure 21.2). The period of oscillations is denoted by T (s).

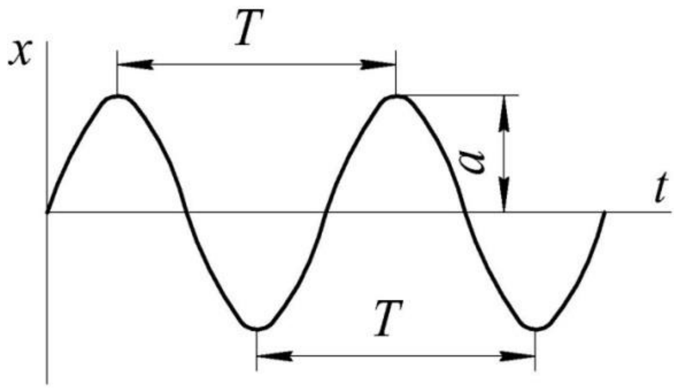


Figure 21.2 - Oscillation parameters

The inverse of the period is called the frequency of oscillations and is equal to the number of oscillations per unit of time:

$$\nu = \frac{1}{T}, \qquad \left(\frac{1}{s}\right).$$

In technology, instead of the frequency ν **the circular frequency** $\boldsymbol{\omega}$, is used in most cases, which is the number of oscillations per 2π seconds:

$$\omega = 2\pi\nu = \frac{2\pi}{T}, \quad \left(\frac{\text{рад}}{s}\right).$$

The largest deviation of the system from the equilibrium position is called the amplitude *a* (see Fig. 21.2).

Unlike natural oscillations, forced oscillations do not dampen, and if their frequencies coincide, the amplitude of oscillations can increase significantly and lead to structural failure. This phenomenon is called **resonance**.

It is known that military units are instructed to "kick their feet" when crossing bridges and walk at a free pace rather than in formation. For example, on April 12, 1831, the Broughton Suspension Bridge over the Irwell River in England collapsed when a military unit was walking across it. The frequency of the soldiers' footsteps coincided with the frequency of the bridge's own vibrations, which caused their amplitude to increase rapidly, the chains broke, and the bridge collapsed into the river. It was this incident, which resulted in two dozen injuries, that contributed to the adoption of the rule "keep pace" in the British army when troops crossed bridges. For the same reason, in 1850, near the French city of Angers, a suspension chain bridge over the Min River, more than a hundred meters long, was destroyed, resulting in numerous casualties.

Not only pedestrians, but also railroad trains can cause the destruction of bridges due to resonance. To avoid bridge resonance, the train must move either slowly or at maximum speed (remember how subway trains slow down when they cross the Metro Bridge in Kyiv). This is usually done to prevent the frequency of wheel impacts on rail joints from coinciding with the natural frequency of the bridge (for the same reason, a section of rail on a bridge is often made solid, i.e. without joints).

Catastrophic consequences for bridges can also occur due to wind. For example, on November 7, 1940, the Tacoma Suspension Bridge with a total length of 1800 m and a central span of 850 m collapsed due to ignoring the effect of wind load on the bridge during its design and as a result of resonance (USA).

Resonance can be encountered not only on land, but also at sea and in the air. For example, at certain propeller shaft speeds, even ships have been known to resonate. And at the dawn of aviation, some aircraft engines caused such strong resonant vibrations of aircraft components that the aircraft completely collapsed in the air.

The cause of resonance of aircraft elements and their destruction can also be caused by flutter, a combination of self-excited, unattenuated bending and torsional self-oscillations of structural elements (mainly an airplane wing or a helicopter rotor). One of the ways to combat this phenomenon is to use so-called anti-flutter weights.

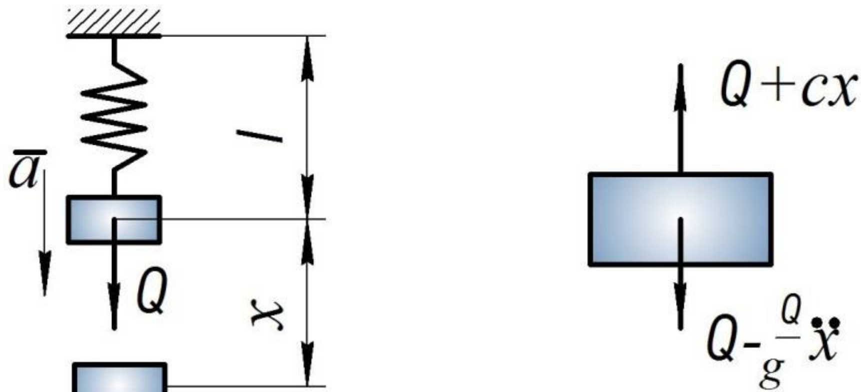
Interestingly, mounting engines on aircraft wing pylons is not a whim of designers and engineers, but an urgent need, as engines dampen wing vibrations during aircraft flight, being a kind of anti-flutter weight.

Parametric oscillations are those of an elastic system in which its physical parameters (mass, stiffness, length, etc.) change periodically.

According to the type of elastic vibrations, we distinguish between **longitudinal, transverse, and torsional** vibrations of systems, as well as their combinations.

21.2 Free (harmonic) vibrations of an elastic system with one degree of freedom

Let's consider the free oscillations of an elastic system consisting of a mass and a spring (Figure 21.3). Using D'Alembert's principle, we equate to zero the sum of projections of all forces acting on the load (Fig. 21.3, b):



a) **b)**

Figure 21.3 - Free oscillation load

$$\sum P_{kx} = 0, \quad or \quad Q + cx - (Q - \frac{Q}{g}\ddot{x}) = 0,$$

Whence

$$\frac{Q}{g}\ddot{x} + cx = 0 \quad or \quad m\ddot{x} + cx = 0, \qquad (21.1)$$

where

$\frac{Q}{g}\ddot{x} = m\frac{d^2x}{dt^2}$ – the force of inertia;

$\ddot{x} = \frac{d^2x}{dt^2} = a$ – acceleration of load;

$m = \frac{Q}{g}$ – load weight;

cx – restorative power;

c – spring stiffness (numerically equal to the force that causes the spring to stretch per unit length);

$c = \frac{EF}{\ell}$;

t – time.

The final differential equation of free oscillations of a load of weight Q will be as follows:

$$\ddot{x} + \omega^2 x = 0, \qquad (21.2)$$

where

$$\omega^2 = \frac{c}{m} = \frac{cg}{Q}, \text{ or}$$

$$\omega = \sqrt{\frac{c}{m}} = \sqrt{\frac{EF}{\ell m}} = \sqrt{\frac{g}{\Delta st}}, \qquad (21.4)$$

$\Delta_{st} = \frac{Q}{c}$ – static deformation of the spring.

The differential equation of free oscillations (21.4) has the following general solution, which determines the relationship between the vertical displacements *x* of the load *Q* and time *t*:

$$x = A cos\omega t + B sin\omega t, \qquad (21.5)$$

and the projection of the load velocity

$$v_x = \dot{x} = -A\omega \sin \omega t + B\omega cos\omega t, \qquad (20.5')$$

where ω – the circular frequency of natural oscillations;

A and B – integration constants that depend on the initial conditions.

If the initial coordinate of the load x_0 and the initial velocity $v_0 = \dot{x}$ at $t = 0$, are given, then the integration constants are determined from equations (21.5) and (21.5′):

$$A = x_0; \quad B = \frac{v_0}{\omega}.$$

Putting

$$x_0 = asin\alpha; \text{ and } \frac{v_0}{\omega} = acos\alpha,$$

equation (21.5) can also be rewritten as

$$x = a\sin(\omega t + \alpha),$$

where

$a = \sqrt{A^2 + B^2} = \sqrt{{x_0}^2 + \frac{{v_0}^2}{\omega^2}}$ – oscillation amplitude;

$\omega t + \alpha$ – the oscillation phase; α – the phase shift $\left(tg\alpha = \frac{x_0\omega}{v_0}\right)$.

Knowing the circular frequency of oscillations, you can find the oscillation period T (the time of one complete oscillation)

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\Delta\ell_{st}}{g}} = 2\pi\sqrt{\frac{m}{c}}. \qquad (21.6)$$

21.3 Forced vibrations of systems with one degree of freedom

Forced oscillations are oscillations of an elastic system that occur when the system is subjected to a given external excitatory force (during the entire period of oscillation).

The equation of these oscillations is obtained from formula (21.1) by adding the exciting force *P(t)* to its right-hand side:

$$\frac{Q}{g}\ddot{x} + cx = P(t). \qquad (21.7)$$

Dividing all terms in (21.7) by $\frac{Q}{g}$, we find

$$\ddot{x} + \omega^2 x = \frac{P(t)}{Q}. \qquad (21.8)$$

Let us consider a special case when $P(t) = P_1 cospt$, i.e., when the period of the force $T_1 = \frac{2\pi}{p}$, the frequency of forced oscillations

$$v_1 = \frac{1}{T_1} = \frac{p}{2\pi}.$$

Marking

$$\frac{P(t)\mathrm{g}}{Q} = \frac{\mathrm{g}}{Q} P_1 cospt = \mathrm{g}cospt,$$

rewrite equation (21.8) in the form

$$\ddot{x} + \omega^2 x = \mathrm{g}cospt. \qquad (21.9)$$

With a slow change in the external force *P*(*t*), i.e., when *p* is small compared to ω, we can neglect the inertial term $\ddot{x}$ and use (21.9) to find the **static strain**

$$x_C = \frac{\mathrm{g}cospt}{\omega^2} \qquad (21.10)$$

with amplitude $\frac{\mathrm{g}}{\omega^2}$.

To determine the dynamic strain, we need to solve the differential equation (21.9). This solution consists of the solution of the homogeneous equation (21.1) and the solution of equation (21.9) in the form of the right-hand side:

$$x = x_{hom} + x_{side} = Acos\omega t + Bsin\omega t + Ccospt.$$

Substituting the partial solution in (21.1), we find

$$-p^2 Ccospt + \omega^2 Ccospt = \mathrm{g}cospt.$$

Hence **the amplitude of forced oscillations**

$$\mathrm{C} = \frac{\mathrm{g}}{\omega^2 - p^2}. \tag{9.11}$$

Then the general solution of the **forced oscillation equation** (21.9) will take the form

$$x = Acos\omega t + Bsin\omega t + \frac{\mathrm{g}}{\omega^2 - p^2}cospt. \tag{21.12}$$

The first two terms in (9.12) characterize the natural oscillations, which usually decay quickly, and the last term characterizes the forced oscillations of the system, which occur with the frequency of the excitation force $P(t)$.

The amplitude *C* of the forced oscillations depends **on the frequency** of these oscillations p. The ratio of the amplitude *C* to the amplitude of the static strain (21.10) determines **the coefficient of oscillation increase *β*:**

$$\beta = \frac{\mathrm{C}}{\mathrm{g}/\omega^2} = \frac{\mathrm{g}}{\omega^2 - p^2} : \frac{\mathrm{g}}{\omega^2} = \frac{\omega^2}{\omega^2 - p^2} = \frac{1}{1 - \frac{p^2}{\omega^2}} \tag{21.13}$$

or

$$\beta = \frac{1}{1 - {T_1}^2/T^2}, \tag{9.14}$$

where

$$T_1 = \frac{2\pi}{p}; \quad T = \frac{2\pi}{\omega}.$$

From formula (21.13) it follows that at a small ratio $\frac{p}{\omega}$ the coefficient β is close to unity and the amplitudes of forced and static deformation differ slightly. If the frequency *p* of the forced vibrations approaches the frequency ω of the system's natural vibrations, the amplitude *C* tends to infinity, i.e., when $p/\omega \to 1$ the amplitude $C \to \infty$. When $C = \infty$ **the state of resonance** occurs. The corresponding frequency of the exciting force is called the **critical** frequency.

21.4 Resonance phenomenon and measures to prevent it

The phenomenon of amplitude increase when the frequencies of natural oscillations and the exciting force coincide is called resonance, and the coincidence of frequencies is called the resonance condition.

In the practice of engineering calculations for dynamic strength, resonance issues are one of the most important in terms of their importance. The fact is that in most cases, the laws of change in the excitatory forces are periodic. For example, unbalanced moving parts of a running engine generate periodically changing forces. A train traveling at a constant speed receives periodic jolts at the joints of the rails. Parts of devices mounted on a vibrating base (on an airplane, car) receive shocks with the frequency of the base, which fluctuates. In all these cases, the question arises as to how dangerous force excitations are for the operation of an elastic system and whether they will lead to its excessive swaying and premature destruction.

This task is solved, first of all, by comparing the frequencies of natural oscillations and the exciting force. If these frequencies differ greatly from each other, it is possible to be sure that the resonance phenomenon does not occur and that the operating conditions for elastic elements are favorable. In this case, it is possible to determine the amplitude of the forced oscillations and the maximum value of the effective cycle stress.

The phenomenon of resonance is associated with an unlimited increase in strain, which can lead to system failure.

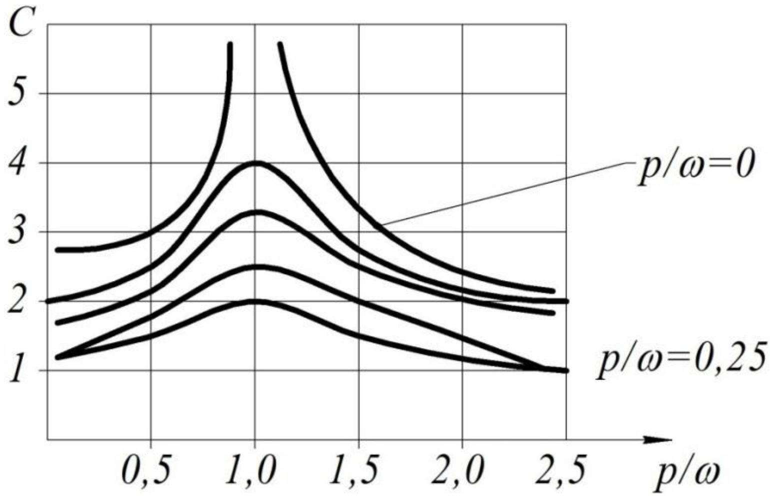


Figure 21.4 - Resonance conditions

Figure 21.4 shows that resonance occurs when the frequency of the exciting force p and the natural oscillations ω coincide, and the system has little strength to oscillations.

If the structure is calculated to operate in the near-resonance zone and we have the following relations

$$\frac{p}{\omega} \leq 0{,}7 \;\; or \;\; \frac{p}{\omega} \geq 1{,}3\,,$$

resonance is avoided by changing the stiffness of the structure. For example, the stiffness of a shaft can be changed by increasing its diameter or installing intermediate supports.

21.5 Determining stresses and calculating vibration strength

Vibration strength calculations are performed in the following sequence:

a) check the absence of possible resonance during the operation of the structure;

b) check the strength in dangerous sections:

c) if the dynamic stresses are comparable to the static stresses, the system is tested for fatigue strength.

If the coefficient β is found, it is easy to determine the stresses in the elastic elements of the oscillating system:

$$\sigma_d = \sigma_{st} \cdot \beta, \tag{21.15}$$

where σ_{st} – the stresses that would arise in the system under the static application of the maximum value of the exciting force P_{max}.

Let's define $1 + \frac{P_{max}\beta}{Q} = K_d$ – is **the dynamism coefficient**, then dynamic deformation

$$\delta_d = \delta_{st} \cdot K_d\,, \tag{21.16}$$

where δ_{st} – the deformation of the system under a static force P_{max}.

By analogy **(vibration strength condition):**

$$\sigma_{dmax} = \sigma_{st} \cdot K_d \le [\sigma]\,. \tag{21.17}$$

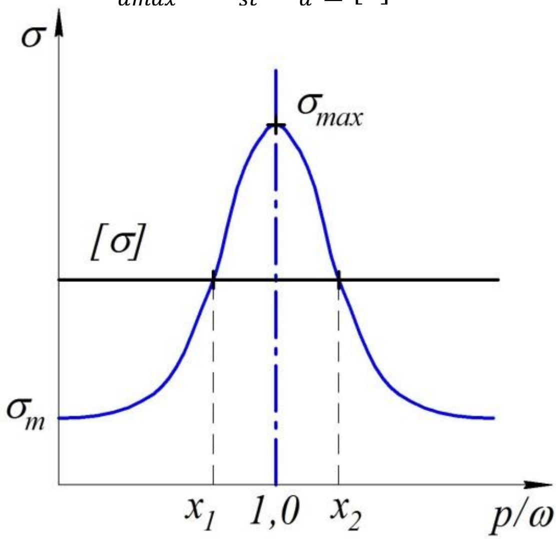


Figure 21.5 - Vibration strength condition

21.6 Control questions

1. What is the number of degrees of freedom of a system?
2. What is called natural (free) oscillations of the system?
3. What is called forced oscillations of the system?
4. How is the frequency of forced oscillations determined?
5. What is called the period of natural and forced oscillations?
6. hat is the frequency of oscillations and what is it equal to?
7. What is the circular frequency of oscillations?
8. What is called the amplitude of oscillations?
9. The phenomenon of resonance (give examples)?
10. The conditions of resonance.
11. Parametric oscillations.
12. What is called spring stiffness and how is it determined?
13. Write the differential equations of free and forced oscillations.
14. How is the amplitude of forced oscillations determined?
15. The conditions of strength in oscillations.
16. Dynamic stresses in the elements of the oscillatory system.

22. STRESS DETERMINATION AND STRENGTH CALCULATIONS UNDER SLOWLY VARYING LOADS

22.1 Phenomenon of fatigue of materials. Definition

Under alternating loads, parts fracture under stresses that do not exceed the yield strength σ_y of the material, without producing noticeable residual deformations, even when the material is highly plastic. At the same time, under the action of variable loads, structural elements collapse at much lower loads than under the action of static loads.

It has been established in practice that if a structural element is repeatedly subjected to alternating loads, then after a certain number of changes (cycles) of stresses, a crack will appear in it, which will gradually develop until the part is destroyed.

Failure of a material under the action of repeatedly alternating stresses is called fatigue failure.

In general, **fatigue is the phenomenon of fracture due to the gradual accumulation of damage in them, which leads to the appearance of a crack under repeated loads.**

The ability of metals to resist fracture when subjected to repeatedly alternating stresses is called **material fatigue strength.**

In addition to this hypothesis, there is a slightly different approach to explaining the physical nature of the fatigue phenomenon. In particular, fatigue cracks can be explained by the exhaustion of the ability of crystalline grains to resist shear.

The study of material fatigue was intensively developed in the 1945-1960s in connection with fatigue failures of aircraft structures, primarily those with catastrophic consequences (the crash of the first civilian jet aircraft "Comet", 1954). At the same time, the scattering patterns of data from the experimental determination of the endurance limit and the number of cycles before material failure were studied in detail, and methods for taking them into account in the design of machines and structures were developed, the basic concepts of failure under low-cycle loading were formulated, and new approaches to assessing the durability of materials and structures were developed when the prediction of failure was based on strain, not stress, in particular its plastic component.

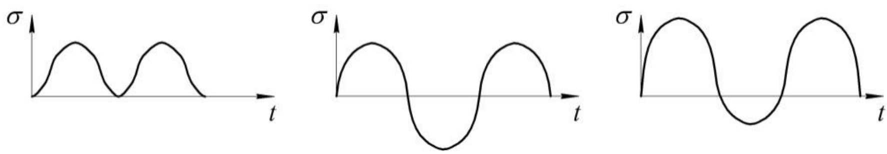
Examples of accidents caused by material fatigue include the Versailles train accident returning to Paris in May 1842 (the lead locomotive broke an axle) and the accident of a semi-submersible drilling rig that capsized while operating in Norwegian waters in March 1980, killing 123 people.

Thus, the mechanism of crack formation under repeatedly alternating loads is very complex and cannot be considered fully understood at present.

Fatigue cracking occurs depending on the specifics of the load application:

1) repeated application of a load of the same sign, for example, periodically varying from zero to maximum (Fig. 22.1, a);

2) repeated repetition of a load that varies not only in modulus but also in sign (alternating loads). In this case, a distinction is made between a change in load in a symmetrical cycle (Fig. 22.1, b) and asymmetric (Fig. 22.1, c).



a) b) c)

Figure 22.1 - Types of load

For fatigue failure to occur, the loads must not only be variable, but also have certain values.

The maximum stress at which a material can resist without fracture for any arbitrarily large number of repetitions of alternating stresses is called the endurance limit (fatigue limit).

Fracture of a part due to fatigue has a characteristic appearance (Figure 22.2). It almost always has two zones. One of them (I) is smooth, ground, formed as a result of the gradual development of a crack;

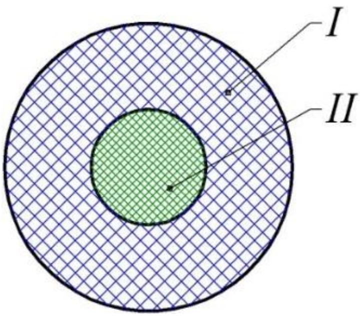


Figure 22.2 - Fracture zones of a part due to fatigue

The second (II) is coarse-grained, formed at the final fracture of a part section that was weakened during crack development. Its zone in brittle materials has a coarse crystalline structure, and in ductile materials - a fibrous structure.

22.2 Parameters of the alternating stress cycle

In engineering practice, it is assumed that the stresses in the material under the action of repeatedly variable loads are periodic functions of time $\sigma = f(t)$ with a period T. During this period, the stresses are characterized by the following parameters,

σ_{max} – the highest stresses of the cycle;

σ_{min} – the smallest stresses of the cycle;

$\sigma_{av} = \frac{\sigma_{max} + \sigma_{min}}{2}$ – average cycle stress;

$\sigma_{a} = \frac{\sigma_{max} - \sigma_{min}}{2}$ – amplitude stress of the cycle;

$r = \frac{\sigma_{min}}{\sigma_{max}}$ – cycle asymmetry coefficient.

Of all the possible cases of material loading encountered in engineering practice, let's consider three cases of loading called a stress cycle.

Symmetrical cycle (the most dangerous cycle).

$$\sigma_{max} = -\sigma_{min} = \sigma_a$$
$$r = -\frac{\sigma_{min}}{\sigma_{max}} = -1 = const$$
$$\sigma_{av} = 0$$

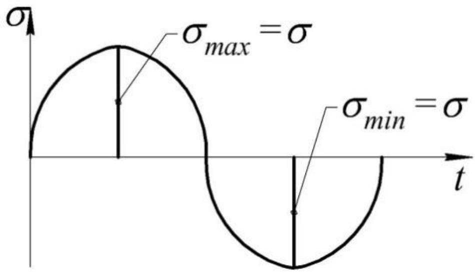


Figure 22.3 - Symmetrical stress cycle

Asymmetric cycle.

If the change in stresses is asymmetric with respect to the time axis t, then such a cycle is called asymmetric.

Pulsating cycle.

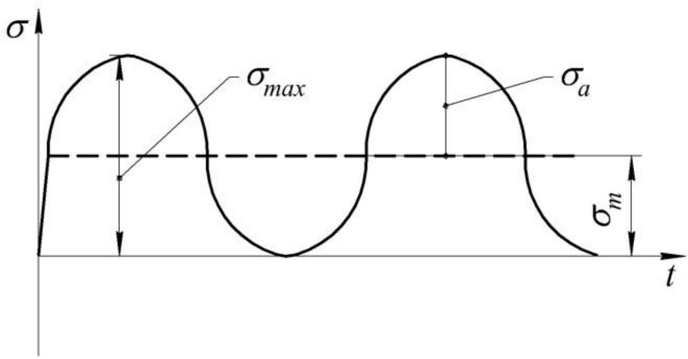


Figure 22.4 - Pulsating stress cycle

$$\sigma_m = \sigma_{av} = \frac{\sigma_{max}}{2} \qquad \sigma_{min} = 0 \quad r = \frac{0}{\sigma_{max}} = 0 = const.$$

The maximum stress at which a material can resist without breaking under any number of alternating loads is called **the endurance limit (fatigue limit). The cycle asymmetry factor indicates at which load cycle this value was obtained.**

For example:

σ_{-1} – the endurance limit for normal stresses under a symmetrical load cycle.

τ_0 – the endurance limit for tangential stresses under a pulsating cycle.

$\sigma_{0,3}$ – the limit of endurance under normal stresses for a given material obtained during an asymmetric cycle with an asymmetry coefficient equal to 0,3, i.e.

$$\frac{\sigma_{min}}{\sigma_{max}} = 0{,}3 = r.$$

22.3 Methods for determining the fatigue limit

The fatigue limit of a particular material under different load cycles is determined on a testing machine. In accordance with the requirements, a suitable test machine is selected (Figure 22.5).

The processing of the obtained experimental data is usually accompanied by the construction of a fatigue curve, which is often referred to in the literature as **Weller curve** (Figure 22.6). The fatigue curve is drawn at points in the coordinates of the number of cycles N and the stress σ_{max}. For each point, 6...12 samples are taken. Most often, these are smooth cylindrical samples with a diameter of 7...10 mm.

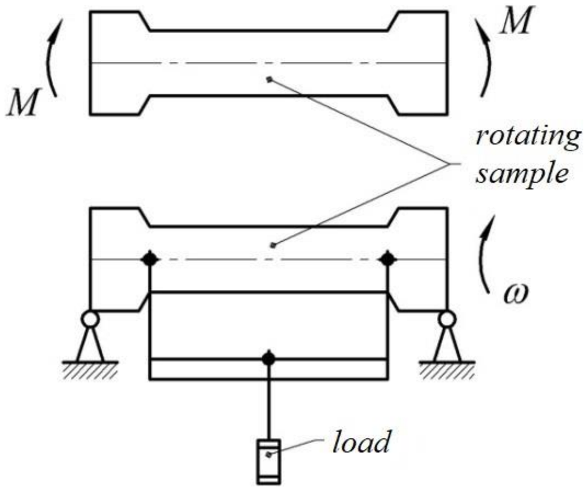


Figure 22.5 - The scheme of the test machine

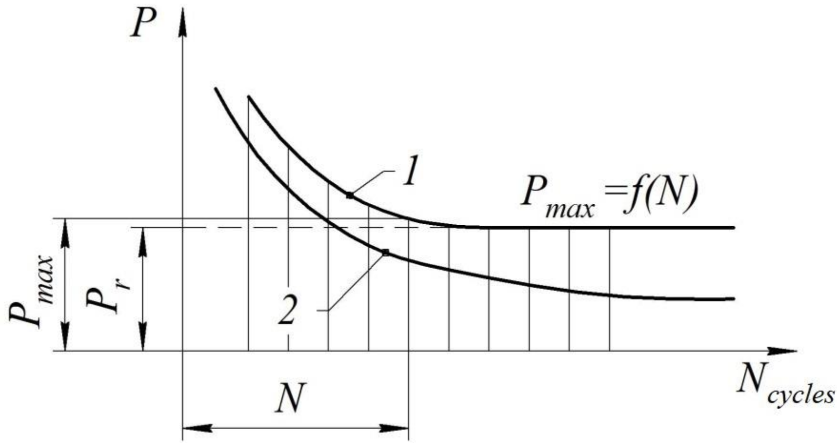


Figure 22.6 - Fatigue curves (Weller curves)

For example, for steel, at high stress levels, the fatigue curve drops sharply (upper curve, Figure 22.6). As the stresses decrease, the slope decreases and the curve asymmetrically approaches a certain horizontal line. This line cuts off a segment on the ordinate axis (σ). This segment illustrates the value of the endurance limit. The ordinate of the point at which the curve begins to coincide with the asymptote (endurance limit) corresponds to the stress at which the sample will not fracture. However, the sample will undergo a predetermined number of cycles, which corresponds to the so-called N_6 test base.

The stress at which the sample will withstand the basic number of cycles without breaking is called **the fatigue limit,** or **endurance limit,** and is denoted by σ_{-1}, σ_0, τ_r.

The test base for ferrous metals is 10 million cycles, and for non-ferrous metals (copper, aluminum, etc.) $N_6 = 5 \cdot 10^8$ cycles.

According to numerous experimental data, for some materials, certain correlations between the endurance limit σ_e at different types of deformation can be observed, i.e., empirical dependencies:

$$\begin{aligned} &\text{for stretching} - \text{compression } \sigma_{-1} = 0{,}28\sigma_e; \\ &\text{for bending } \sigma_{-1} = 0{,}40\sigma_e; \\ &\text{for torsion } \sigma_{-1} = 0{,}22\sigma_e. \end{aligned} \qquad (22.1)$$

If the material operates at cycles other than standard, the endurance limit is determined by the ultimate stress diagrams.

The most complete is **the Smith-Hay diagram**. However, they require a lot of initial data.

22.4 Influence of design and process factors

A number of different factors, in addition to the cycle characteristics, influence the endurance value of specimens or parts made of a material. These include the shape of the sample, dimensions, surface condition, test environment, test temperature, cyclic force mode (training, pauses, overloads, load frequency, etc.), preliminary internal stress of the material, etc. To determine the influence of a particular factor, the endurance limit σ_{-1}, obtained by air testing during a symmetrical cycle of a batch of smooth polished samples with a diameter of 7-10 mmis used as a reference. Then the influence of various factors on fatigue strength can be assessed by the deviation of the endurance limit σ_{-1} of the reference samples.

The effect of stress concentration.

The most important factor that reduces the endurance limit is stress concentration, which is accounted for by the stress concentration factor. A distinction is made between **the theoretical stress concentration factor** α_T and **the effective or real stress concentration factor,** which is defined as the ratio of the endurance limit of a smooth specimen without a concentrator σ_{-1} to the endurance limit of a specimen with a stress concentration.

For normal stresses

$$К_\sigma = \frac{\sigma_{-1}}{\sigma_{-1К}}. \tag{22.2}$$

For tangential stresses

$$К_\sigma = \frac{\tau_{-1}}{\tau_{-1К}}.$$

$$g_\sigma = \frac{К_\sigma - 1}{\alpha_T - 1}, \tag{22.3}$$

where g_σ – the coefficient of material sensitivity to stress concentration.

Knowing g_σ we can calculate $К_\sigma$.

$$К_\sigma = 1 + g_\sigma(\sigma_T - 1). \tag{22.4}$$

The value of g_σ is available in additional literature and depends on the steel strength σ_e

$$g = \frac{\sigma_e - 400}{1100}(1 - \beta) + \beta, \tag{22.5}$$

where $\beta = f(\alpha_T)$ – the higher the steel strength, the higher its sensitivity to stress concentration. Therefore, the use of high-strength steels at variable stresses is not always advisable. The method's sensitivity to concentration is lower for coarse-grained steels than for fine-grained steels

$$g_{steel} = 0{,}6 \div 0{,}8; \qquad g_{cast\ iron} = 0.$$

The influence of the scale factor on the endurance limit

The larger the size of the part (sample), the lower its strength. This is explained by the fact that more imperfections (gas vapors, dislocations, etc.) accumulate in a larger volume. Therefore, as the absolute size of the sample increases, the endurance limit decreases. The ratio of the endurance limit of a part of size $\frac{d}{\sigma_{-1}}$ to the endurance limit of a laboratory specimen of a similar configuration with small dimensions (d_0=7...10 mm) σ_{-1d_0} is called the coefficient of influence of the absolute dimensions of the section ε_m.

$$\varepsilon_{ave} = \frac{\sigma_{-1d}}{\sigma_{-1d_0}}. \tag{22.6}$$

Table 22.1 - Dependence of ε_{ave} on the sample diameter

d, mm	10	20	30	40	50	60	80	100	150	200
$\boldsymbol{\varepsilon_{ave}}$	1	0,93	0,87	0,82	0,78	0,75	0,7	0,65	0,58	0,55

The fatigue strength is also affected by the length of the specimens. However, this influence is secondary to the influence of the absolute cross-sectional dimensions. In addition, in the presence of stress concentration, the fracture point is localized along the length of the part, so the effect of length on fatigue strength is not taken into account in practical calculations.

The main reasons that lead to a decrease in the endurance limits with an increase in the size of the part are as follows:

1) deterioration in the quality of the metal of the casting or forging - **metallurgical factor.** The metallurgical factor is related to the fact that with an increase in the size of the casting or forging, the heterogeneity of the metal increases, the degree of forging decreases, high-quality heat treatment becomes more difficult, etc. This leads to a decrease in the characteristics of mechanical properties, such as σ_{-1}, determined on standard laboratory samples cut from billets of various sizes. For example,

an increase in the size of a steel billet from 20 – 30 mm to 200 mm results in a 10-15% decrease in the strength limits.

2) the impact of heat and mechanical treatment in the manufacture of parts of different sizes **- a technological factor.**

The influence of the second, technological, factor is associated with the fact that during machining, a scum forms in the surface layer of the samples, which increases the endurance limit. However, the impact of this factor is insignificant and can be eliminated by a special technology for manufacturing samples, consisting in the sequential removal of increasingly thin layers of metal on the final passes during manufacturing or by annealing in a vacuum.

3) an increase in the probability of dangerous defects and overstressed grains, which, due to the statistical nature of the fatigue fracture process, leads to an increase in the probability of fracture **- a statistical factor.** The third statistical factor is related to the statistical nature of the fatigue fracture process. Due to the different orientation and shapes of grains, the presence of different phases, inclusions, defects, etc., metal grains are stressed unequally. With an increase in the stressed volume, the number of defects and dangerously stressed grains increases, which leads to an increase in the probability of fracture, and, consequently, to an actual decrease in strength, as follows from the statistical theory of fatigue strength.

22.5 Strength calculations for repeatedly varying loads

The dimensions of the cross-section of the part, which were determined in static calculations:

$$d = d_0 \sqrt{\frac{|n|}{n}}\,, \tag{22.7}$$

where d_0 – the diameter calculated in static calculations;
$|n|$– standard safety margin =1,4...3,0;
n – the actual safety margin obtained in static calculations;
d– the specified diameter of the part.

Determination of safety factors.

In **static tension:**

$$\sigma = \frac{N_z}{F} \leq [\sigma] = \frac{\sigma_T}{n}. \tag{22.8}$$

With **repeated alternating stretching:**

$$\sigma_a = \frac{N_z}{F} \leq [\sigma] = \frac{\sigma_{-1}}{n_\sigma}. \tag{22.9}$$

where n_σ – **the safety factor for normal stresses under repeatedly alternating loading.**

Similarly for **tangential stresses:**

$$n_\tau = \frac{\tau_{-1}}{\tau_a}. \tag{22.10}$$

Under static complex load.

According to the third theory of strength:

$$\sigma_{equi}^{\text{III}} = \sqrt{\sigma^2 + 4\tau^2} \leq [\sigma], \qquad where\ \frac{\sigma_T}{\tau_{\text{T}}} = 2,\ \ \tau_{\text{T}} = 0{,}5\sigma_T.$$

According to the fourth theory of strength:

$$\sigma_{equi}^{IV} = \sqrt{\sigma^2 + 3\tau^2} \leq [\sigma], \qquad where\ \frac{\sigma_T}{\tau_{\text{T}}} = \sqrt{3}.$$

$$\sigma_{equi}^{\text{III}} = \sqrt{\sigma^2 + \left(\frac{\sigma_T}{\tau_{\text{T}}}\right)^2} \leq [\sigma] = \frac{\sigma_T}{\tau_{\text{T}}}. \tag{22.11}$$

Under **repeatedly variable complex load:**

$$\sigma = \sqrt{\sigma_a{}^2 + \left(\frac{\sigma_{-1}}{\tau_{-1}}\right)^2 \cdot \tau_a^2} \leq [\sigma] = \frac{\sigma_{-1}}{n}. \tag{22.12}$$

Let's square the right and left sides of equation (22.12)

$$\sigma_a{}^2 + \left(\frac{\sigma_{-1}}{\tau_{-1}}\right)^2 \cdot \tau_a^2 = \frac{\sigma_{-1}^2}{n^2}.$$

We divide it into σ_{-1}^2

$$\frac{\sigma_a{}^2}{\sigma_{-1}{}^2} + \frac{\tau_a{}^2}{\tau_{-1}{}^2} = \frac{1}{n^2}. \qquad (22.13)$$

Taking into account (22.9) and (22.10), (22.13) will be rewritten

$$\frac{1}{n_\sigma^2} + \frac{1}{n_\tau^2} = \frac{1}{n^2}. \qquad (22.14)$$

Then

$$n = \frac{n_\sigma \cdot n_\tau}{\sqrt{n_\sigma{}^2 + n_\tau{}^2}}, \qquad (22.15)$$

where n – **the overall safety factor,** which depends on the material;
n =1,3-1,4 – for highly homogeneous materials;
n =1,4-1,7 – for medium-homogeneous materials;
n =1,7-3,0 – for low homogeneity of the material.

The coefficient n in formula (22.15) can be used if normal and tangential stresses in the part appear simultaneously (synchronously). If, under complex loading, normal and tangential stresses do not appear simultaneously, then the safety factor is determined separately for normal stresses (taking into account the main factors affecting the endurance limit):

$$n_\sigma = \frac{\sigma_{-1}}{\dfrac{\sigma_a K_\sigma}{\varepsilon_\Pi \varepsilon_\mu} + \sigma_m \psi_\sigma}, \qquad (22.16)$$

by tangential stresses

$$n_\tau = \frac{\tau_{-1}}{\dfrac{\tau_a K_\tau}{\varepsilon_\Pi \varepsilon_\mu} + \tau_m \psi_\sigma}. \qquad (22.17)$$

Formulas 22.16 and 22.17 do not take into account the effect of temperature, aggressiveness of the environment, radioactive radiation, the time factor of stress action, etc.

Measures to increase the endurance limit of the material

Parts operating under repeatedly variable loads must be made of high-quality fine-grained steel.

Structurally, the part should not have stress concentrators.

To increase the endurance limit, various methods of mechanical and thermochemical hardening of the surface of the part can be used. These include nitriding or surface boronizing. This includes roller rolling and shot blasting.

Parts that operate under repeatedly variable loads must have a surface finish up to and including electropolishing. Such measures can increase the endurance limit of the material and structure.

22.6 Control questions

1. What is called material fatigue?
2. What is the fatigue strength of materials?
3. What is the endurance limit (fatigue limit)?
4. Methods of increasing the fatigue limit of materials.

23. CONTACT STRESSES

23.1 Basic concepts

The deformations and stresses that occur when two contacting bodies bounded by curved surfaces are subjected to mutual pressure are called contact deformations. As a result of deformations at the points of contact of structural elements, pressure is transferred over very small areas. The material in the vicinity of such a site, unable to deform freely, experiences a volumetric stress state (Figure 23.1).

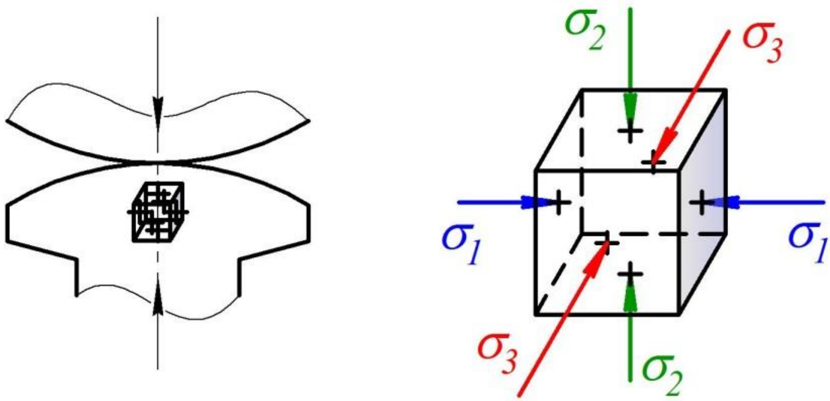


Figure 23.1 - Volumetric stress state in the contact load zone

The study of contact stresses and strains is required to address the strength of many important parts. These parts include, for example, ball and roller bearings, gears, cam elements, rolling stock wheels, rails, ball and cylindrical rollers, and more.

When two bodies with smooth curved surfaces are compressed, the points of the surfaces of these bodies are connected in the contact zone. As a result, a surface is formed, called the pressure surface (contact patch), and its contour is called the pressure contour. A compressive stress acts along the pressure surface. The calculation of the total strength of bodies in sections remote from the contact point is performed in accordance with the Saint-Venant principle without taking into account the peculiarities of stress distribution in the contact zone. To determine the local strength of a body near the point of contact, the law of stress distribution in the contact zone and the value of the maximum contact stress play a primary role. The material located directly under the contact zone is in a volumetric stress state, since compression occurs in the direction normal to the pad.

For the first time, the correct solution of the basic cases of compression of elastic bodies was given by the methods of elasticity theory in the works of the German physicist H. Hertz.

Below are some results obtained by the methods of elasticity theory under the following assumptions:

- - the loads cause only elastic deformations in the contact zone, which correspond to Hooke's law;
- - the contact areas are small compared to the surfaces of the contacting bodies;
- - pressure forces distributed over the contact surfaces are normal to these surfaces.

23.2 Formulas for determining contact stresses

Compression of balls. When two balls with radii R_1 and R_2 (Fig. 11.2) are mutually compressed by forces P, a circular contact area is formed, the radius of which is determined by the formula

$$a = 0{,}88\sqrt[3]{P\frac{\frac{1}{E_1}+\frac{1}{E_2}}{\frac{1}{R_1}+\frac{1}{R_2}}}, \tag{23.1}$$

where E_1, and E_2 – elastic moduli of the ball materials.

Normal (compressive) stresses on the contact pad are distributed over the hemisphere. The greatest stress is in the center of the contact area:

$$\sigma_3 = -|\sigma_{max}| = -1{,}5\frac{P}{\pi a^2} =$$
$$= -0{,}388\sqrt[3]{4P\frac{E_1^2E_2^2}{(E_1+E_2)^2}\frac{(R_1+R_2)^2}{R_1^2R_2^2}}; \qquad (23.2)$$

Two other main tensions in the center of the site

$$\sigma_1 = \sigma_2 \approx -0{,}8\,|\sigma_{max}|.$$

This means that at the most stressful point in the contact area, the material is subjected to a stress state close to uniform compression. As a result, the material can withstand very high pressures in the contact area without residual deformation. Let's calculate, for example, the stress in the center of the contact area at which residual deformations first appear. Let us use IV Theory of Strength for this purpose:

$$\sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = \sigma_\tau.$$

Substituting the values of the principal stresses, we find

$$0{,}2\sigma_{max} = \sigma_\tau \ \ or \ \ \sigma_{max} = 5\,\sigma_\tau.$$

For hardened chromium steel, which is used to make ball bearings, instead of the yield strength, we take the proportionality limit $\sigma_{pr} = 1000$ МПa. So, $\sigma_{max} = 5000$ MPa.

The most dangerous point lies on the *z*-axis at a depth approximately equal to half the radius of the contact pad. The main stresses at this point are

$$\sigma_1 = \sigma_2 = -0{,}18\sigma_{max}; \quad \sigma_3 = -0{,}8\sigma_{max}\ , \qquad (23.3)$$

where σ_{max}- the greatest stress in the center of the contact area, determined by formula (23.2).

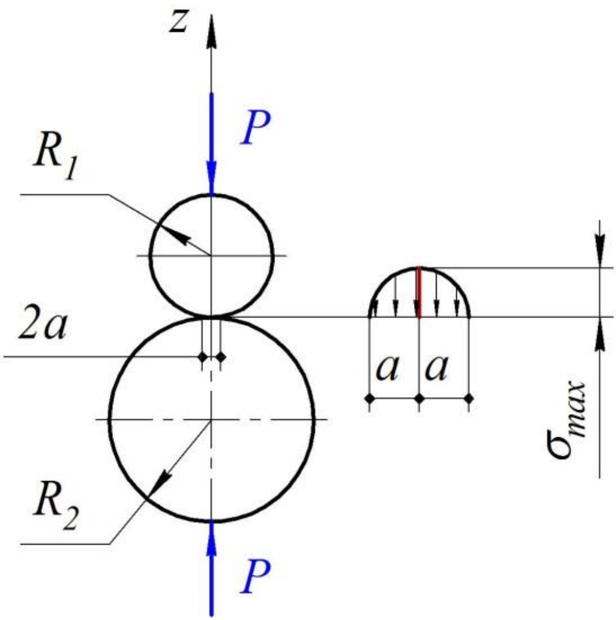


Figure 23.2 - Compression of balls

Highest tangential stress at the danger point

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 0{,}31\sigma_{max}. \qquad (23.4)$$

By changing the sign of R_2 in formula (23.2) to the opposite, we obtain the value of σ_{max} in the case of a ball pressure on a concave spherical surface (Fig. 23.4):

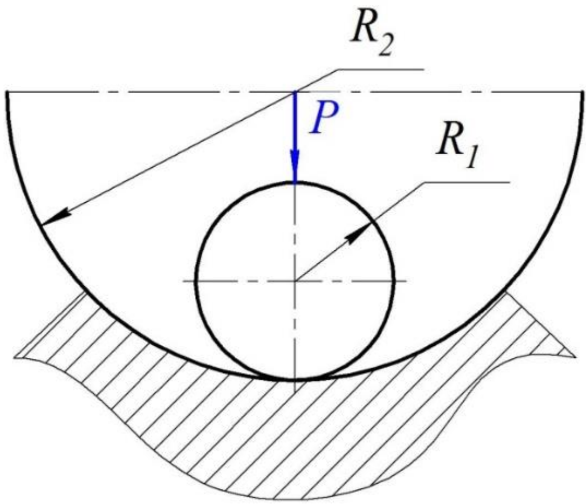


Figure 23.3 - Ball pressure on a concave spherical surface

$$\sigma_{max} = 0{,}388\sqrt[3]{4P\frac{E_1^2E_2^2}{(E_1+E_2)^2}\frac{(R_1-R_2)^2}{R_1^2R_2^2}}. \qquad (23.5)$$

At the mutual pressing of the ball and the plane (Fig. 23.4), taking $R_2 = \infty$, we find

$$\sigma_{max} = 0{,}388\sqrt[3]{4P\frac{E_1^2E_2^2}{(E_1+E_2)^2}\frac{1}{R^2}}. \qquad (23.6)$$

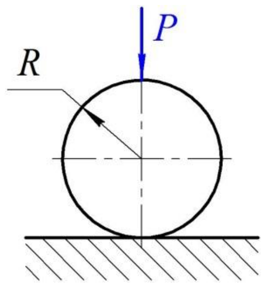


Figure 23.4 - The case of mutual compression of a ball and a plane

Compression of cylinders. When two cylinders with parallel faces are pressed together by a uniformly distributed load of intensity q, Н/м (Fig. 23.5), the contact area has the form of a narrow rectangle, the width of which is determined by the formula:

$$b = 2{,}15\sqrt{q\frac{\frac{1}{E_1}+\frac{1}{E_2}}{\frac{1}{R_1}+\frac{1}{R_2}}}. \tag{23.7}$$

The highest compressive stress is applied at the points of the contact pad axis,

$$\sigma_{max} = 1{,}27\frac{q}{b} = 0{,}418\sqrt{2q\frac{E_1\ E_2}{E_1+E_2}\frac{R_1+R_2}{R_1R_2}}. \tag{23.8}$$

The stress state analysis shows that the danger point lies on the z-axis at a depth equal to 0.4 times the width of the contact pad. The principal stresses at this point have the following values:

$$\sigma_1 = -0{,}18\sigma_{max};\quad \sigma_2 = -0{,}288\sigma_{max};\quad \sigma_3 = -0{,}78\sigma_{max}. \tag{23.9}$$

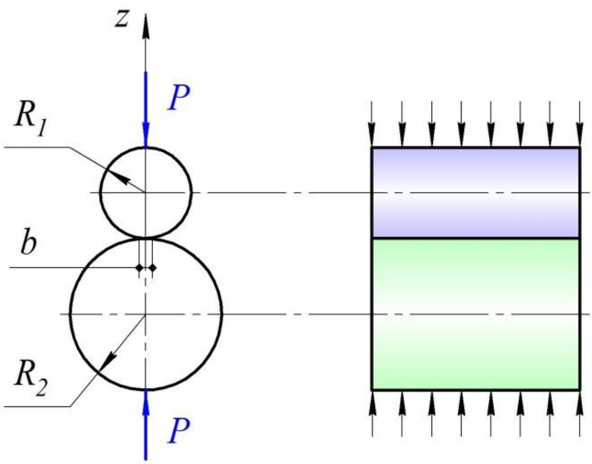


Figure 23.5 - Mutual pressing of two cylinders with parallel faces by an evenly distributed load

The highest tangential stress at the danger point

$$\tau_{max} = 0{,}3\sigma_{max}. \qquad (23.10)$$

By reversing the sign of R_2 in formula (23.8), we can find the stresses in the case of cylinder pressure on a part with a concave cylindrical surface. Such stresses act at the points of contact between the cylindrical joint and the balancers (Fig. 23.6).

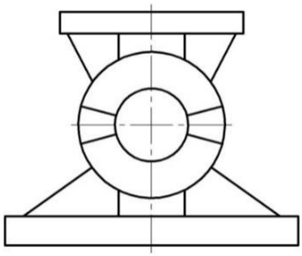


Figure 23.6 - Contact points of the cylindrical joint and balancers

At the mutual pressure of the cylinder and the plane, taking $R_2 = \infty$, in the formula, we find

$$\sigma_{max} = 0{,}418\sqrt{\frac{2q}{R}\cdot\frac{E_1\ E_2}{E_1 + E_2}}. \qquad (23.11)$$

The above formulas were obtained with $\mu = 0{,}3$. However, for practical calculations, they can be used for other values of the Poisson's ratio.

The general case of contact between two bodies. Here are the formulas for the general case of contact between two bodies of the same material.

It is assumed that both bodies at the point of contact have a common tangent plane AB and a common normal z, along which the force P is directed (Figure 23.7). Recall that the principal curvatures are the largest and smallest curvatures located in two mutually perpendicular planes passing through the center of curvature. Radii of curvature are considered positive if the centers of curvature are located inside the body. Let us denote by φ the angle between the principal curvature planes of the bodies in which the smaller radii lie.

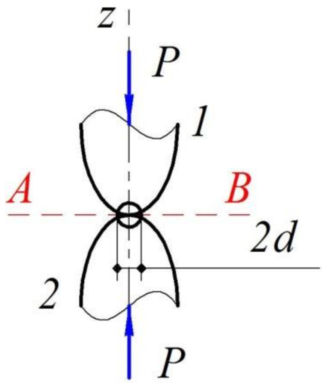


Figure 23.7 - The case of contact between two bodies of the same material

In general, the contact pad is an ellipse with semi-axes:

$$a = \alpha \cdot \sqrt[3]{\frac{3P(1-\mu^2)}{E\left(\frac{1}{\rho_1}+\frac{1}{\rho_1{}'}+\frac{1}{\rho_2}+\frac{1}{\rho_2{}'}\right)}}; \qquad (23.12)$$

$$b = \beta \cdot \sqrt[3]{\frac{3P(1-\mu^2)}{E\left(\frac{1}{\rho_1}+\frac{1}{\rho_1{}'}+\frac{1}{\rho_2}-\frac{1}{\rho_2{}'}\right)}}, \qquad (23.13)$$

where μ – Poisson's ratio.

The values of the coefficients α and β are given in Table 23.1 as a function of the auxiliary angle ψ, which is determined by the formula

$$cos\,\psi = \frac{\pm\sqrt{\left(\frac{1}{\rho_1}-\frac{1}{\rho_1{}'}\right)^2+\left(\frac{1}{\rho_2}-\frac{1}{\rho_2{}'}\right)^2+2\left(\frac{1}{\rho_1}-\frac{1}{\rho_1{}'}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho_2{}'}\right)cos2\varphi}}{\frac{1}{\rho_1}+\frac{1}{\rho_1{}'}+\frac{1}{\rho_2}+\frac{1}{\rho_2{}'}}. \qquad (23.14)$$

Table 23.1 - Values of the coefficients α and β

ψ, °	α	β	ψ, °	α	β
20	**3,778**	**0,408**	**60**	**1,486**	**0,717**
30	**2,731**	**0,493**	**65**	**1,378**	**0,759**
35	**2,397**	**0,530**	**70**	**1,284**	**0,802**
40	**2,136**	**0,567**	**75**	**1,202**	**0,846**
45	**1,926**	**0,604**	**80**	**1,128**	**0,893**
50	**1,754**	**0,641**	**85**	**1,061**	**0,944**
55	**1,611**	**0,678**	**90**	**1,000**	**1,000**

The sign of the numerator in formula (23.14) is chosen so that $cos\,\psi$ is positive.

The highest compressive stress is in the center of the contact pad

$$\sigma_{max} = 1{,}5\frac{P}{\pi ab}. \tag{23.15}$$

The largest tangential stress at the danger point is almost independent of the ratio of the site dimensions:

$$\tau_{max} \approx 0{,}32\sigma_{max}\,. \tag{23.16}$$

Contact stresses do not depend on the elastic properties of the materials. As the load increases, the rate of stress growth decreases, which is due to an increase in the size of the contact area.

We also give the values of the maximum compressive stress for the most common cases of contact between two bodies (Table 23.2).

Taking into account the "softness" of the stress state at dangerous points (all three main stresses are compressive), the strength at contact stresses should be tested according to the third or fourth theories of strength:

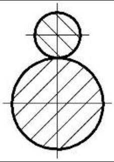
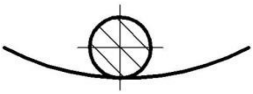
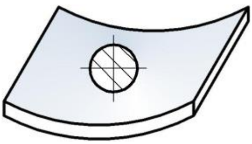
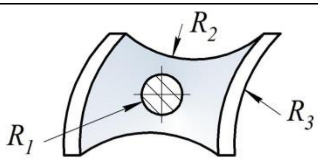
$$\sigma_{equi} = m\sigma_{max} \leq [\sigma],$$

whence

$$\sigma_{max} \leq \frac{1}{m}[\sigma] = [\sigma]_{cont,} \tag{23.18}$$

where m – safety factor.

Table 23.2 - Maximum compressive stress for different contact cases

The shape of bodies		σ_{max}
scheme	**parameters**	
	**two balls of radii R_1, R_2**	$0{,}388\sqrt[3]{PE^2\frac{(R_1+R_2)^2}{R_1^2R_2^2}}$
	**ball of radius R_1 in the hemisphere of radius R_2**	$0{,}388\sqrt[3]{PE^2\frac{(R_2-R_1)^2}{R_1^2R_2^2}}$
	**radius ball R_1 in a cylindrical radius trough R_2**	$\alpha\sqrt[3]{PE^2\frac{(R_2-R_1)^2}{R_1^2R_2^2}}$
	**bearing**	$\alpha\sqrt[3]{PE^2\frac{(R_2-R_1)^2}{R_1^2R_2^2}}$

Here $[\sigma]_{cont} = \frac{[\sigma]}{m}$ **– are the permissible values for the highest stress at the contact point.**

23.3 Strength conditions for contact stresses

The values of the coefficient ***m***, depending on the ratio of the semi-axes of the elliptical contact pad and the chosen strength theory, are given in Table 23.3. The values of the maximum allowable pressure on the contact pad are given in Table 23.4.

Table 23.3 - Values of the coefficient *m* depending on the ratio of the semi-axes of the elliptical contact pad and the selected strength theory

$\frac{b}{a}$	$m = \frac{\sigma_{equi\ \mathrm{III}}}{\sigma_{max}}$	$m = \frac{\sigma_{equi\ \mathrm{IV}}}{\sigma_{max}}$
1 (коло)	0,620	0,620
0,75	0,625	0,617
0,50	0,649	0,611
0,25	0,646	0,587
0 (strip)	0,600	0,557

Table 23.4 - Permissible maximum pressure on the contact pad

Metal grade	Temporary strength, σ_B, MPa	Brinell's hardness, HB	Maximum allowable pressure on the contact pad $[\sigma]_{cont}$, MPa
Steel:			
30	480-600	180	850-1050
40	570-700	200	1000-1350
50	630-800	300	1050-1400
50Г	650-850	240	1100-1450
15Х	620-750	240	1050-1600
20Х	700-850	240	1200-1450
15ХФ	1600-1800	240	1350-1600
ШХ15	–	–	
Cast iron:			
CI 21		180-207	800-900
CI 24		187-217	900-1000
CI 28		170-241	1000-1100
CI 32		170-241	1100-1200
CI 35		197-255	1200-1300
CI 38		197-255	1300-1400

The following procedure can be recommended for **calculating the strength of structural elements in contact:**

– determine the main radii of curvature of the contacting bodies and the angle φ between the main planes of curvature of one and the other body;

– calculate the dimensions of the semi-axes of the elliptical contact pad using formulas (23.12) and (23.13), taking into account formula (23.14);

– determine by formula (23.15) the greatest compressive stress in the center of the contact area. In the case of circular and rectangular contact pads, the dimensions are found directly from formulas (23.2) or (23.8) without determining the dimensions of the pad;

– the strength calculation is performed according to formula (23.18). The value of the coefficient m is taken from Table 23.3. It is recommended to proceed from the fourth theory of strength.

–

23.4 Control questions

1. What are contact strains and stresses? Give some examples.
2. Condition of strength under contact stresses.

RECOMMENDED LITERATURE

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Technical editor O. S. Omelchenko
Computer layout O. S. Omelchenko
Computer graphics A. A. Skrebtsov
Responsible for the publication V. G. Shevchenko

Publisher:
Publishing house “STATUS” Ltd.
Address of the editorial office: Ukraine, 69035, Zaporizhzhia, 158 Soborny avenue, office 249. tel. +38 (068) 448-11-28, mail@status.zp.ua http://status.zp.ua

Certificate of making a publishing business subject to the state register of publishers, manufacturers and distributors of publishing products series DK № 5316 dated 03/04/2017.

Sent to typesetting 12 • IX • 2024. Sent to the press 20 • XI • 2024.
Size 60x84 1/16. Digital printing. Conventional printed sheets 17,9.
Standard publisher's signatures 1,76. Printers sheet-copies 1790,25.
Print run 100 copies. Order No. 12 145/09.2024-A.

ISBN 978-617-8040-84-0

Printing by:
by the printing company FOP Yandola O.V.
Zaporizhzhia, st. Zhukovsky, 51, tel. 067-270-6000, www.copy.zp.ua

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Адреса редакції: Україна, 69057, м. Запоріжжя,
Соборний просп., буд. 158, оф. 249.
моб. +38 (068) 448-11-28, mail@status.zp.ua http://status.zp.ua

Свідоцтво про внесення суб'єкта видавничої справи до державного реєстру видавців, виготовлювачів і розповсюджувачів видавничої продукції ***серія ДК № 5316 від 03.04.2017***

Здано в набір 12•IX•2024. Підписано до друку 20•XI•2024. Формат 60х84 1/16.
Папір офсетний № 1. Гарнітура Таймз. Друк цифровий. Ум. друк. арк. 17,9.
Обл.-вид. арк. 1,76. Друк. арк. відбиток 1790,25. Наклад 100 прим.
Замовлення № 12 145/09.2024-А.

ISBN 978-617-8040-84-0

Віддруковано:
поліграфічним підприємством ФОП Яндола О.В.
м. Запоріжжя, вул. Жуковського, 51 , т. 067-270-6000, www.copy.zp.ua