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# APPROXIMATION OF FUNCTIONS OF TWO VARIABLES ON THE UNIT BICIRCLE IN GENERALIZED HÖLDER SPACES

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The approximative properties of interpolational Lagrange polynomials, Fourier series of functions of two variables which are defined on the unit bicircle, is stated. For functions from generalized Hölder spaces $H_\omega$ the estimations are conducted for norms of $C$, $H_\omega$, $L_p$, $p>1$ spaces.

Let $\gamma_{01}$ and $\gamma_{02}$ denote the unit circles with centers in the origin; and $\gamma_0=\gamma_{01}\times\gamma_{02}$ is the unit bicircle. Let $\omega(\delta_1,\delta_2)$ be the some modulus of continuity, and $\Omega_1(\delta)$, $\Omega_2(\delta)$ are the simple moduli of continuity, which respect to that one. Let $H_\omega$ denotes the space of continuous functions $x(t,\tau)$ at $\gamma_0$, which satisfy the conditions:

$$H(x;\omega)=\sup_{\delta_1^2+\delta_2^2\neq 0}\frac{\omega(\delta_1,\delta_2;x)}{\omega(\delta_1,\delta_2)}\leq c_1,$$

$$H^{t\tau}(x;\omega)=\sup_{\delta_1^2+\delta_2^2\neq 0}\frac{\omega_{1,1}(\delta_1,\delta_2;x)}{\Omega_1(\delta_1)\Omega_2(\delta_2)}\leq c_2.$$

Here $c_1$ and $c_2$ are the some constants; $\omega_{1,1}$ denotes the mixed modulus of continuity of the second order [1]. In the space $H_\omega$ the norm is defined as:

$$\|x(t,\tau)\|_{H_\omega}=\|x(t,\tau)\|_C+H(x;\omega)+H^{t\tau}(x;\omega).$$

Besides, we'll consider the other kinds of moduli of continuity (such as full, mixed and partial ones) and constants $H^t(x;\Omega_1)=\sup\limits_{\delta_1>0}\frac{\omega(\delta_1,0;x)}{\Omega_1(\delta_1)}$, $H^\tau(x;\Omega_2)=\sup\limits_{\delta_2>0}\frac{\omega(0,\delta_2;x)}{\Omega_2(\delta_2)}$, which are defined in [2]. Let $X_{mn}$ be the set of all polynomials of the form $x_{mn}(t,\tau)=\sum\limits_{k=-m}^{m}\sum\limits_{l=-n}^{n}c_{kl}t^k\tau^l$; $E_{mn}(x)$ is the best uniform approximation of $x(t,\tau)$ by the polynomials from $X_{mn}$. Let

$$\left(L_{mn}g\right)(t,\tau)=\sum_{p=0}^{2m}\sum_{q=0}^{2n}g\left(t_p,\tau_q\right)l_p(t)l_q(\tau)=\sum_{k=-m}^{m}\sum_{l=-n}^{n}\lambda_{kl}t^k\tau^l,$$

$$l_p(t)=\prod_{\substack{k=0\\k\neq p}}^{2m}\frac{t-t_k}{t_p-t_k}\left(\frac{t_p}{t}\right)^m=\sum_{k=-m}^{m}\lambda_{1,k}^{(p)}t^k,$$

$$l_q(\tau)=\prod_{\substack{j=0\\j\neq q}}^{2n}\frac{\tau-\tau_j}{\tau_q-\tau_j}\left(\frac{\tau_q}{\tau}\right)^n=\sum_{j=-n}^{n}\lambda_{2,j}^{(q)}\tau^j,$$

denotes interpolational Lagrange polynomial of the function $g(t,\tau)$ with respect to the system of equidistant points:

$$t_p=exp\left\{\frac{2\pi i}{2m+1}(p-m)\right\},\ i^2=-1,\ p=0,...,2m;$$

$$\tau_q=exp\left\{\frac{2\pi i}{2n+1}(q-n)\right\},\ i^2=-1,\ q=0,...,2n$$

at $\gamma_0$. Beside that we use the following estimations [1] for the functions $x(t,\tau)\in H_\omega$:

$$E_{mn}(x)\leq c\left[H^t(x;\Omega_1)\Omega_1\left(\frac{1}{m}\right)+H^\tau(x;\Omega_2)\Omega_2\left(\frac{1}{n}\right)\right]\leq cH(x;\omega)\omega\left(\frac{1}{m},\frac{1}{n}\right)$$

and for the derivatives of the polynomials $P_{mn}(t,\tau)\in X_{mn}$:

a) $\left\|\left(P_{mn}\right)_t'\right\|_C\leq cm\left\|P_{mn}\right\|_C$;

b) $\left\|\left(P_{mn}\right)_\tau'\right\|_C\leq cn\left\|P_{mn}\right\|_C$.

***Theorem 1.*** *Let $x(t,\tau)\in H_\omega$. Then:*

$$\left\|x-L_{mn}x\right\|_C\leq clnmlnnH(x;\omega)\omega\left(\frac{1}{m},\frac{1}{n}\right).$$

Let operator $\Phi_{mn}$ assignes to any function $g(t,\tau)\in H_\omega$ the $mn$-th partial sum of its Fourier series, that is

$$(\Phi_{mn}g)(t,\tau)=\sum_{k=-m}^{m}\sum_{l=-n}^{n}g_{kl}t^k\tau^l,$$

where

$$g_{kl} = \frac{1}{4\pi^2} \int_{\gamma_{01}} \int_{\gamma_{02}} \frac{g(t,\tau)}{t^{k+1}\tau^{l+1}} dt d\tau \,,\ k,l = 0, \pm 1, \pm 2, \ldots .$$

***Theorem 2.*** *For any function $g(t,\tau) \in H_\omega$ and for arbitrary natural $m$, $n$ the unequality is fulfilled:*

$$\left\| g - \Phi_{mn} g \right\|_C \le \left(1 + c ln(2m+1) ln(2n+1)\right) H(g;\omega)\omega\left(\frac{1}{m}, \frac{1}{n}\right).$$

***Lemma 1.*** *Let $P_{mn} \in X_{mn}$ at $\gamma_0$. Then for arbitrary pair $\Omega_1(\delta)$ and $\Omega_2(\delta)$ the estimations are true:*

a) $H^t\left(P_{mn};\Omega_1\right) \le c \dfrac{\left\| P_{mn} \right\|_C}{\Omega_1\left(\frac{1}{m}\right)}$,

b) $H^\tau\left(P_{mn};\Omega_2\right) \le c \dfrac{\left\| P_{mn} \right\|_C}{\Omega_2\left(\frac{1}{n}\right)}$.

In the future we'll assume that $m$, $n$ satisfy the conditions:

$$M_1 \le \frac{m}{n} \le M_2 ,$$

where $M_1$, $M_2$ are constants, $0 < M_1$, $M_2 < \infty$.

***Lemma 2.*** *Let $x^*_{mn}(t,\tau)$ be the polynomial of the best uniform approximation of the function $x(t,\tau) \in H_{\omega^{(1)}}$. Then for arbitrary $\omega^{(2)}(\delta_1,\delta_2)$ such that $H_{\omega^{(1)}} \subset H_{\omega^{(2)}}$ and $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$, $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ are increasing functions, the estimations are true:*

$$a)\ H^t\left(x - x^*_{mn};\Omega_1^{(2)}\right) \le c\left[ H^t\left(x;\Omega_1^{(1)}\right) \frac{\Omega_1^{(1)}\left(\frac{1}{m}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)} + H^\tau\left(x;\Omega_2^{(1)}\right) \frac{\Omega_2^{(1)}\left(\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{n}\right)} \right];$$

$$b)\ H^\tau\left(x - x^*_{mn};\Omega_2^{(2)}\right) \le c\left[ H^t\left(x;\Omega_1^{(1)}\right) \frac{\Omega_1^{(1)}\left(\frac{1}{m}\right)}{\Omega_2^{(2)}\left(\frac{1}{m}\right)} + H^\tau\left(x;\Omega_2^{(1)}\right) \frac{\Omega_2^{(1)}\left(\frac{1}{n}\right)}{\Omega_2^{(2)}\left(\frac{1}{n}\right)} \right].$$

***Lemma 3.*** *The unequality is true:*

$$\left\| L_{mn} \right\|_{H_\omega \to H_\omega} \le c\ln m \ln n\,.$$

The proof of this statement follows from [4] on the basis of the estimation:

$$\left\| L_{mn} \right\|_{H_\omega} \le c \left\| L_m \right\|_{H_{\Omega_1}} \cdot \left\| L_n \right\|_{H_{\Omega_2}} \le c\ln m \ln n\,.$$

***Lemma 4.*** *Under the conditions of Lemma 2 the estimations are true:*

$$a)\ H^t\left(x - L_{mn}x;\Omega_1^{(2)}\right) \le c\ln m\left[ H^t\left(x;\Omega_1^{(1)}\right)\frac{\Omega_1^{(1)}\left(\frac{1}{m}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)} + H^\tau\left(x;\Omega_2^{(1)}\right)\frac{\Omega_2^{(1)}\left(\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{n}\right)} \right];$$

$$b)\ H^\tau\left(x - L_{mn}x;\Omega_2^{(2)}\right) \le c\ln n\left[ H^t\left(x;\Omega_1^{(1)}\right)\frac{\Omega_1^{(1)}\left(\frac{1}{m}\right)}{\Omega_2^{(2)}\left(\frac{1}{m}\right)} + H^\tau\left(x;\Omega_2^{(1)}\right)\frac{\Omega_2^{(1)}\left(\frac{1}{n}\right)}{\Omega_2^{(2)}\left(\frac{1}{n}\right)} \right].$$

***Lemma 5.*** *Under the conditions of Lemma 2 the estimation is fulfilled:*

$$H^{t\tau}\left(x - x_{mn}^*;\omega^{(2)}\right) \le c\frac{\omega^{(1)}\left(\frac{1}{m},\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)\Omega_2^{(2)}\left(\frac{1}{n}\right)}\left\| x \right\|_{H_{\omega^{(1)}}}.$$

***Lemma 6.*** *Under the conditions of Lemma 2 the estimation is true:*

$$H^{t\tau}\left(x - L_{mn}x;\omega^{(2)}\right) \le c\ln m\ln n\frac{\omega^{(1)}\left(\frac{1}{m},\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)\Omega_2^{(2)}\left(\frac{1}{n}\right)}.$$

***Theorem 3.*** *Let* $x(t,\tau) \in H_{\omega^{(1)}}$; $\omega^{(2)}(\delta_1,\delta_2)$ *is such that* $H_{\omega^{(1)}} \subset H_{\omega^{(2)}}$ *and* $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$, $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ *are increasing functions. Then:*

$$\left\| x - L_{mn}x \right\|_{H_{\omega^{(2)}}} \le c\ln m\ln n\frac{\omega^{(1)}\left(\frac{1}{m},\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)\Omega_2^{(2)}\left(\frac{1}{n}\right)}\left\| x \right\|_{H_{\omega^{(1)}}}.$$

The proof of this statement follows directly from Lemmas 4, 6 and Theorem 1.

***Theorem 4.*** *Let* $x(t,\tau)\in H_{\omega^{(1)}}$*;* $\omega^{(2)}(\delta_1,\delta_2)$ *is such that* $H_{\omega^{(1)}}\subset H_{\omega^{(2)}}$ *and* $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$*,* $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ *are increasing functions.Then:*

$$\left\|x-\Phi_{mn}x\right\|_{H_{\omega^{(2)}}}\leq c\ln m\ln n\frac{\omega^{(1)}\left(\frac{1}{m},\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)\Omega_2^{(2)}\left(\frac{1}{n}\right)}\left\|x\right\|_{H_{\omega^{(1)}}}.$$

Let us now consider the space $L_p, p>1$. If put that Lagrange operator works from the space $C$ to the space $L_p$, then respectively to [3] the estimation $\left\|L_{mn}\right\|_{C\to L_p}\leq c(p)$ is true, where $c(p)$ is completely defined constant, which depends on $p$ and doesn't depend on $m$, $n$.

***Theorem 5.*** *Let* $g(t,\tau)\in H_\omega$*. Then the estimation is fulfilled :*

$$\left\|g-L_{mn}g\right\|_{L_p}\leq(1+c(p))H(g;\omega)\omega\left(\frac{1}{m},\frac{1}{n}\right).$$

***Remark 1.*** If under the conditions of Theorems 1 – 5 the partial derivatives of the approximated functions of orders $p$ and $q$ with respect to $t$ and $\tau$ belong to the space $H_\omega$ (it means $x(t,\tau)\in H_\omega^{(p,q)}$), that the estimations in the mentioned theorems can be considerably improved, namely by the value $\frac{1}{m^p}+\frac{1}{n^q}$. This conclusion follows directly from the unequalities [1]:

$$E_{mn}(x)\leq c\left[\frac{1}{m^p}H^t\left(\frac{\partial^p x}{\partial t^p};\Omega_1\right)\Omega_1\left(\frac{1}{m}\right)+\frac{1}{n^q}H^\tau\left(\frac{\partial^q x}{\partial\tau^q};\Omega_2\right)\Omega_2\left(\frac{1}{n}\right)\right]\leq$$

$$\leq c\left(\frac{1}{m^p}+\frac{1}{n^q}\right)H\left(\frac{\partial^{p+q}x}{\partial t^p\partial\tau^q};\omega\right)\omega\left(\frac{1}{m},\frac{1}{n}\right).$$

For example, in this case the theorem which is respected to Theorem 3 can be formulated as follows:

***Theorem 6.*** *Let* $x(t,\tau)\in H_{\omega^{(1)}}^{(p,q)}$*;* $\omega^{(2)}(\delta_1,\delta_2)$ *is such that* $H_{\omega^{(1)}}\subset H_{\omega^{(2)}}$ *and* $\Omega_1^{(1)}(\delta)/\Omega_1^{(2)}(\delta)$*,* $\Omega_2^{(1)}(\delta)/\Omega_2^{(2)}(\delta)$ *are increasing functions. Then the estimation is true*

$$\left\|x-L_{mn}x\right\|_{H_{\omega^{(2)}}} \le clnmlnn\left(\frac{1}{m^{p}}+\frac{1}{n^{q}}\right)\frac{\omega^{(1)}\left(\frac{1}{m},\frac{1}{n}\right)}{\Omega_1^{(2)}\left(\frac{1}{m}\right)\Omega_2^{(2)}\left(\frac{1}{n}\right)}\left\|x^{(p,q)}\right\|_{H_{\omega^{(1)}}}.$$

## References


1. Тиман А.Ф. Теория приближения функций действительного переменного. – М.: ГИФМЛ, 1965. – 624 с.
2. Сніжко Н.В. Класифікація узагальнених просторів Гельдера функцій двох змінних//Вісник Київського університету. Сер. фіз.-мат. науки. – 1999. – Вип № 3. – С. 124 – 128.
3. Иванов В.В. Теория приближенных методов и ее применение к численному решению сингулярных интегральных уравнений. – К.: Наукова думка, 1968. – 287 с.
4. Пилиди В.С., Сазонов Л.И. О бисингулярных интегральных операторах в простраствах гёльдеровских функций //ДАН СССР. – 1979. – Т. 246. – № 2. – С. 278 – 282